Jana Tomanová

A note on link graphs

Mathematica Slovaca, Vol. 39 (1989), No. 3, 225--231

Math. Slovaca 39, 1989, No. 3, 225—231

A NOTE ON LINK GRAPHS

JANA TOMANOVÁ

We consider finite, loopless, undirected graphs without multiple edges. The *link of a vertex v* of a graph G is the subgraph induced by all vertices adjacent to v; following the terminology of [1] we denote it by link(v, G).

If all the links of G are isomorphic to the same graph L, then we say that G has a constant link L. The graph L is called a link graph of G (see Fig. 1).

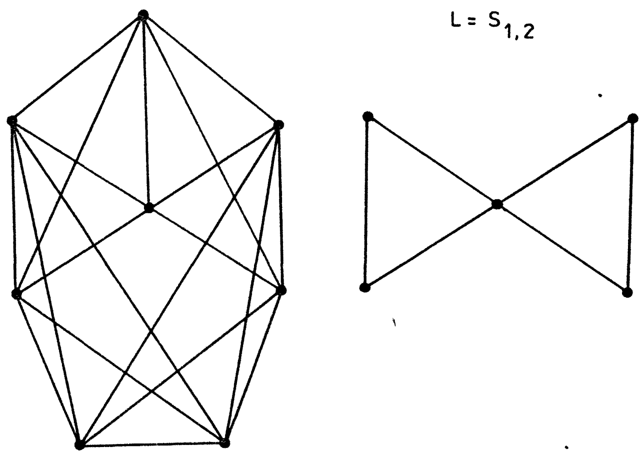

Fig. 1. A graph G with the constant link L.

Zykov [6] posed what has become a well-known open problem to characterize link graphs. It appeared that the problem is algorithmically unsolvable in the class of all graphs, see Bulitko [3]. However, the solution of Zykov's problem is known for certain classes of graphs, namely paths, cycles, linear forests and graphs homeomorphic to a star [2]. The survey of these and similar results for unions of stars and for unions of paths and cycles is given in Hell [5]. There are also investigated some operations on graphs which enlarge the class of link graphs. In [1] group theoretic methods are used to obtain constructions of graphs with a constant link isomorphic to a certain tree. The idea of this construction is based on the following facts.

Recall that a permutation group H is said to be *sharply transitive* on a set S if H is transitive on S and each permutation in H is uniquely determined by its action on any single element of S.

Let H be a group and Z a generating set of H closed under inverses and not containing the identity. The *Cayley graph* $[H, Z]$ for this group and the generating set has as vertices the elements of H, with u and v adjacent when $u^{-1} v$ belongs to Z. Figure 1 shows the Cayley graph for the group all elements of which are of order two and for $Z = \{x, a, xa, b, xb\}$. For any $u, v \in H$ the permutation $g \to vu^{-1}g$ induces an automorphism of $[H, Z]$ which sends u on v. Moreover, this permutation is uniquely determined by the pair (u, v). It means that H is a sharply transitive group of automorphisms of $[H, Z]$, hence $[H, Z]$ is a graph with a constant link.

Conversely, let H be a sharply transitive group of automorphisms of a connected graph G. Then there is a generating set Z such that $[H, Z]$ is isomorphic to G. Indeed, for each vertex v of G one can identify vertices of the link(v, G) with automorphisms which send v on the link(v, G). It means that we can take $Z = \{h \in H : h(v) \in \text{link}(v, G)\}$. The fact that automorphisms preserve adjacency is used to compute relations between the elements of Z, thus the group H is completely determined.

In our paper we shall use this method to construct graphs with a constant link isomorphic to the so-called treelike graphs, see [4, p. 68—73].

Definition 1. *The graph G is said to be treelike if*

(A) *G does not contain any cycle of the length greater than three as an induced subgraph.*

(B) *Maximal cliques of G are of the same size n, $n \geqslant 3$ and they are edge disjoint (see Fig. 2).*

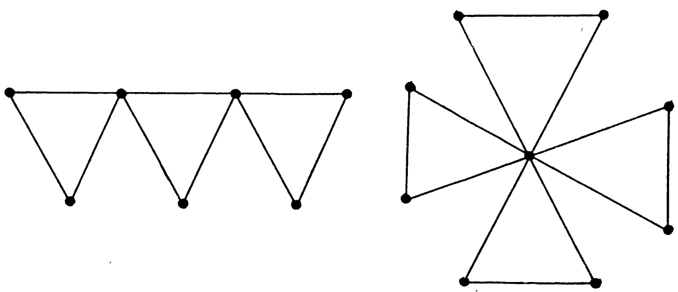

Fig. 2. Examples of treelike graphs.

