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ON SOME APPROXIMATION PROPERTIES OF THE METRIC DIMENSION

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ABSTRACT. In the paper some estimations of the lower and upper metric dimension of a compact subset A in $\mathbb{R}^n$ are obtained. These estimations are given by properties of sets near to A with respect to the Hausdorff metric.

The notion of (lower) metric dimension was first given in [P–S] and then studied (both, lower and upper cases) in [K–T]. The relationship between metric dimensions $\underline{\dim}$ and $\overline{\dim}$ and the Hausdorff dimension hd is given in [V]. The definition of metric dimension needs some of integer valued covering functions, which are also called ε-entropy and ε-capacity; this is the reason why the metric dimension ([C–S], [H], [K–T], [M–Z], [V],) is also called an entropy dimension ([B], [P], [Y]) or a limit capacity ([M], [P–T]).

The metric dimension as well as the Hausdorff dimension can be defined in metric spaces, but only for totally bounded subsets. In this case the main difference between hd and $\underline{\dim}$ consists in the fact that hd $X = 0$ for a countable set X , while $\underline{\dim}\ X$ can be positive, so $\underline{\dim}$ and $\overline{\dim}$ can better control the partition of points of these subsets. The aim of this paper is to derive some estimations for metric dimensions of a compact subset A of a metric space X, by properties of subsets near to A in the Hausdorff metric. Simple examples show that it is not possible to obtain the estimation directly from the metric dimension of these sets, so another kind of properties must be taken into account.

Let (X, d) be a metric space and $K \subset X$ nonempty compact subspace, let $\mathbb{N}$ and $\mathbb{R}$ be the set of all natural and real numbers, respectively. Denote by $B(p, r)$ an open ball centered at $p \in X$ with radius $r > 0$. Then $N(r, K)$ means the least number of open balls with radius $r > 0$ covering K. This number is

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well defined by compactness of K. Furthermore, put

$$\underline{\dim}\ K = \liminf_{r \to 0+} \frac{\log N(r,K)}{-\log r},$$
$$\overline{\dim}\ K = \limsup_{r \to 0+} \frac{\log N(r,K)}{-\log r}.$$

The base for the following considerations is a lemma, which allows to compare functions $N(\,.\,,A)$ and $N(\,.\,,B)$, where A, B are subsets of a metric space X, near with respect to the Hausdorff metric. Recall that if (X,d) is a metric space and $\mathcal{K}$ is the system of all its non empty compact subsets, then the Hausdorff metric h on $\mathcal{K}$ is defined in the following way: for $A, B \in \mathcal{K}$

$$h(A,B) = \sup\{d(a,B), d(A,b); a \in A,\ b \in B\}. \tag{1}$$

The space $(\mathcal{K}, h)$ is compact metric space provided (X,d) is compact.

LEMMA A. *Let A, B be non empty compact subsets of the metric space (X,d) and $\varepsilon > 0$. Assuming $h(A,B) \le \varepsilon$,*

$$N(r+\varepsilon, A) \le N(r,B) \tag{2}$$

for each $r > 0$.

Proof. Let $\{B(x_i, r)\}_{i=1}^m$ be a covering of B by open balls with radius $r > 0$, and let $a \in A$. Since $h(A,B) \le \varepsilon$, we have by (1) that $d(a,B) \le \varepsilon$ and hence there is a $y \in B$, $y \in B(x_j, r)$ for some $1 \le j \le m$, such that $d(a,y) \le \varepsilon$. Then

$$d(a,x_j) \le d(a,y) + d(y,x_j) < \varepsilon + r,$$

and therefore $a \in B(x_j, r+\varepsilon)$. This means that $\{B(x_i, r+\varepsilon)\}_{i=1}^m$ is the covering of A, thus the inequality (2) is true. □

Consider continuous functions $p, q, r \colon \mathbb{R}^+ \to \mathbb{R}^+$ for which

$$\lim_{\varepsilon \to 0+} p(\varepsilon) = \lim_{\varepsilon \to 0+} q(\varepsilon) = \lim_{\varepsilon \to 0+} r(\varepsilon) = 0.$$

The following general approximation lemma holds.

