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RADICAL CLASSES OF CYCLICALLY ORDERED GROUPS

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The investigation of cyclically ordered groups was begun by L. Rieger [14]. Further results in this field were obtained in the papers [9], [15], [16], [17], [18] and [10]. For the basic notations, cf. also L. Fuchs [3]. Cyclically ordered groups can be viewed as being a natural generalizations of linearly ordered groups.

The notions of radical class and semisimple class of linearly ordered groups were introduced and studied by C. G. Chehata and R. Wiegandt [1]. Some further aspects of the radical theory for linearly ordered groups were investigated in the papers [5], [6], [11] in the general case, and in the papers [4], [7], [8], [12], [13] in the case of abelian linearly ordered groups. In particular, in the paper [8] K-radical classes of abelian linearly ordered groups were dealt with (an analogous notion for radical classes of lattice ordered groups was introduced by P. Conrad [2]).

In the present paper the Chehata—Wiegandt notion of a radical class and the notion of a K-radical class of abelian linearly ordered groups are extended for the case of abelian cyclically ordered groups.

We denote by

$\mathscr{R}$ — the lattice of all radical classes of abelian linearly ordered groups;

$\mathscr{R}_k$ — the lattice of all K-radical classes of abelian linearly ordered groups;

$\mathscr{R}_c$ — the lattice of all radical classes of abelian cyclically ordered groups;

$\mathscr{R}_{kc}$ — the lattice of all K-radical classes of abelian cyclically ordered groups. (For definitions, cf. below.)

In fact, some results (including the proofs) for the lattice $\mathscr{R}_c$ are analogous to those for the lattice $\mathscr{R}$ (cf., e.g., Theorem 2.4). But there are also some rather strong distinctions (cf., e.g., the existence of atoms and dual atoms in $\mathscr{R}_c$, while the lattice $\mathscr{R}$ has no atom and no dual atom). Hence the lattices $\mathscr{R}$ and $\mathscr{R}_c$ fail to be isomorphic. On the other hand, the lattice $\mathscr{R}_k$ is isomorphic to the lattice $\mathscr{R}_{kc}$.

The lattice $\mathscr{R}$ is a closed convex sublattice of $\mathscr{R}_c$. Moreover, the partially ordered collection $\mathscr{R}$ is a retract of $\mathscr{R}_c$. The lattice $\mathscr{R}_k$ is not a sublattice of $\mathscr{R}_{kc}$.

1. Preliminaries on cyclically ordered groups

For the sake of completeness, we recall some definitions concerning cyclically ordered groups.

Let G be a group. The group operation will be denoted additively, the commutativity of this operation will not be assumed. Suppose that a ternary relation $[x, y, z]$ is defined on G such that the following conditions I—IV are satisfied for all $x, y, z, a, b \in G$:

I. If $[x, y, z]$ holds then x, y and z are distinct; if x, y and z are distinct, then either $[x, y, z]$ or $[z, y, x]$.

II. $[x, y, z]$ implies $[y, z, x]$.

III. If $[x, y, z]$ and $[y, u, z]$, then $[x, u, z]$.

IV. $[x, y, z]$ implies $[a + x + b, a + y + b, a + z + b]$.

Under these assumptions G is said to be a cyclically ordered group. The ternary relation under consideration is said to be a cyclic order on G. When speaking of this relation we often denote it by $[\,]$.

If H is a subgroup of G, then H is viewed as being cyclically ordered by the original cyclic order reduced to H.

Isomorphisms of cyclically ordered groups are defined in the obvious way. A mapping f of a cyclically ordered group G into a cyclically ordered group G' is a *homomorphism* if the following conditions are satisfied:

(i) f is a homomorphism with respect to the group operation;

(ii) whenever x, y and z are elements of G such that $f(x), f(y)$ and $f(z)$ are distinct, and $[x, y, z]$ holds, then $[f(x), f(y), f(z)]$ is valid.

Let L be a linearly ordered group. For distinct elements x, y and z of L we put $[x, y, z]$ if

$$(1) \qquad x < y < z \quad \text{or} \quad y < z < x \quad \text{or} \quad z < x < y$$

is valied. Then $[\,]$ is a cyclic order on G *induced* by the linear order $\leqq$.

