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A NOTE ON DIAGONAL AND MATRIX PROPERTIES OF CONVERGENCE SPACES

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1. Introduction

In the convergence spaces theory and its applications to analysis, probability theory and functional analysis an important role is played by the so-called diagonal and matrix properties. Usually we have in mind properties of the following form: Let X be a convergence space. If a sequence (A_n) of subsets A_n of X and a point x in X satisfy a certain condition (in terms of closures, convergent sequences, infinite series), then there exists a certain class of sequences (x_n) each of which converges to x. In case each A_n is a countably infinite set, the sequence (A_n) can be visualized as an infinite matrix and the sequences (x_n) are then usually taking at most finitely many points from each row and each column of the matrix.

The interested reader is referred to a survey [3], where several classification schemes for diagonal conditions are considered and some of the applications, as well as further references, are given.

In the present paper we investigate the condition

(+) *If* (A_n) *is a sequence of nonempty subsets and* x *is a point such that* $x \notin \bigcup_{n=1}^{\infty} \operatorname{cl} A_n$ *and each neighborhood of* x *contains points of* A_n *for all but finitely many* n, *then there exists a sequence of points* (x_n) *such that* $x_n \in A_n$ *and* (x_n) *converges to* x.

This condition was introduced by J. Novák in [6], where he put forward the following

Problem 1. *Let* X *be a convergence space.*

a) What are the necessary and sufficient conditions such that (+) *is true?*

b) Does there exist a convergence space such that its convergence is a star convergence and such that (+) *is not true?*

c) Is (+) *true if the convergence is a star convergence and* X *is first countable?*

Part *b*) of Problem 1 has been solved by M. Contessa and F. Zanolin in [1], via a rather complicated construction involving maximal almost disjoint families of infinite subsets of natural numbers. In the third section we present a "naive" example answering *b*) positively. Further, we prove that the answer to *c*) is "YES". These two results were already announced in [2]. We do not know about any solution of *a*).

The last section contains some additional results concerning condition (+).

2. Basics

For the reader's conven ence, we recall in this section some of the basic notions used throughout the paper.

Let X be a nonempty set and X^N the set of all sequences ranging in X. By (x_n) we denote the sequence the n-th term of which is x_n, for each $x \in X$ we denote by (x) the constant sequence generated by x and by $\{x\}$ we denote the one-point set containing x. Recall that a multivalued sequential convergence for X is a subset $\mathfrak{L} \subset X^N \times X$. We assume the following two axioms of convergence:

($\mathscr{L}_1$) *For each* $x \in X$ *we have* $((x), x) \in \mathfrak{L}$;

($\mathscr{L}_2$) *If* $((x_n), x) \in \mathfrak{L}$, *then* $((x_{i_n}), x) \in \mathfrak{L}$ *for each subsequence* (x_{i_n}) *of* (x_n).

We say that a sequence (x_n) $\mathfrak{L}$-converges (or simply converges) to a point x whenever $((x_n), x) \in \mathfrak{L}$. For each subset A of X its $\mathfrak{L}$-closure (or simply closure) $\operatorname{cl} A$ is defined as the set of all points of X to which some sequence ranging in A $\mathfrak{L}$-converges. The closure operator induced by $\mathfrak{L}$ need not be idempotent. The set X equipped with $\mathfrak{L}$ and the induced closure operator is called a convergence space. If the closure is idempotent (i.e. $\operatorname{cl} A = \operatorname{cl}(\operatorname{cl} A)$ for each subset A of X), then we speak of a Fréchet space.

Let X be a closure space. We say that a sequence (x_n) converges to a point x iff each neighborhood of x contains x_n for all but finitely many $n \in N$. The resulting convergence $\mathfrak{L}$ (ca led associated) satisfies axioms ($\mathscr{L}_1$), ($\mathscr{L}_2$) and the axiom

($\mathscr{L}_3$). *Let* (x_n) *be a sequence and let* x *be a point of* X. *If for each subsequence* (x'_n) *of* (x_n) *there exists a subsequence* (x''_n) *of* (x'_n) *such that* $((x''_n), x) \in \mathfrak{L}$, *then* $((x_n), x) \in \mathfrak{L}$.

It may happen that a sequence in X converges to two different points.

Throughout the paper, in every convergence or closure space, the convergence of sequences will satisfy the following axiom

($\mathscr{L}_0$) *If* $((x_n), x) \in \mathfrak{L}$ *and* $((x_n), y) \in \mathfrak{L}$, *then* $x = y$.

Let $\mathfrak{L}$ be a convergence for X. Then $\mathfrak{L}$ can be enlarged to a convergence $\mathfrak{L}^*$ satisfying ($\mathscr{L}_3$), the $\mathfrak{L}$-closure and the $\mathfrak{L}^*$-closure are identical and, in fact, $\mathfrak{L}^*$ is the convergence associated with the $\mathfrak{L}$-closure. If $\mathfrak{L} = \mathfrak{L}^*$, then we speak of a star convergence and the resulting convergence is said to be a star convergence space.

