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INEQUALITIES FOR THE LANDAU CONSTANTS

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ABSTRACT. A new expansion for the nth Landau constant G_n, involving the digamma function ψ, leads to the sharp double inequality

$$1.0663 < G_n - (1/\pi)\psi(n+5/4) < 1.0724\,.$$

1. Introduction

The normalized binomial middle coefficients, μ_i, can be variously defined in terms of the familiar binomial coefficients and factorials as

$$\mu_i = \frac{1}{2^{2i}}\binom{2i}{i} = (-1)^i\binom{-1/2}{i} = \frac{(2i)!}{(2^i i!)^2}\,, \qquad i = 0,1,2,\dots\,, \tag{1a}$$

or as

$$\mu_0 = 1\,, \qquad \mu_i = \frac{(2i-1)!!}{(2i)!!}\,, \quad i = 1,2,3,\dots\,, \tag{1b}$$

in terms of double factorials, which for $n = 1,2,3,\dots$ are given by

$$(2n)!! = 2\cdot 4\cdot 6\cdots(2n)\,, \qquad (2n-1)!! = 1\cdot 3\cdot 5\cdots(2n-1)$$

with $0!! = 1$. The sum

$$G_n = \sum_{i=0}^{n}\mu_i^2 = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 + \dots + \left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right)^2 \tag{2}$$

is known as the nth Landau constant ([2]–[3], [7]–[8]). It was proved ([7]) that, if a function $f(z)$, which is analytic throughout the interior of the unit circle and expandable in the Taylor series

$$f(z) = \sum_{i=0}^{\infty} a_i z^i$$

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with $|f(z)| < 1$ whenever $|z| < 1$, then

$$\left|\sum_{i=0}^{n} a_i\right| \le G_n .$$

Moreover, if $T_n(f)$ is a polynomial operator associated to $f(z)$, then its norm is given by $\|T_n\| = G_n$.

An investigation of the asymptotic behaviour of G_n was begun by Landau [7], who established that $G_n \sim (1/\pi)\log n$. Further, Watson [9] proved that

$$G_n = \frac{1}{\pi}\log(n+1) + A - \varepsilon_n \tag{3a}$$

where

$$A = \frac{1}{\pi}(\gamma + 4\log 2) \approx 1.0663 \qquad \varepsilon_n \to 0 \quad (n \to \infty) \tag{3b}$$

and $\gamma \approx 0.5772$ is Euler's constant. This expansion was used in obtaining the double inequality ([2])

$$1 + \frac{1}{\pi}\log(n+1) < G_n \le 1.0663 + \frac{1}{\pi}\log(n+1), \qquad n = 0, 1, 2, \dots , \tag{4a}$$

which was sharpened [3] to

$$1.0663 < G_n - \frac{1}{\pi}\log(n+0.75) \le 1.0916, \qquad n = 0, 1, 2, \dots . \tag{4b}$$

In this note we give a new expansion of G_n which allows a new sharp estimate of the Landau constants.

2. Expansion and inequalities for G_n

In what follows, ψ and ${}_2F_1$ designate the digamma function and the hypergeometric function, respectively. The digamma function is given by [1; p. 258, Eq. 6.3.1]

$$\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

where Γ is the gamma function. The hypergeometric function has the following series representation ([1; p. 556, Eq. 15.1.1])

$${}_2F_1\left[\begin{matrix} a, & b; & \\ & & x \\ & c; & \end{matrix}\right] = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r} \frac{x^r}{r!} \tag{5a}$$

where Pochhammer's symbol is defined by ([1; p. 256, Eq. 6.1.22])

$$(x)_n = \begin{cases} 1\,, & n = 0\,, \\ x(x+1)\cdots(x+n-1)\,, & n = 1, 2, 3, \ldots \end{cases} \\ = \frac{\Gamma(x+n)}{\Gamma(x)} \tag{5b}$$

and where the denominator parameter c is not allowed to be zero or a negative integer. Then, the series concerned converges absolutely for $|x| < 1$ for all values of its parameters, and also when $x = 1$, provided that $c - a - b > 0$.