A cyclically ordered group $(G; [\,])$ is said to be linearly ordered if there exists a linear order $\leqq$ on G such that $(G, \leqq)$ is a linearly ordered group and the cyclic order $[\,]$ is induced by $\leqq$. If such a linear order does exist, then it is uniquely determined (cf. [10], Lemma 3.1).

Let K be the set of all real numbers x with $0 \leqq x < 1$. The operation $+$ in K is defined to be the addition mod 1. For distinct elements x, y and z of K we put $[x, y, z]$ if (1) is valid. Then K is a cyclically ordered group.

Let G_1 be a cyclically ordered group and let L be a linearly ordered group. We denote by $G_1 \otimes L$ the direct product of groups G_1 and L with a ternary relation $[\,]$ which is defined as follows. For distinct elements $u = (a, x)$, $v = (b, y)$ and $w = (c, z)$ of the set $G_1 \times L$ we put $[u, v, w]$ if some of the following conditions is satisfied:

(i) $[a, b, c]$;
(ii) $a = b \neq c$ and $x < y$;
(iii) $b = c \neq a$ and $y < z$;
(iv) $c = a \neq b$ and $z < x$;
(v) $a = b = c$ and $[x, y, z]$.

Then $G_1 \otimes L$ is a cyclically ordered group.

For $M \subseteq G_1 \times L$ we denote by $M(G_1)$ the natural projection of M into G_1, i.e., the set of all elements $a \in G_1$ such that there is $x \in L$ with $(a, x) \in M$. The set $M(L)$ is defined analogously.

Let K be as above and let K_1 be a subgroup of K. A mapping φ of a cyclically ordered group G into $K_1 \times L$ is said to be a representation of G if the following conditions are satisfied:

(i) φ is an isomorphism of G into $K_1 \otimes L$;
(ii) $(\varphi(G))(K_1) = K_1$ and $(\varphi(G))(L) = L$.

1.1. Theorem. (Swierczkowski [15].) *Each cyclically ordered group possesses a representation.*

A subgroup H of a cyclically ordered group G will be said to be c-convex (cf. [10]) if some of the following conditions is fulfilled:

(i) $H = G$;
(ii) for each $h \in H$ with $h \neq 0$ we have $2h \neq 0$; if $h \in H$, $g \in G$, $[-h, 0, h]$, $[-h, g, h]$, then $g \in H$.

Let φ be a representation of a cyclically ordered group G. Under the denotations as above, let $G^0(\varphi)$ be the set of all $g \in G$ such that $\{\varphi(g)\}(K_1) = \{0\}$. Then we have

(i) $G^0(\varphi)$ is the largest linearly ordered subgroup of G;
(ii) if G_1 is a c-convex subgroup of G and $G_1 \neq G$, then $G_1 \subseteq G^0(\varphi)$.

(Cf. [10], 3.5 and 4.6.)

From (i) (or from (ii) we obtain):

(iii) If φ' is another representation of G, then $G^0(\varphi') = G^0(\varphi)$.

In view of (iii) we shall write G^0 instead of $G^0(\varphi)$.

In what follows all groups under consideration are assumed to be abelian (i.e., "group" means "abelian group").

Let H be a c-convex subgroup of G. Let x_1, x_2, x_3 be elements of G such that the classes $x_i + H$ $(i = 1, 2, 3)$ are distinct. It is easy to verify that if $[x_1, x_2, x_3]$ is valid in G, then $[x_1', x_2', x_3']$ holds for each $x_i' \in x_i + H$ $(i = 1, 2, 3)$. In such a case we put $[x_1 + H, x_2 + H, x_3 + H]$. We get a cyclically ordered group which will be denoted by G/H. Moreover, we immediately obtain:

1.2. Lemma. (i) *Let H be a c-convex subgroup of G. Then the mapping $x \to x + H$ is a homomorphism of G onto G/H.* (ii) *Let f be a homomorphism of a cyclically ordered group G onto a cyclically ordered group G_1 and let H be the* *kernel of f. Then H is a c-convex subgroup of G and the mapping $x + H \to f(x)$ is an isomorphism of G/H onto G_1.*

Let K_1 be a subgroup of K and let L' be a homomorphic image of a linearly ordered group L. For each element (a, x) of $K_1 \otimes L$ we put $f_1((a, x)) = (a, x')$, where $x' = h(x)$, h being the homomorphic mapping of L onto L' under consideration. Then f_1 is a homomorphism of $K_1 \otimes L$ onto $K \otimes L'$. (We shall say that f_1 is a natural homomorphism of $K_1 \otimes L$ onto $K_1 \otimes L'$.)