LEMMA B. *Let* p, q, r *be functions as above,* $A \subset X$ *be a non empty compact subset of a metric space* (X, d) *and* $\{A_\varepsilon\}_{\varepsilon>0}$ *be a system of non empty compact subsets of* X *for which* $h(A, A_\varepsilon) \le p(\varepsilon)$, $\varepsilon > 0$. *Then*

$$\text{(i)} \qquad \underline{l} \cdot \liminf_{\varepsilon \to 0+} \frac{\log N\big(q(\varepsilon)+p(\varepsilon), A_\varepsilon\big)}{-\log r(\varepsilon)} \le \underline{\dim}\ A$$

$$\le \min\left(\underline{k} \cdot \limsup_{\varepsilon\to 0+} \frac{\log N\big(q(\varepsilon), A_\varepsilon\big)}{-\log r(\varepsilon)} \,,\ \overline{k} \cdot \liminf_{\varepsilon\to 0+} \frac{\log N\big(q(\varepsilon), A_\varepsilon\big)}{-\log r(\varepsilon)} \right),$$

$$\text{(ii)} \ \max\left(\overline{l} \cdot \liminf_{\varepsilon\to 0+} \frac{\log N\big(q(\varepsilon)+p(\varepsilon), A_\varepsilon\big)}{-\log r(\varepsilon)} \,,\ \underline{l} \cdot \limsup_{\varepsilon\to 0+} \frac{\log N\big(q(\varepsilon)+p(\varepsilon), A_\varepsilon\big)}{-\log r(\varepsilon)} \right)$$

$$\le\ \overline{\dim}\ A\ \le \overline{k} \cdot \limsup_{\varepsilon\to 0+} \frac{\log N\big(q(\varepsilon), A_\varepsilon\big)}{-\log r(\varepsilon)},$$

where $\underline{l} = \liminf_{\varepsilon\to 0+} \frac{\log r(\varepsilon)}{\log q(\varepsilon)}$, $\overline{l} = \limsup_{\varepsilon\to 0+} \frac{\log r(\varepsilon)}{\log q(\varepsilon)}$, $\underline{k} = \liminf_{\varepsilon\to 0+} \frac{\log r(\varepsilon)}{\log\big(q(\varepsilon)+p(\varepsilon)\big)}$, *and* $\overline{k} = \limsup_{\varepsilon\to 0+} \frac{\log r(\varepsilon)}{\log\big(q(\varepsilon)+p(\varepsilon)\big)}$.

Proof. Since $h(A, A_\varepsilon) \le p(\varepsilon)$, (2) implies $N(q(\varepsilon)+p(\varepsilon), A) \le N(q(\varepsilon), A_\varepsilon)$ for any $\varepsilon > 0$, and this yields

$$\frac{\log N\big(q(\varepsilon)+p(\varepsilon), A\big)}{-\log\big(q(\varepsilon)+p(\varepsilon)\big)} \le \frac{\log N\big(q(\varepsilon), A_\varepsilon\big)}{-\log r(\varepsilon)} \cdot \frac{\log r(\varepsilon)}{\log\big(q(\varepsilon)+p(\varepsilon)\big)}.$$

for sufficiently small ε. Similarly, the inequality $N\big(q(\varepsilon)+p(\varepsilon), A_\varepsilon\big) \le N\big(q(\varepsilon), A\big)$ gives

$$\frac{\log N\big(q(\varepsilon), A\big)}{-\log q(\varepsilon)} \ge \frac{\log N\big(q(\varepsilon)+p(\varepsilon), A_\varepsilon\big)}{-\log r(\varepsilon)} \cdot \frac{\log r(\varepsilon)}{\log q(\varepsilon)},$$

The well-known properties of lim inf and lim sup then imply the required estimations. □

In the case of $X = \mathbb{R}^m$ the situation is much more simple.