In what follows we consider only treelike graphs with maximal cliques isomorphic to the complete graph on $2n + 1$ vertices, $n \geqslant 1$.

A treelike graph G is starlike if all its maximal cliques have exactly one common vertex. If $m \geqslant 2$ is the number of maximal cliques of a starlike graph, then we denote it $S_{1,m}$.

Theorem 1. *Let $m \geqslant 2$, be an integer. The starlike graph $S_{1,m}$ is a link graph.*

Proof. The central vertex has to be of even order [1, Theorem 3]. There are $2n$ vertices to be determined in any branch. In order that no superfluous adjacencies arise, both v and v^{-1} have to belong to the same branch. Therefore it is natural to take for H the group generated by the set

$$Z = \{x, a_1, a_1^2, \dots, a_1^n, xa_1, xa_1^2, \dots, xa_1^n, \dots, a_m, a_m^2, \dots, a_m^n, xa_m, xa_m^2, \dots, xa_m^n\}$$

with the relations

$$x^2 = 1, \; a_i^{n+1} = 1 \qquad i = 1, 2, \dots, m$$

$$(xa_i)^2 = 1, \; a_i a_j = a_j a_i \qquad i, j \in \{1, 2, \dots, m\}.$$

Figure 3 shows the starlike graph $S_{1,3}$ with branches isomorphic to the complete graph on 5 vertices.

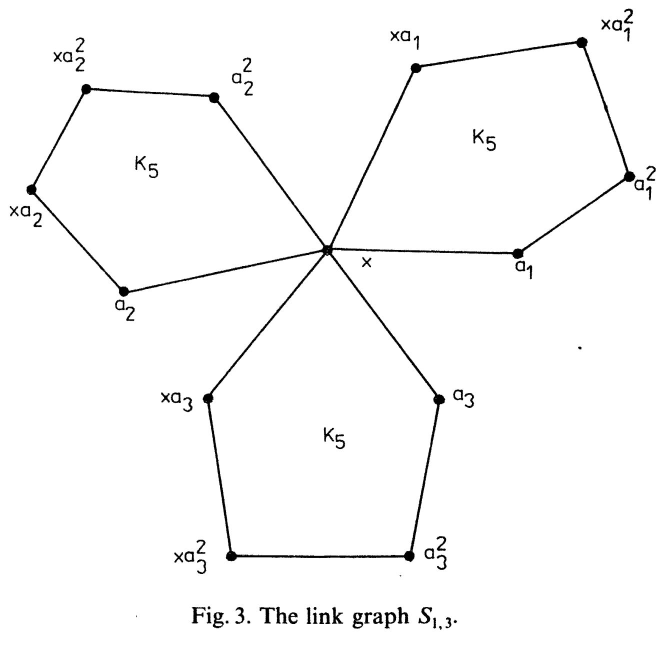

Fig. 3. The link graph $S_{1,3}$.

Since any word in H can be written in the form $x^t a_1^{e_1} a_2^{e_2} \dots a_m^{e_m}$ $t \in \{0, 1\}$, $e_i \in \{0, 1, \dots, n\}$ the group H is finite. Hence the proof.

The generating set Z we have just defined contains $n+1$ elements of order two for any branch of $S_{1,m}$. We will use this fact to construct link graphs isomorphic to treelike graphs satisfying some conditions on the degrees of vertices.

Corollary 1. *Let T be a treelike graph. We denote p the number of vertices which belong to the same maximal clique in T and which have a degree greater than or equal to $4n$. If $p \leqslant n+1$, then T is a link graph.*

Proof. Let H be the group with a generating set Z we defined in Theorem 1, and u an element of order two contained in Z, say xa_1. The complete graph on $2n+1$ vertices can be attached to this element in the following way: We add to Z the elements

$$b_1,\ b_1^2,\ \ldots,\ b_1^n,\ xa_1b_1,\ xa_1b_1^2,\ \ldots,\ xa_1b_1^n$$

and denote this sequence Z_1. Define

$$b_1^{n+1}=1,\ (xb_1)^2=1,\ a_ib_1=b_1a_i \qquad i=1,2,\ldots,m.$$

Figure 4 shows the attachment of K_5 to the vertex xa_1 in $S_{1,2}$.