1.3. Lemma. *Let G be a cyclically ordered group and let H be a c-convex subgroup of G, $H \neq G$. Let $\varphi: G \to K_1 \otimes L$ be a representation of G. Let f be the natural homomorphism of G onto G/H. Put $L_1 = L/H(L)$. Let f_1 be the natural homomorphism of $K_1 \otimes L$ onto $K_1 \otimes L_1$. Then there exists a representation $\varphi': G/H \to K_1 \otimes L_1$ of G/H such that the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & K_1 \otimes L \\ f\downarrow & & \downarrow f_1 \\ G/H & \xrightarrow{\varphi'} & K_1 \otimes L_1 \end{array}$$

is commutative.

Proof. The assertion is an immediate consequence of the fact that $H \neq G$ (and hence $H \subseteq G^0$).

1.4. Lemma. *Let G be a cyclically ordered group. Let $\varphi: G \to K_1 \otimes L$ and $\varphi': G \to K_1' \otimes L'$ be representations of G. Then K_1 is isomorphic to K_1'.*

Proof. This assertion is contained in [10], Theorem 5.3.

2. Radical classes

The class of all cyclically ordered groups will be denoted by $\mathscr{C}_a$. Let $G \in \mathscr{C}_a$. Let β be an ordinal. Let G_α $(\alpha < \beta)$ be c-convex subgroups of G such that

$$\{0\} = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots (\alpha < \beta), \qquad \cup_{\alpha < \beta} G_\alpha = G.$$

For each $\alpha < \beta$ we put $H_\alpha = G_\alpha / \cup_{\gamma < \alpha} G_\gamma$. Then G is said to be a transfinite extension of cyclically ordered groups H_α $(\alpha < \beta)$.

When considering a subclass X of $\mathscr{C}_a$ we always assume that X is closed with respect to isomorphism.

2.1. Definition. *A nonempty subclass X of $\mathscr{C}_a$ is said to be a radical class, if it is closed with respect to homomorphisms and transfinite extensions.*

The definition of a radical class of linearly ordered groups is defined analogously (the c-convexity is replaced by convexity); cf. [7]. The collection of all radical classes of linearly ordered groups is denoted by $\mathscr{R}$; this collection is partially ordered by inclusion. Then $\mathscr{R}$ is a complete lattice.

We denote by $\mathscr{L}_a$ the class of all linearly ordered groups. From Lemma 4.2, [10] we obtain:

2.2. Lemma. *Let $X \subseteq \mathscr{L}_a$. Then the following conditions are equivalent:*

(i) *X is a radical class of linearly ordered groups.*

(ii) *X is a radical class of cyclically ordered groups.*

Let $Y \subseteq \mathscr{C}_a$. We denote by

Ext Y — the class of all cyclically ordered groups which are transfinite extensions of some elements of Y;

Hom Y — the class of all homomorphic images of elements of Y.

The collection of all radical classes of cyclically ordered groups will be denoted by $\mathscr{R}_c$. This collection is partially ordered by inclusion. The greatest element of $\mathscr{R}_c$ is $\mathscr{C}_a$. The least element of $\mathscr{R}_c$ is the class of all one-element cyclically ordered groups; this class will be denoted by 0^-.

From the definition of a radical class we obtain immediately:

2.3. Theorem. *Let A_i $(i \in I)$ be elements of $\mathscr{R}_c$. Then we have $\wedge_{i \in I} A_i = \cap_{i \in I} A_i$. The partially ordered collection $\mathscr{R}_c$ is a complete lattice.*

2.4. Theorem. *Let $\emptyset \neq X \subseteq \mathscr{C}_a$. Put $T(X) = \operatorname{Ext}\operatorname{Hom} X$.*

(i) *$T(X) \in \mathscr{R}_c$.*

(ii) *If $Y \in \mathscr{R}_c$, $X \subseteq Y$, then $T(X) \subseteq Y$.*

The proof is analogous to that of Proposition 2.2 in [7]; it will be omitted.

In view of 2.4, $T(X)$ is said to be the radical class generated by X. If $X = \{G\}$, then we write $T(G)$ instead of $T(X)$.