THEOREM C. *Let* $A \subset \mathbb{R}^m$ *be a non empty compact set and* $\{A_\varepsilon\}$ *be for all sufficiently small* $\varepsilon > 0$ *a system of non empty compact subsets of* $\mathbb{R}^m$ *such that* $h(A, A_\varepsilon) \le \varepsilon$. *Then*

$$\text{(i)} \quad \underline{\dim}\ A = \liminf_{\varepsilon\to 0+} \frac{\log N(\varepsilon, A_\varepsilon)}{-\log \varepsilon},$$

(ii) $\overline{\dim}\ A = \limsup_{\varepsilon \to 0+} \frac{\log N(\varepsilon, A_\varepsilon)}{-\log \varepsilon}$.

Proof. Put $p(\varepsilon) = q(\varepsilon) = r(\varepsilon) = \varepsilon$ in Lemma B. We obtain

$$\liminf_{\varepsilon \to 0+} \frac{\log N(2\varepsilon, A_\varepsilon)}{-\log \varepsilon} \le \underline{\dim}\ A \le \liminf_{\varepsilon \to 0+} \frac{\log N(\varepsilon, A_\varepsilon)}{-\log \varepsilon}.$$

By [M–Z; Proposition 2] there exists a constant $c \in \mathbb{R}^+$ such that $N(\varepsilon, A) \le c \cdot N(2\varepsilon, A)$ for all $\varepsilon > 0$. Then

$$\liminf_{\varepsilon \to 0+} \frac{\log N(2\varepsilon, A_\varepsilon)}{-\log \varepsilon} \ge \liminf_{\varepsilon \to 0+} \frac{\log c^{-1} \cdot N(\varepsilon, A_\varepsilon)}{-\log \varepsilon} = \liminf_{\varepsilon \to 0+} \frac{\log N(\varepsilon, A_\varepsilon)}{-\log \varepsilon}$$

and therefore (i) is valid. The proof of (ii) is similar. □

Frequently we can approximate the set A by a countable system of sets only. In these cases we can apply the following theorem.

THEOREM D. *Let $A \subset \mathbb{R}^m$ be a non empty compact set and let $\{A_n\}_{n=1}^{\infty}$ be a system of non empty compact subsets of $\mathbb{R}^m$ such that $h(A, A_n) \le \varepsilon_n$, where the sequence $\{\varepsilon_n\}$ monotonically converges to 0. Then the following estimations hold.*

(i) $\liminf_{n \to \infty} \frac{\log N(\varepsilon_{n-1}, A_n)}{-\log \varepsilon_n} \le \underline{\dim}\ A \le \liminf_{n \to \infty} \frac{\log N(\varepsilon_n, A_n)}{-\log \varepsilon_n}$,

(ii) $\limsup_{n \to \infty} \frac{\log N(\varepsilon_n, A_n)}{-\log \varepsilon_n} \le \overline{\dim}\ A \le \limsup_{n \to \infty} \frac{\log N(\varepsilon_n, A_n)}{-\log \varepsilon_{n-1}}$.

If, moreover, $\lim_{n \to \infty} \frac{\log \varepsilon_n}{\log \varepsilon_{n-1}} = 1$ *then*

(iii) $\overline{\dim}\ A = \limsup_{n \to \infty} \frac{\log N(\varepsilon_n, A_n)}{-\log \varepsilon_n}$.

Proof. (i) Take $\varepsilon > 0$ and define $A_\varepsilon = A_n$ whenever $\varepsilon \in \langle \varepsilon_n, \varepsilon_{n-1})$. Then

$$h(A, A_\varepsilon) = h(A, A_n) \le \varepsilon_n \le \varepsilon, \quad \varepsilon > 0.$$

Using Theorem C (i) we obtain

$$\underline{\dim}\ A = \liminf_{\varepsilon \to 0+} \frac{\log N(\varepsilon, A_\varepsilon)}{-\log \varepsilon} = \liminf_{\varepsilon \to 0+} \frac{\log N(\varepsilon, A_n)}{-\log \varepsilon} \ge \liminf_{n \to \infty} \frac{\log N(\varepsilon_{n-1}, A_n)}{-\log \varepsilon_n}.$$

The inequality $\underline{\dim}\ A \le \liminf_{n \to \infty} \frac{\log N(\varepsilon_n, A_n)}{-\log \varepsilon_n}$ follows from the fact that $\left\{ \frac{\log N(\varepsilon_n, A_n)}{-\log \varepsilon_n} \right\}_{n=1}^{\infty}$ is the sequence chosen from the family $\left\{ \frac{\log N(\varepsilon, A_\varepsilon)}{-\log \varepsilon} \right\}_{\varepsilon \le \varepsilon_1}$. The inequalities (ii) can be proved in the similar way using Theorem C (ii).

