Ján Jakubík

Graph isomorphisms of semimodular lattices


Persistent URL: http://dml.cz/dmlcz/129308

Terms of use:

© Mathematical Institute of the Slovak Academy of Sciences, 1985

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
GRAPH ISOMORPHISMS OF SEMIMODULAR LATTICES

JÁN JAKUBÍK

This note is a continuation of a former paper of the author [4], where it was proved that a condition concerning sublattices of type \( C \) (for denotations, cf. below) is sufficient for semimodular lattices \( \mathcal{L} \) and \( \mathcal{L}_1 \) of locally finite length with isomorphic graphs to have direct product representations \( f: \mathcal{L} \rightarrow \mathcal{L} \times \mathcal{L} \) and \( g: \mathcal{L}_1 \rightarrow \mathcal{L}_1 \times \mathcal{L}_1 \) such that \( h = g^{-1}f \) (where \( \mathcal{L} \) is dual to \( \mathcal{L} \) and \( h \) is the given graph isomorphism of \( \mathcal{L} \) onto \( \mathcal{L}_1 \)).

In the present paper it will be shown that the condition concerning sublattices of type \( C \) is also necessary for the existence of such direct product representations. A further result on graph isomorphisms of semimodular lattices (dealing with sublattices of type \( C_1 \)) is established.

Graph isomorphisms of distributive lattices were studied in [7]; for the case of modular lattices cf. Birkhoff [1] and the author [3], [5].

We recall some notions of graphs of lattices. Let \( \mathcal{L} = (L; \leq) \) be a lattice. \( \mathcal{L} \) is said to be of locally finite length if each bounded chain in \( \mathcal{L} \) is finite. In what follows all lattices are assumed to be of locally finite length. If \( a, b \in L \) and \( a \) is covered by \( b \) (i.e., \( a < b \) and the interval \([a, b]\) is prime), then we write \( a < b \) or \( b > a \). The lattice \( \mathcal{L} \) is called semimodular if and only if its elements satisfy

\[(\xi') \text{ If } x \text{ and } y \text{ cover } a, \text{ and } x \neq y, \text{ then } x \vee y \text{ covers } x \text{ and } y. \text{ (Cf. [2a], p. 100; in [2b], p. 15, the term 'semimodularity' has a different meaning.)}\]

By the graph \( G(\mathcal{L}) \) we mean the undirected graph whose set of vertices is \( L \) and whose edges are those pairs \( \{a, b\} \) which satisfy either \( a < b \) or \( b < a \). If \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are graphs with sets of vertices \( G_1 \) and \( G_2 \) and if \( h: G_1 \rightarrow G_2 \) is a bijection such that, for any \( x \) and \( y \) from \( G_1 \) the pair \( \{x, y\} \) is an edge in \( \mathcal{G}_1 \) if and only if \( \{h(x), h(y)\} \) is an edge in \( \mathcal{G}_2 \), then \( h \) is said to be an isomorphism of \( \mathcal{G}_1 \) onto \( \mathcal{G}_2 \).

If \( \mathcal{L}_1 = (\wedge_1; \vee_1) \) is a lattice and \( h \) is an isomorphism of \( G(\mathcal{L}) \) onto \( G(\mathcal{L}_1) \), then \( h \) is called a graph isomorphism of the lattice \( \mathcal{L} \) onto \( \mathcal{L}_1 \). The covering relation in \( \mathcal{L}_1 \) is denoted by \( \prec_1 \).

Now let \( h: \mathcal{L} \rightarrow \mathcal{L}_1 \) be any bijection and let \( T \subseteq L \). The subset \( T \) is said to be preserved (reversed) under \( h \) if, whenever \( t_1, t_2 \in T, x_1, x_2 \in L \) and \( t_1 \leq x_1 < x_2 \leq t_2 \), then \( h(x_1) \prec_1 h(x_2) \) (or \( h(x_1) >_{1nT(x_2)} \), respectively).
Let \( C \) be the lattice in Fig. 1. A lattice is said to be of type \( C \) if it is isomorphic to \( C \). Consider the following conditions for the lattices \( L \) and \( L_1 \) and for the mapping \( h \):

(a_1) All sublattices of type \( C \) of \( L \) are preserved under \( h \) and all sublattices of type \( C \) of \( L_1 \) are preserved under \( h^{-1} \).

(a_2) There are lattices \( A \) and \( B \) and direct product representations \( f: L \to A \times B, g: L_1 \to A \times B \) such that \( h = g^{-1}f \).

The following result was proved in [4]:

\( A \) ([4], Theorem 2.) Let \( L \) and \( L_1 \) be semimodular lattices and let \( h \) be a graph isomorphism of \( L \) onto \( L_1 \). Then \( (a_1) \Rightarrow (a_2) \).