THEOREM 1. *The Landau constants* G_n *have the following expansion*

$$G_n = \frac{1}{\pi}\psi\left(n + \frac{3}{2}\right) + A - \alpha_n \tag{6a}$$

where A *is a constant defined by* (3b) *and*

$$\alpha_n = \frac{1}{\pi}\sum_{r=1}^{\infty} \frac{(1/2)_r (1/2)_r}{r(n+3/2)_r r!}\,. \tag{6b}$$

P r o o f. Making use of the following relationship ([1; p. 256, Eq. 6.1.21]) between the binomial coefficients and the gamma function

$$\binom{x}{n} = \frac{1}{\Gamma(n+1)} \frac{\Gamma(x+1)}{\Gamma(x-n+1)}$$

the duplication formula for the gamma function ([1; p. 256, Eq. 6.1.18])

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma(x+1/2)$$

it follows from the first expression in the definition (1a) of μ_i, that

$$\mu_i = \frac{1}{\sqrt{\pi}} \frac{\Gamma(i+1/2)}{\Gamma(i+1)}\,, \qquad i = 0, 1, 2, \ldots\,.$$

Further, this result readily leads to

$$\mu_i^2 = \frac{1}{\pi} \frac{1}{i+1/2}\, {}_2F_1\left[\begin{matrix} 1/2, & 1/2; & \\ & & 1 \\ & i+3/2; & \end{matrix}\right], \qquad i = 0, 1, 2, \ldots\,, \tag{7a}$$

which can be verified by Gauss's summation formula for the hypergeometric function of unit argument ([1; p. 556, Eq. 15.1.20])

$${}_2F_1\left[\begin{matrix} a, & b; & \\ & & 1 \\ & c; & \end{matrix}\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,,$$

$$c \neq 0, -1, -2, \ldots\,, \qquad c - a - b > 0\,.$$

Equivalently, after replacing ${}_2F_1$ with its series representation (5a) we have

$$\mu_i^2 = \frac{1}{\pi}\frac{1}{i+1/2}\sum_{r=0}^{\infty}\frac{(1/2)_r(1/2)_r}{(i+3/2)_r}\frac{1}{r!} = \frac{1}{\pi}\frac{1}{i+1/2}\left[1+\sum_{r=1}^{\infty}\frac{(1/2)_r(1/2)_r}{(i+3/2)_r}\frac{1}{r!}\right]. \tag{7b}$$

Note that, in view of the convergence conditions mentioned above, the absolute convergence of the series in (7) is assured in the case under consideration (i.e. when $i = 0, 1, 2, \dots$). Thus, in view of (2), we find that the nth Landau constant is given by

$$G_n = \frac{1}{\pi}(S_1 + S_2)$$

where

$$S_1 = \sum_{s=0}^{n}\frac{1}{s+1/2}, \qquad S_2 = \sum_{r=1}^{\infty}\frac{(1/2)_r(1/2)_r}{r!}S(r)$$

and

$$S(r) = \sum_{s=0}^{n}\frac{1}{(s+1/2)(s+3/2)_r}.$$

However, $S(r)$ can be rewritten as

$$\begin{aligned} S(r) &= \sum_{s=0}^{n}\frac{1}{(s+1/2)_r(s+1/2+r)_r} \\ &= \sum_{s=0}^{n}\frac{1}{(s+1/2)(s+1/2+1)\cdots(s+1/2+r)} \end{aligned}$$

since the definition of Pochhammer's symbol (5b) allows us to conclude that

$$x(x+1)_n = (x)_n(x+n)$$

holds. After summing S_1 and $S(r)$ in closed-form by using [4; p. 945, Eq. 8.365.3]

$$\sum_{k=0}^{n}\frac{1}{k+x} = \psi(x+n+1) - \psi(x)$$

and [6; p. 114, 6.1.192]