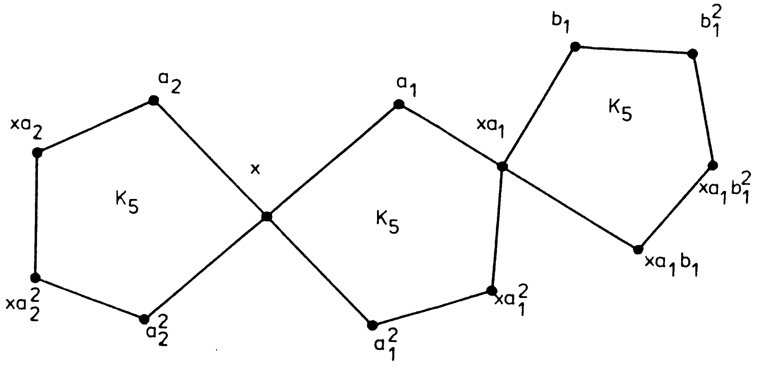

Fig. 4. The attachment of K_5 to the vertex xa_1 in $S_{1,2}$.

Since any word in the group H' generated by the set $Z'=Z\cup Z_1$ is of form $x^ta_1^{e_1}a_2^{e_2}\ldots a_m^{e_m}$, $t\in\{0,1\}$, $e_i\in\{0,1,\ldots,n\}$ the group H' is finite. Moreover no vertex $v\in Z'-Z_1$, except the case $v=xa_1$, is adjacent to any vertex contained in Z_1. Using the operation of attachment of the complete graph to the vertex of order two we can increase the length of any branch in $S_{1,m}$ and the degree of any vertex of order two as much as we want. This completes the proof.

Finally we will construct link graphs isomorphic to treelike graphs without any condition on the degrees.

Theorem 2. *Let G be a treelike graph with maximal cliques isomorphic to the complete graph on $2^n - 1$ vertices, $n \geqslant 2$. Then G is a link graph.*

Proof. First we will find the group H with the generating set Z and the following properties:

(1) The graph $[H, Z]$ has a constant link isomorphic to $S_{1,2}$.

(2) All elements in H are of order two.

Owing to property (2) we can use the operation of attachment of the complete graph to any vertex in $S_{1,2}$ and obtain a new group which satisfies the property (2) again.

Let H_1 be the group with a generating set

$$Z_1 = \{x, a_1, xa_1, b_1, xb_1\} \quad \text{with a single relation}$$
$$z^2 = 1,\ z \in Z_1$$

It is evident that $[H_1, Z_1]$ is a graph with a constant link isomorphic to $S_{1,2}$ (see Fig. 1). We denote by

$$A_1 = \{x, a_1, xa_1\} \quad \text{and} \quad B_1 = \{x, b_1, xb_1\} \quad \text{and define}$$
$$Z_2 = Z_1 \cup a_2A_1 \cup b_2B_1 \cup \{a_2, b_2\},\ z^2 = 1 \quad z \in Z_2\,.$$

Then the Cayley graph $[H_2, Z_2]$ has a constant link isomorphic to $S_{1,2}$ with branches equal to the complete graph on 7 vertices (see Fig. 5).

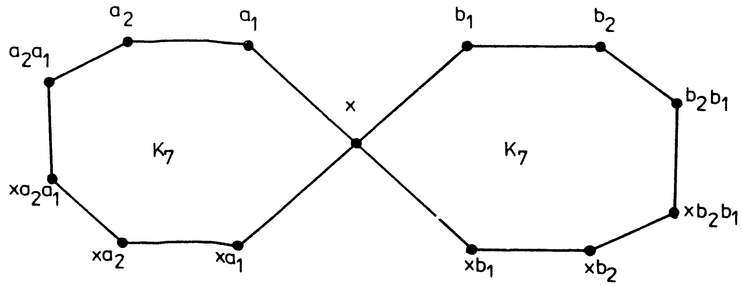

Fig. 5. A link graph of the Cayley graph $[H_2, Z_2]$.