Let X be a star convergence space. As proved in [4], the free commutative group FC(X), with points of X as its generators, can be equipped with a star convergence such that the group operations are sequentially continuous and X is a closed subspace of FC(X). We say that FC(X) is the free commutative convergence group over X.

3. Main results

Example 3.1. Let X be a countably infinite set. Consider X as a disjoint union of a point x and two infinite sets A and B. Arrange A into a one-to-one double sequence (x_{mn}) and B into a one-to-one double sequence (y_{mn}). We equip X with a topology as follows. All points of $A \cup B$ are isolated. For each natural number k and for each mapping f of the set N of all natural numbers into N define

$$U(f, k) = \{x\} \cup \left(\bigcup_{m=1}^{\infty} \bigcup_{n=f(m)}^{\infty} \{x_{mn}\} \right) \cup \left(\bigcup_{m=k}^{\infty} \bigcup_{n=1}^{\infty} \{y_{mn}\} \right).$$

All sets $U(f, k)$ form a local base at x. It is easy to see that X is a normal topological space. Further, let $\mathfrak{L}$ be the associated convergence of sequences. We omit the easy proof of the next proposition.

Proposition 3.1.1. *(i) $\mathfrak{L}$ satisfies the axioms $(\mathscr{L}_i)$, $i = 0, 1, 2, 3$.*

(ii) The $\mathfrak{L}$-closure is idempotent and coincides with the original topological closure of the space X.

Observe that for each natural number m the sequence (x_{mn}) converges to x and for each mapping f of N into N the sequence $(y_{mf(m)})$ converges to x. The space X is in fact the quotient of the disjoint topological sum of the spaces L_1 and L_2 from Example 5 in [5], where we identify points $x_0 \in L_1$ and $y_0 \in L_2$ into x.

Proposition 3.1.2. *The space X does not satisfy condition* $(+)$.

Proof. Consider the two-point sets $\{x_{ij}, y_{ij}\}$ and arrange them into a equence (A_n) in a diagonal way, i.e., $A_1 = \{x_{11}, y_{11}\}$, $A_2 = \{x_{21}, y_{21}\}$, $A_3 = \{x_{12}, y_{12}\}$, $A_4 = \{x_{31}, y_{31}\}$, $A_5 = \{x_{22}, y_{22}\}$, $A_6 = \{x_{13}, y_{13}\}, \ldots$. Each set A_k is closed and hence $x \notin \bigcup_{n=1}^{\infty} \operatorname{cl} A_n$. Clearly, each neighborhood of x contains points of A_n for all but finitely many $n \in N$. Let (x_n) be a sequence of points of X such that $x_n \in A_n$. Then either there exists a natural number m such that the set $\left(\bigcup_{n=1}^{\infty} \{y_{mn}\} \right) \cap \left(\bigcup_{n=1}^{\infty} \{x_n\} \right)$ is infinite, or there exists a mapping f of N into N such that the set $\left(\bigcup_{m=1}^{\infty}\{x_{mf(m)}\}\right) \cap \left(\bigcup_{n=1}^{\infty}\{x_n\}\right)$ is infinite. Consequently, $x_n \notin U(f+1, m+1)$ for infinitely many $n \in N$. Hence the sequence (x_n) cannot converge in X to x. This completes the proof.

Corollary 3.2. *There exists a star convergence space which does not satisfy condition* $(+)$.

Proof. The assertion follows directly from Proposition 3.1.1 and Proposition 3.1.2.

Corollary 3.3. *There exists a commutative star convergence group which does not satisfy condition* $(+)$.

Proof. Let X be any star convergence space which does not satisfy condition $(+)$. Let $\mathrm{FC}(X)$ be the free commutative convergence group over X. Since X is a closed subspace of $\mathrm{FC}(X)$, the convergence group $\mathrm{FC}(X)$ has the properties claimed in Corollary 3.3.

Proposition 3.4. *Let X be a first countable star convergence space. Then X satisfies condition* $(+)$.