(iii) Observe, that

$$\limsup_{n\to\infty} \frac{\log N(\varepsilon_n, A_n)}{-\log \varepsilon_{n-1}} = \limsup_{n\to\infty} \left(\frac{\log N(\varepsilon_n, A_n)}{-\log \varepsilon_n} \cdot \frac{\log \varepsilon_n}{\log \varepsilon_{n-1}} \right)$$
$$= \limsup_{n\to\infty} \frac{\log N(\varepsilon_n, A_n)}{-\log \varepsilon_n} \cdot \lim_{n\to\infty} \frac{\log \varepsilon_n}{\log \varepsilon_{n-1}} = \limsup_{n\to\infty} \frac{\log N(\varepsilon_n, A_n)}{-\log \varepsilon_n} .$$

□

Example 1. Let $A = \{0\} \cup \left\{\frac{1}{a^k}\right\}_{k=0}^{\infty}$, where $a \in \mathbb{R}$, $a > 1$. If we take $A_n = \{1, \frac{1}{a}, \dots, \frac{1}{a^n}\}$, then $h(A, A_n) = \frac{1}{a^n}$. Using (ii) of the previous theorem we obtain

$$\overline{\dim}\ A \le \limsup_{n\to\infty} \frac{\log N(a^{-n}, A_n)}{-\log a^{1-n}} \le \lim_{n\to\infty} \frac{\log(n+1)}{(n-1)\cdot \log a} = 0 ,$$

from which $\underline{\dim}\ A = \overline{\dim}\ A = 0$.

The following theorem presents estimations of metric dimensions in the case when the set A can be approximated by finite sets. For a set $K \subset X$ denote $\mu(K) = \inf\{d(x,y); x, y \in K, x \neq y\}$, $|K|$ means the cardinality of K.

THEOREM E. *Let (X,d) be a compact metric space, let A be its infinite compact subspace and $\{A_n\}_{n=1}^{\infty}$ be a sequence of finite subets of X such that $h(A_n, A) \xrightarrow{n\to\infty} 0$. Then the following holds:*

(i) *$A_n \subset A$ for all $n > n_0$ implies*

$$\overline{\dim}\ A \ge \limsup_{n\to\infty} \frac{\log |A_n|}{-\log \mu(A_n)} .$$

(ii) *If $h(A_n, A) \le \mu(A_n)$ for each $n > n_0$, then*

$$\underline{\dim}\ A \le \liminf_{n\to\infty} \frac{\log |A_n|}{-\log \mu(A_n)} ,$$

and

$$\overline{\dim}\ A \ge \limsup_{n\to\infty} \frac{\log |A_n|}{-\log \mu(A_n)} .$$

P r o o f. (i) Denote $\mu_n = \mu(A_n)$. As A is infinite, $\mu_n \xrightarrow{n\to\infty} 0$. Hence

$$\overline{\dim}\ A = \limsup_{\varepsilon\to 0+} \frac{\log N(\varepsilon, A)}{-\log \varepsilon} \ge \limsup_{n\to\infty} \frac{\log N(\mu_n/2, A)}{-\log \mu_n/2}$$
$$\ge \limsup_{n\to\infty} \frac{\log N(\mu_n, A_n)}{-\log \mu_n + 1} = \limsup_{n\to\infty} \frac{\log |A_n|}{-\log \mu_n} ,$$

since $A \subset B$ implies $N(2r, A) \le N(r, B)$.