$$\sum_{k=0}^{n}\frac{1}{(k+x)(k+x+1)(k+x+2)\cdots(k+x+r)} = \frac{1}{r}\left[\frac{1}{(x)_r} - \frac{1}{(x+n+1)_r}\right]$$

respectively, we have

$$G_n = \frac{1}{\pi}\left[\psi(n+3/2) - \psi(1/2) + \sum_{r=1}^{\infty}\frac{(1/2)_r}{rr!} - \sum_{r=1}^{\infty}\frac{(1/2)_r(1/2)_r}{r(n+3/2)_r r!}\right]. \tag{8}$$

Finally, since $\psi(1/2) = -(\gamma + 2\log 2)$ ([4; p. 945, 8.366.2]) and

$$\sum_{r=1}^{\infty} \frac{(1/2)_r}{r r!} = 2\log 2$$

([6; p. 126, 6.6.34]), the result (6) follows from (8). □

THEOREM 2. *The sequence* $\{\delta_n\}$ *where* $\delta_n = G_n - (1/\pi)\psi(n+5/4) - A$ *and* A *is the constant defined by* (3b), *strictly decreases. Moreover,*

$$A < G_n - \frac{1}{\pi}\psi\left(n + \frac{5}{4}\right) < A + \delta_0\,, \qquad n = 1,2,3,\dots, \tag{9a}$$

where $\delta_0 = \frac{3}{2} - \frac{1}{\pi}(4 + \log 2) \approx 0.006125$. *In other words, the following inequalities hold*

$$1.0663 < G_n - \frac{1}{\pi}\psi\left(n + \frac{5}{4}\right) < 1.0724\,. \tag{9b}$$

Proof. First, observe that since ([1; p. 258, Eq. 6.3.5])

$$\psi(x+1) = \psi(x) + \frac{1}{x} \tag{10}$$

we have for $n = 1,2,3,\dots$

$$\psi\left(n + \frac{5}{4}\right) - \psi\left(n + \frac{1}{4}\right) = \frac{4}{4n+1}$$

while

$$G_n - G_{n-1} = \left[\frac{(2n-1)!!}{(2n)!!}\right]^2$$

follows from (2). Thus, in order to prove that $\{\delta_n\}$ strictly decreases, i.e. $\delta_n - \delta_{n-1} < 0$, we need to verify that

$$\pi < \frac{4}{4n+1}\left[\frac{(2n)!!}{(2n-1)!!}\right]^2, \qquad n = 1,2,3,\dots.$$

To do that it is enough to appeal to the following stronger Wallis's formula established by Gurland [5]

$$\frac{4n+3}{(2n+1)^2}\left[\frac{(2n)!!}{(2n-1)!!}\right]^2 < \pi < \frac{4}{4n+1}\left[\frac{(2n)!!}{(2n-1)!!}\right]^2, \qquad n = 1,2,3,\dots.$$

Finally, the value of δ_0 can be easily obtained from (3b) and (10) knowing that ([4; p. 945, 8.366.4])

$$\psi(1/4) = -(\gamma + \pi/2 + 3\log 2)\,.$$

This completes the proof of the theorem. □

3. Concluding remarks

Making use of the well-known Wallis's formula

$$\frac{2}{2n+1}\left[\frac{(2n)!!}{(2n-1)!!}\right]^2 < \pi < \frac{1}{n}\left[\frac{(2n)!!}{(2n-1)!!}\right]^2, \qquad n = 1,2,3,\dots,$$

it is not difficult to show that the sequence $\{\alpha_n\}$ defined in (6a) strictly decreases, and

$$A - \alpha_0 < G_n - \frac{1}{\pi}\psi\left(n + \frac{3}{2}\right) < A, \qquad n = 1,2,3,\dots,$$

where $\alpha_0 = (2/\pi)(1 + \log 2) - 1 \approx 0.07789$.

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