Proof. let x be a point of X and let (A_n) be a sequence of nonempty subsets of X such that $x \notin \bigcup_{n=1}^{\infty} \mathrm{cl}\, A_n$ and each neighborhood of x contains points of A_n for all but finitely many $n \in N$. Let $\{U_n;\ n \in N\}$ be a nonincreasing local base at x such that $U_1 \subset X \setminus A_1$. Put $I_n = \{i \in N;\ A_i \cap U_n = \emptyset\}$. Then $\emptyset \neq \neq I_n \subset I_{n+1} \neq N$ and $N = \bigcup_{n=1}^{\infty} I_n$. Let (V_n) be a subsequence of (U_n), which we obtain by leaving out those U_n for which $I_n = I_{n-1}$, $n > 1$. Then also $\{V_n;\ n \in N\}$ is a local base at x. Put $J_1 = I_1$ and $J_n = \{i \in N;\ A_i \cap V_n = \emptyset\}$ for $n > 1$. Let f be a one-to-one mapping of N onto N such that for each $i \in J_{n+1} \setminus J_n$ and for each $j \in J_n$ we have $f(j) < f(i)$, i.e., elements of J_1 are mapped onto $\{1, \ldots, k\}$, where k is the number of elements of J_1 and then, inductively, if all elements of J_n are already mapped onto $\{1, \ldots, l\}$, where l is the number of elements of J_n, then elements of $J_{n+1} \setminus J_n$ are mapped onto $\{l+1, \ldots, l+m\}$, where m is the number of elements of $J_{n+1} \setminus J_n$. Denote $B_s = A_{f^{-1}(s)}$. Let k_n be the number of elements of J_n. Then (k_n) is an increasing sequence of natural numbers and $V_n \cap B_m \neq \emptyset$ for $m > k_n$. Now we are going to construct a sequence (b_n) of points $b_n \in B_n$ inductively. Choose $b_1 \in B_1, \ldots, b_{k_1} \in B_{k_1}$ arbitrarily. Assume that points $b_1 \in B_1, \ldots, b_{k_n} \in B_{k_n}$ are already chosen. Then choose points $b_{k_n+1} \in V_n \cap B_{k_n+1} \neq \neq \emptyset, \ldots, b_{k_{n+1}} \in V_n \cap B_{k_{n+1}} \neq \emptyset$. Since each neighborhood V_k of x contains b_n for all but finitely many $n \in N$, the sequence (b_n) converges in X to x. Put $x_n = b_{f(n)}$.

Then $x_n \in A_n$. Since X is a star convergence space, the sequence (x_n) converges in X to x. This completes the proof.

Proposition 3.5. *Let X be a convergence space which satisfies condition $(+)$. Let x be a point of X and let (x_n) be a sequence of points of* X *such that $x \neq x_n$ for all $n \in N$ and let there exist for each subsequence (x'_n) of (x_n) a subsequence (x''_n) of (x'_n) such that (x''_n) converges in X to x. Then (x_n) converges in X to x.*

Proof. Put $A_n = \{x_n\}$. clearly, the assumptions of $(+)$ are satisfied and hence (x_n) converges in X to x.

4. Further results

Example 4.1. Let X be a countably infinite set. Choose $z \in X$ and arrange the set $X \setminus \{z\}$ into a one-to-one sequence (z_n). Define $\mathfrak{L} \subset X^N \times X$ as follows: $((x_n), x) \in \mathfrak{L}$ if either $x_n = x$ for all $n \in N$ or $x = z$ and (x_n) is a finite-to-one sequence of points of $X \setminus \{z\}$.

Proposition 4.1.1. (*i*) *$\mathfrak{L}$ satisfies axioms $(\mathscr{L}_i)$, $i = 0, 1, 2$.*

(*ii*) *The $\mathfrak{L}$-closure is idempotent.*

(*iii*) *X equipped with $\mathfrak{L}$ and the $\mathfrak{L}$-closure is a first countable normal Fréchet space.*

(*iv*) *$\mathfrak{L}$ does not satisfy axiom $(\mathscr{L}_3)$.*

Proof. A straightforward proof of (*i*), (*ii*) and (*iii*) is omitted. To prove (*iv*), consider the sequence (x_n) defined as follows: $x_{2n-1} = z$ and $x_{2n} = z_n$ for all $n \in N$. Each subsequence of (x_n) contains a sequence $\mathfrak{L}$-converging to z but the sequence (x_n) does not converge in X to z.

Proposition 4.1.2. *X satisfies condition $(+)$.*

Proof. Let x be a point of X and let (A_n) be a sequence of nonempty subsets of X such that $x \notin \bigcup_{n=1}^{\infty} \operatorname{cl} A_n$ and each neighborhood of x contains points of A_n for all but finitely many $n \in N$. Since z is the only nonisolated point of X, we have $x = z$. Further, each A_n is a finite subset of $X \setminus \{z\}$. For each $k \in N$, the set $\bigcup_{i=1}^{k} A_i$ is a finite subset of $X \setminus \{z\}$. Since $X \setminus \bigcup_{i=1}^{k} A_i$ is a neighborhood of x and contains points of A_n for all but finitely myny $n \in N$, there exists a finite-to-one sequence (x_n) of points $x_n \in A_n$. clearly, the sequence (x_n) converges in X to x. This completes the proof.

It would be interesting to find more about the relationship between condition $(+)$ and other diagonal conditions listed in [3]. This could lead to the solution of part *a*) of Problem 1.

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ЗАМЕЧАНИЕ ОБ ДИАГОНАЛЬНЫХ И МАТРИЧНЫХ СВОЙСТВАХ ПРОСТРАНСТВ СХОДИМОСТИ

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Резюме

В работе исследуется некоторое диагональное условие в пространствах сходимости. Частично решается некоторая проблема, заданная Й. Новаком на Канпурской топологической конференции в 1968 г. и касающаяся этого условия.