2.5. Theorem. *Let A_i $(i \in I)$ be elements of $\mathscr{R}_c$. Then we have $\vee_{i \in I} A_i = \operatorname{Ext} \cup_{i \in I} A_i$.*

Proof. This is a consequence of 2.3 and 2.4.

From 2.3, 2.5 and [7], Corollary 2.3 we obtain:

2.6. Proposition. *$\mathscr{R}$ is a closed ideal of the lattice $\mathscr{R}_c$.*

In view of 2.6 and [7], Example 2.7 we have

2.7. Proposition. *The lattice $\mathscr{R}_c$ is not modular.*

2.8. Lemma. *Let G, G_α and H_α $(\alpha < \beta)$ be as in the definition of the transfinite extension. Then some of the following conditions is valid:*

(i) *There exists $\alpha < \beta$ such that H_α is not linearly ordered; in such a case $H_{\alpha(1)}$ is linearly ordered for each $\alpha(1) < \alpha$, and $H_{\alpha(2)} = \{0\}$ for each $\alpha(2)$ with $\alpha < \alpha(2) < \beta$.*

(ii) *All H_α $(\alpha < \beta)$ are linearly ordered (and thus G is linearly ordered).*

Proof. Let $\alpha < \beta$ and suppose that H_α is not linearly ordered. Since H_α is isomorphic to $G_\alpha / \cup_{\gamma < \alpha} G_\gamma$, we infer that G_α is not linearly ordered. But G_α is a c-convex subgroup of G, and hence in view of $G_\alpha \nsubseteq G^0$ we obtain $G_\alpha = G$. Thus $G_{\alpha(2)} = G$ whenever $\alpha < \alpha(2) < \beta$, therefore $H_{\alpha(2)} = \{0\}$ for such an $\alpha(2)$. Let $\alpha(1) < \alpha$. If $H_{\alpha(1)}$ is not linearly ordered, then we would have $H_\alpha = \{0\}$, which is a contradiction.

Now let X be a nonempty subclass of $\mathscr{C}_a$ such that X contains a zero group. We denote by X_l the class of all linearly ordered groups belonging to X. Put $X_0 = (X \setminus X_l) \cup 0^-$. Since X is closed with respect to isomorphisms, we infer that $0^- \subseteq X$. Hence we have

$$X = X_l \cup X_0, \quad X_l \cap X_0 = 0^-.$$

We apply the just introduced denotations also for the class $\mathscr{C}_a$. Hence $(\mathscr{C}_a)_l$ is the class $\mathscr{L}_a$ of all linearly ordered groups and $(\mathscr{C}_a)_0$ is the class of all cyclically ordered groups which are either zero groups or fail to be linearly ordered.

We infer that

$$\operatorname{Hom} X = \operatorname{Hom} X_l \cup \operatorname{Hom} X_0, \quad \operatorname{Hom} X_l \subseteq (\mathscr{C}_a)_l, \quad \operatorname{Hom} X_0 \subseteq (\mathscr{C}_a)_0.$$

Now let $\{0\} \neq G \in \operatorname{Ext} \operatorname{Hom} X = T(X)$. If $G \in (T(X))_l$, then according to 2.8 we must have $G \in \operatorname{Ext} \operatorname{Hom} X_l$. Conversely, if $G \in \operatorname{Ext} \operatorname{Hom} X_l$, then clearly $G \in (T(X))_l$.

Next let $G \in (T(X))_0$. Since $G \neq \{0\}$, G fails to be linearly ordered. In view of 2.8 there exists a linearly ordered c-convex subgroup H of G such that $H \in \operatorname{Ext} \operatorname{Hom} X_l$ and that $\{0\} \neq G/H \in \operatorname{Hom} X_0$. Conversely, if this condition is satisfied, then clearly $G \in (T(X))_0$.

By summarizing, we obtain:

2.9. Theorem. *Let $X \subseteq \mathscr{C}_a$. Assume that X contains a zero group.*

(i) $(T(X))_l = \operatorname{Ext} \operatorname{Hom} X_l = T(X_l)$.