(In [4] it was assumed that \( L \) and \( L_1 \) are finite, but the proof established in [4] remains valid in the case when \( L \) and \( L_1 \) are of locally finite length. Also, in Thm. 2 of [4] it was asserted only that there are lattices \( A \) and \( B \) such that \( L \cong A \times B \) and \( L_1 \cong A \times B^{-} \); but, in fact, the stronger result \( (a_1) \Rightarrow (a_2) \) was proved in [4]. If \( (a_2) \) holds, then \( h \) is a graph isomorphism of \( L \) onto \( L_1 \).)

1. **Lemma.** Let \( \mathcal{L} = (T; \leq) \) be a lattice of type \( C \). Then \( \mathcal{L} \) is subdirectly irreducible.

The proof is simple; it will be omitted.

Now let \( L, L_1 \) and \( h \) be as above. Assume that \( (a_2) \) holds. We denote \( A = (A; \leq), B = (B; \leq) \). In view of the assumption, there exists an isomorphism \( f \) of \( L \) onto \( A \times B \). If \( x \in L \) and \( f(x) = (a, b) \), then we write also \( a = x(A), b = x(B) \). For \( M \subseteq L \) we put \( M(A) = \{ x(A): x \in M \}, M(B) = \{ x(B): x \in M \} \).

2. **Lemma.** Let \( \mathcal{L} = (T; \leq) \) be a sublattice of \( \mathcal{L} \) and suppose that \( \mathcal{L} \) is of \( \mathcal{C} \). Then we have either (i) \( \text{card} T(A) = 1 \), or (ii) \( \text{card} T(B) = 1 \).

Proof. Put \( \mathcal{T}_1 = (T(A); \leq), \mathcal{T}_2 = (T(B); \leq) \). The injection defined by \( f|_{\mathcal{T}}: \mathcal{T} \to \mathcal{T}_1 \times \mathcal{T}_2 \) is a subdirect product representation of \( \mathcal{T} \); in view of Lemma 1 we infer that either (i) of (ii) is valid.
If (i) holds, then clearly $T$ is reversed under $f$; if (ii) is valid, then $T$ is preserved under $f$.

3. Lemma. Let $\mathcal{L}$ and $\mathcal{L}_1$ be semimodular lattices. Then $(\alpha_2) \Rightarrow (\alpha_1)$.

Proof. Let $h: \mathcal{L} \rightarrow \mathcal{L}_1$ be a bijection. Assume that $(\alpha_2)$ is valid. Then $h = g^{-1}f$, and as already remarked above, $h$ is a graph isomorphism. By way of contradiction, suppose that there is a sublattice $\mathcal{T}$ in $\mathcal{L}$ such that $\mathcal{T}$ is of type $C$ and $T$ is not preserved under $h$. (If in this supposition $\mathcal{L}$ and $\mathcal{H}$ are replaced by $\mathcal{L}_1$ and $h^{-1}$, then we proceed analogously.) Thus the condition (i) of Lemma 2 holds and hence $\mathcal{T}$ is reversed under $h$. Also, from $(\alpha_2)$ we easily obtain that $(h(T); \preceq_1) = (\mathcal{T}_1)$ is a sublattice of $\mathcal{L}_1$ which is dually isomorphic to $C$. By using [8], § 45 it is easy to verify that $\mathcal{L}_1$ is not semimodular, which is a contradiction.

Theorem (A) and Lemma 3 yield:

4. Theorem. Let $\mathcal{L}$ and $\mathcal{L}_1$ be semimodular lattices and let $h$ be a graph isomorphism of $\mathcal{L}$ onto $\mathcal{L}_1$. Then the conditions $(\alpha_1)$ and $(\alpha_2)$ are equivalent.

Let $\mathcal{T} = (T; \preceq)$ be a sublattice of a lattice $\mathcal{L} = (L; \preceq)$. Assume that there exists an isomorphism $\varphi$ of $\mathcal{C}$ onto $\mathcal{T}$ such that $\varphi(u) < \varphi(x_1) < \varphi(v)$, $\varphi(u) < \varphi(y_1) < \varphi(v)$, $\varphi(x) < \varphi(z)$ and $\varphi(y) < \varphi(z)$. Then $\mathcal{T}$ will be called a $C_1$-sublattice of $\mathcal{L}$. If, moreover, $\varphi(x_1) < \varphi(x)$, $\varphi(v) < \varphi(z)$ and $\varphi(y_1) < \varphi(y)$, then $\mathcal{T}$ is said to be a $C_2$-sublattice of $\mathcal{L}$.

Let $\mathcal{L}_1 = (L_1; \preceq_1)$ be a lattice and let $h: \mathcal{L} \rightarrow \mathcal{L}_1$ be a bijection. Consider the following conditions $(i = 1, 2)$:

$(\alpha_{1i})$ All $C_i$-sublattices of $\mathcal{L}$ are preserved under $h$ and all $C_i$-sublattices of $\mathcal{L}_1$ are preserved under $h^{-1}$.