(ii) It follows from (2) and the inequality $h(A_n, A) < \mu_n$ that $N(2\mu_n, A) \le N(\mu_n, A_n)$. Therefore

$$\underline{\dim}\ A = \liminf_{\varepsilon\to 0+} \frac{\log N(\varepsilon, A)}{-\log\varepsilon} \le \liminf_{n\to\infty} \frac{\log N(2\mu_n, A)}{-\log 2\mu_n}$$

$$\le \liminf_{n\to\infty} \frac{\log N(\mu_n, A_n)}{-\log\mu_n - 1} = \liminf_{n\to\infty} \frac{\log|A_n|}{-\log\mu_n},$$

and

$$\overline{\dim}\ A = \limsup_{\varepsilon\to 0+} \frac{\log N(\varepsilon, A)}{-\log\varepsilon} \ge \limsup_{n\to\infty} \frac{\log N(\mu_n/2, A)}{-\log \mu_n/2}$$

$$\ge \limsup_{n\to\infty} \frac{\log N(\mu_n, A_n)}{-\log\mu_n + 1} = \limsup_{n\to\infty} \frac{\log|A_n|}{-\log\mu_n}.$$

□

Example 2. Let $A = \{0\} \cup \left\{\frac{1}{k^a}\right\}_{k=1}^{\infty}$, where $a > 0$, and let $A_n = \left\{\frac{1}{k^a}\right\}_{k=1}^{n}$ for $n \in \mathbb{N}$. Then by Theorem E (i)

$$\overline{\dim}\ A \ge \limsup_{n\to\infty} \frac{\log n}{\log \frac{n^a(n-1)^a}{n^a-(n-1)^a}}.$$

But, for $a = 1$ we have directly $\lim\limits_{n\to\infty} \frac{\log n}{\log n(n-1)} = \frac{1}{2}$, and, for $a \ne 1$,

$$\lim_{n\to\infty} \frac{\log n}{\log \frac{n^a(n-1)^a}{n^a-(n-1)^a}} = \lim_{n\to\infty} \frac{\log n}{a\log n + a\log(n-1) - \log\bigl(n^a - (n-1)^a\bigr)}$$

$$= \lim_{n\to\infty} \frac{1}{a + a\frac{\log(n-1)}{\log n} - \frac{\log\left(n^a-(n-1)^a\right)}{\log n}}$$

Here $\lim\limits_{n\to\infty} \frac{\log(n-1)}{\log n} = 1$ and

$$\lim_{n\to\infty} \frac{\log\bigl(n^a - (n-1)^a\bigr)}{\log n} = \lim_{n\to\infty} a\frac{1-\left(\frac{n-1}{n}\right)^{a-1}}{1-\left(\frac{n-1}{n}\right)^{a}},$$

by the L'Hospital rule, so the limit is the same as

$$a \lim_{x\to 1} \frac{1-x^{a-1}}{1-x^a},$$

which is, repeating the L'Hospital rule, $a \cdot \frac{a-1}{a} = a - 1$. From this

$$\overline{\dim}\ A \ge \frac{1}{a + a - (a-1)} = \frac{1}{a+1}.$$

Example 3. Let $A = \{0\} \cup \left\{\frac{1}{\log k}\right\}_{k=2}^{\infty}$ and take $A_n = \left\{\frac{1}{\log k}\right\}_{k=2}^{n}$ for $n \geq 2$. Since $A \subset \mathbb{R}^1$, then $\overline{\dim}\ A \leq 1$. On the other hand, by Theorem E (i) we have

$$\overline{\dim}\ A \geq \limsup_{n\to\infty} \frac{\log(n-1)}{\log \frac{\log n \cdot \log(n-1)}{\log n - \log(n-1)}}.$$

But

$$\lim_{n\to\infty} \frac{\log(n-1)}{\log \frac{\log n \cdot \log(n-1)}{\log n - \log(n-1)}} = \lim_{n\to\infty} \frac{1}{\frac{\log\log n}{\log(n-1)} + \frac{\log\log(n-1)}{\log(n-1)} - \frac{\log\left(\log n - \log(n-1)\right)}{\log(n-1)}}.$$

Moreover, $\frac{\log\log n}{\log(n-1)}$ and $\frac{\log\log(n-1)}{\log(n-1)}$ tend to 0 and $\lim\limits_{n\to\infty} \frac{\log\left(\log n - \log(n-1)\right)}{\log(n-1)} = -1$, repeating the L'Hospital rule.

So $\frac{1}{0+0-(-1)} \leq \overline{\dim}\ A \leq 1$, i.e., $\overline{\dim}\ A = 1$.

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