(ii) *$(T(X))_0$ is the class of all $G \in \mathscr{C}_a$ such that either $G = \{0\}$ or there exists a c-convex subgroup H of G such that $H \in \operatorname{Ext} \operatorname{Hom} X_l$, $\{0\} \neq G/H \in \operatorname{Hom} X_0$.*

Denote

$$\mathscr{R}_{c0} = \{T(X) \colon X \subseteq (\mathscr{C}_a)_0 \text{ and } \{0\} \in X\}.$$

From 2.9 and 2.5 we obtain:

2.10. Corollary. *$\mathscr{R}_{c0}$ is a closed ideal of the lattice $\mathscr{R}_c$.*

2.11. Lemma. *Assume that $\{0\} \neq G' \in \operatorname{Hom}\{G\}$. Let $\varphi\colon G \to K_1 \otimes L$ be a representation of G and let $\varphi'\colon G' \to K_1' \otimes L'$ be a representation of G'. Then*

(i) *the cyclically ordered groups K_1 and K_1' are isomorphic;*

(ii) *the linearly ordered group $(G')^0$ is a homomorphic image of the linearly ordered group G^0.*

Proof. Let $f\colon G \to G'$ be a homomorphism and let H be a kernel of f. In view of 1.2 we can assume (without loss of generality) that $G' = G/H$ and that f is the natural homomorphism of G onto G/H. Since $G/H \neq \{0\}$, we must have $H \subset G$. Thus $H \subseteq G^0$. Therefore in view of 1.3 and 1.4 we obtain that (i) and (ii) are valid.

From 2.8 and 2.11 we infer:

2.12. Lemma. *Let $X \subseteq (\mathscr{C}_a)_0$, $\{0\} \in X$.*

(i) *If* $G \in \operatorname{Hom} X$, *then* $G \in (\mathscr{C}_a)_0$.

(ii) $\operatorname{Ext} X = X$.

As a corollary we obtain:

2.13. Theorem. *Let* $X \subseteq (\mathscr{C}_a)_0$, $\{0\} \in X$. *Then* $T(X) = \operatorname{Hom} X$.

Now, 2.13 and 2.5 yield:

2.14. Corollary. *Let* A_i $(i \in I)$ *be elements of* $(\mathscr{C}_a)_0$. *Then in the lattice* $\mathscr{R}_c$ *the relation* $\vee_{i \in I} A_i = \cup_{i \in I} A_i$ *is valid.*

From 2.10 and 2.14 we infer:

2.15. Corollary. *The lattice* $\mathscr{C}_{a0}$ *is completely distributive.*

If A and B are elements of $\mathscr{R}_c$, then the relation $A \leqq B$ is valid in $\mathscr{R}_c$ if and only if $A_l \leqq B_l$ holds in $\mathscr{R}$ and $A_0 \leqq B_0$ holds in $\mathscr{R}_{c0}$. Thus we have

2.16. Proposition. *The mapping*

$$f: A \to (A_l, A_0) \qquad (A \in \mathscr{R}_c)$$

is an isomorphism of the partially ordered collection $\mathscr{R}_c$ *into the direct product* $\mathscr{R} \times \mathscr{R}_{c0}$.

Let us remark that the mapping f from 2.16 fails to be an isomorphism of the *lattice* $\mathscr{R}_c$ into $\mathscr{R} \times \mathscr{R}_{c0}$.

Let P be a partially ordered class and let Q be a nonempty subclass of P. Assume that there exists a mapping $\psi: P \to Q$ such that

(i) if $p_1, p_2 \in P$ and $p_1 \leqq p_2$, then $\psi(p_1) \leqq \psi(p_2)$;

(ii) if $q \in Q$, then $\psi(q) = q$.

Under these assumptions Q is said to be a retract of P.

From 2.16 we obtain

2.17. Corollary. *Both the partially ordered collections* $\mathscr{R}$ *and* $\mathscr{R}_{c0}$ *are retracts of* $\mathscr{R}_c$.

3. Atoms of $\mathscr{R}_c$

For $X, Y \in \mathscr{R}_c$ we write $X \prec Y$ if $X \subset Y$ and if there does not exist any $Z \in \mathscr{R}_c$ with $X \subset Z \subset Y$. In such a case we also say that Y covers X. If $0^- \prec X$ $(X \prec \mathscr{C}_a)$, then X is said to be an atom in $\mathscr{R}_c$ (or dual atom in $\mathscr{R}_c$, respectively). Analogous denotations are applied for $\mathscr{R}$ and $\mathscr{R}_{c0}$.