Analogously we define group H_i, $i > 1$ with the generating set

$$Z_i = Z_{i-1} \cup a_iA_{i-1} \cup b_iB_{i-1} \cup \{a_i, b_i\}, \quad \text{where}$$
$$A_{i-1} = \{x^{e_0}a_1^{e_1}a_2^{e_2} \ldots a_{i-1}^{e_{i-1}} : e_j \in \{0, 1\}, \quad j = 0, 1, \ldots i-1\}$$
$$1 \notin A_{i-1}$$

Substituting b for a we obtain the definition of B_{i-1}. The elements of Z_i satisfy the condition $z^2 = 1$, $z \in Z_i$.
In such a way we have $[H_i, Z_i]$ with the property:

$$\mathrm{link}(1, [H_i, Z_i]) \cong S_{1,2}, \; S_{1,2} \text{ is a starlike graph}$$

with branches equal to the complete graph on $2^{i+1} - 1$ verticses.

In what follows we will use the operation of attachment. Let u be an arbitrary element in Z_i. We add to Z_i all words

$$u^{e_0} y_1^{e_1} y_2^{e_2} \dots y_i^{e_i}, \quad e_j \in \{0, 1\}, \qquad j = 0, 1, \dots, i$$

except the case $e_j = 0$ for all j.
Figure 6 shows the attachment of the complete graph on 7 vertices to $S_{1,2}$.

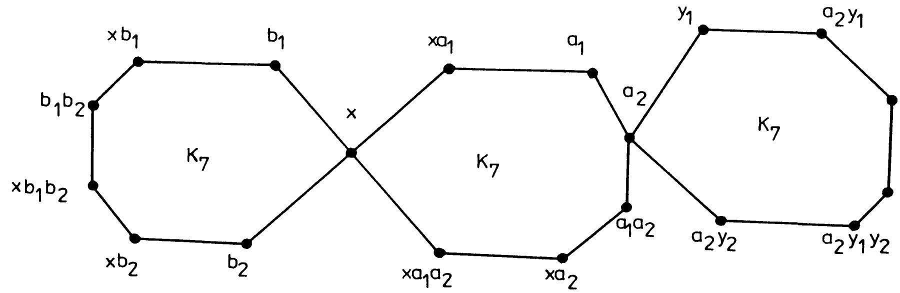

Fig. 6. An attachment of K_7 to the vertex a_2 in $S_{1,2}$.

Since all elements of the group we have just defined are of order two the operation of attachment of the complete graph to any element in Z_i can be used again. This completes the proof.

REFERENCES

[1] BLASS, A.—HARARY, F.—MILLER, Z.: Which trees are link graphs? J. Com. Theory B 29, 1980, 277—292.

[2] BROWN, M.—CONNELLY, R.: On graphs with constant link. New Directions in the Theory of Graphs (F. Harary Ed.). Academic Press, New York, 1973, 19—51.

[3] BULITKO, V. K.: О графах с заданными окружениями вершин. Trudy matematič. inst. im. V. A. Steklova 133, 1973, 78—94.

[4] HARARY, F.—PALMER, E. M.: Graphical Enumeration. Academic Press, New York and London, 1973.

[5] HELL, P.: Graphs with given neighborhoods I. Proc. Colloque Inter. C.N.R.S., Orsay, 1976, 219—223.

[6] ZYKOV, A. A.: Problem 30, Theory of Graphs and Applications (M. Fiedler Ed.) Academia, Praha, 1964, 164.

Received November 26, 1987

Katedra algebry a teórie čísel
Matematicko-fyzikálna fakulta UK
Mlynská dolina
842 15 Bratislava

ЗАМЕТКА О ГРАФАХ С ОКРУЖЕНИЕМ ИЗОМОРФНЫМ ДАННОМУ ГРАФУ

Jana Tomanová

Резюме

Пусть v — вершина графа G, ее окружение в G — это подграф, порожденный всеми смежными с ней вершинами. В статье приводится конструкция графов G, обладающих тем свойством, что окружение каждой вершины графа G изоморфно данному графу L; L принадлежит классу древовидных графов.