Let $u, v, x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ be distinct elements of $L$ such that $u < x_1 < x_2 < \ldots < x_m < v$, $u < y_1 < y_2 < \ldots < y_n < v$ and either (i) $x_1 \vee y_1 = v$, or (ii) $x_m \land y_n = u$. Then the set $\{u, v, x_1, \ldots, x_m, y_1, \ldots, y_n\}$ is said to be a cycle in $\mathcal{L}$; if moreover, $m > 1$ or $n > 1$, then this cycle is called proper.

From [6] (Thm. 3.7 and Lemma 2.3) we obtain:

5. Lemma. Let $\mathcal{L}$ and $\mathcal{L}_1$ be lattices and let $h$ be a graph isomorphism of $\mathcal{L}$ onto $\mathcal{L}_1$. Then the condition $(\alpha_2)$ is equivalent with the condition

$(\alpha_3)$ if $C_0$ is a proper cycle of $\mathcal{L}$ (of $\mathcal{L}_1$), then $C_0$ is either preserved or reversed under $h$ (or $h^{-1}$, respectively).

6. Lemma. Let $\mathcal{L}$ and $\mathcal{L}_1$ be semimodular lattices and let $h$ be a graph isomorphism of $\mathcal{L}$ onto $\mathcal{L}_1$. Then $(\alpha_{1i}) \Rightarrow (\alpha_2)$.

Proof. In establishing the proof of Theorem 2 in [4] the condition $(\alpha_1)$ was used in the proofs of the lemmas 9 and 10 only; now for proving that $(\alpha_{1i}) \Rightarrow (\alpha_2)$ is valid it suffices to replace the expression ‘a lattice of type $C$’ by ‘a $C_1$-sublattice’ in these lemmas.

7. Lemma. Let $\mathcal{L}$ and $\mathcal{L}_1$ be semimodular lattices and let $h$ be a graph isomorphism of $\mathcal{L}$ onto $\mathcal{L}_1$. Then $(\alpha_2) \Rightarrow (\alpha_{1i})$.

231
Proof. According to Lemma 3 we have \((\alpha_2) \Rightarrow (\alpha_1)\), and clearly \((\alpha_1) \Rightarrow (\alpha_{11})\).

Alternative proof: Let \(\mathcal{I}\) be a \(C_1\)-sublattice of \(\mathcal{L}\). Under the denotations as above, there exist elements \(a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n \in L\) such that \(q(x_1) = a_0 < a_1 < \ldots < a_m = q(x), q(y_i) = b_0 < b_1 < b_2 < \ldots < b_n = q(y)\). Then \(\{q(u), q(z), a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n\}\) is a proper cycle in \(\mathcal{L}\) (because \(a_m \land b_n = q(u)\)). Hence in view of Lemma 5, the interval \(J = [q(u), q(z)]\) is either preserved or reversed under \(h\). If \(J\) is reversed under \(h\), then we easily obtain from \((\alpha_2)\) that \(h^{-1}\) is a dual isomorphism of \(J\) onto the interval \([h(q(z)), h(q(u))]\) of \(\mathcal{L}_1\), but this interval fails to be semimodular; thus \(\mathcal{L}_1\) is not semimodular, which is a contradiction. Hence \(T\) is preserved under \(h\). Analogously we verify that each \(C_1\)-sublattice of \(\mathcal{I}_1\) is preserved under \(h^{-1}\).

Theorem 4, Lemma 6 and Lemma 7 yield:

8. Corollary. Let \(\mathcal{L}\) and \(\mathcal{L}_1\) be semimodular lattices and let \(h\) be a graph isomorphism of \(\mathcal{L}\) onto \(\mathcal{L}_1\). Then \((\alpha_2) \iff (\alpha_{11}) \iff (\alpha_1)\).

The following question remains open:
Let \(\mathcal{L}, \mathcal{L}_1\) and \(h\) be as in Corollary 8; are the conditions \((\alpha_2)\) and \((\alpha_{12})\) equivalent?

REFERENCES


Received February 9, 1983

Katedra matematiky VŠT
Švermova 9
040 02 Košice

ИЗОМОРИФИЗМЫ ГРАФОВ ПОЛУДЕКИНДОВЫХ РЕШЕТОК

Ján Jakubík

На странице автора [4] найдено достаточное условие, при котором полу декиндины решетки \(\mathcal{L}\) и \(\mathcal{L}_1\) локально конечной длины с изоморфными графами отличаются только двойственностью некоторого прямого сомножителя; в предлагаемой заметке доказано, что это условие является тоже необходимым.