3.1. Lemma. (Cf. [7], 4.3 and 4.12.) *In the lattice* $\mathscr{R}$ *there exist no atoms and no dual atoms.*

3.2. Lemma. *If* X *is an atom in* $\mathscr{R}_c$, *then either* $X \in \mathscr{R}$ *or* $X \in \mathscr{R}_{c0}$.

Proof. Let X be an atom in $\mathscr{R}_c$. Then $X_0 \in \mathscr{R}_{c0}$ and $X_l \in \mathscr{R}$. Moreover, we have $X = X_0 \vee X_l$. Since X is an atom in $\mathscr{R}_c$, we infer that either $X = X_0$ or $X = X_l$.

3.3. Lemma. *Let $X \in \mathcal{R}_c$. Then the following conditions are equivalent:*

(i) *X is an atom in $\mathcal{R}_c$.*

(ii) *X is an atom in $\mathcal{R}_{c0}$.*

Proof. This is an immediate consequence of 2.10.

Let K be as in Section 1 and let G be a subgroup of K. We denote by $T[G]$ the class of all cyclically ordered groups isomorphic to G. Put $T_0[G] = T[G] \cup \cup 0^-$. We have

$$\operatorname{Hom} T_0[G] = T_0[G] = \operatorname{Ext} T_0[G].$$

Thus $T_0[G] \in \mathcal{R}_c$. Moreover, $T(G) = T_0[G]$.

3.4. Lemma. *Let $X \in \mathcal{R}_c$. Then the following conditions are equivalent:*

(i) *X is an atom of $\mathcal{R}_c$.*

(ii) *There exists a subgroup $G \neq \{0\}$ of K such that $X = T_0[G]$.*

Proof. Let (ii) be valid. Let $Y \in \mathcal{R}_c$, $0^- \subset Y \subseteq X$. Then there is $G_1 \in Y$, $G_1 \neq \{0\}$. In view of the definition of $T_0[G]$, the cyclically ordered group G_1 must be isomorphic to G, hence

$$0^- \subset X = T(G) = T(G_1) \subseteq Y \subseteq X,$$

implying $X = Y$. Hence (i) holds.

Conversely, assume that (i) is valid. There exists $G \in X$, $G \neq \{0\}$. In view of 3.3 we have $X \in \mathcal{R}_{c0}$, hence $G \in (\mathscr{C}_a)_0$. Then there exists $\{0\} \neq G_1 \in \mathscr{C}_a$ such that $G_1 \in \operatorname{Hom}\{G\}$ and G_1 is isomorphic to a subgroup K_1' of the cyclically ordered group K. Thus $G_1 \in X$, whence $0^- \neq T_0[G_1] \subseteq X$, which infers $X = T_0[G_1]$.

3.5. Proposition. *Let $Y \in \mathcal{R}_{c0}$, $Y \neq 0^-$. Then there is an atom X in $\mathcal{R}_{c0}$ such that $X \leqq Y$.*

The method of proof is analogous to that applied in the second part of the proof of 3.4.

From 3.4 we also obtain:

3.6. Proposition. *Let $\mathscr{A}_1$ be the collection of all atoms of the lattice $\mathcal{R}_c$. Then $\mathscr{A}_1$ is infinite, but it fails to be a proper collection.*

The elements of $\mathscr{A}_1$ are of height 1 in the lattice $\mathcal{R}_c$. We can define an element Y to be of height 2 if there exists $X \in \mathscr{A}_1$ such that X is covered by Y. We denote by $\mathscr{A}_2$ the collection of all elements of height 2 in the lattice $\mathcal{R}_c$. The natural question arises whether $\mathscr{A}_2$ is nonempty.

We can construct an element of $\mathscr{A}_2$ as follows. Let α be an infinite cardinal. We denote by $\omega(\alpha)$ the first ordinal with cardinality α. Let I be a linearly ordered set dually isomorphic to $\omega(\alpha)$ and for each $i \in I$ let $G_{(i)}$ be a linearly ordered group isomorphic to the additive group of all reals with the natural linear order. Let G_α be a lexicographic product of linearly ordered groups G_i $(i \in I)$. Let K_1 be a nonzero subgroup of the cyclically ordered group K. We put

$$G_\alpha^* = G_\alpha \otimes K_1 .$$

Let H be a homomorphic image of G_α, $H \neq \{0\}$. From the definition of I and from the fact that $G_{(i)}$ has no nontrivial convex subgroup we infer that H is isomorphic to G_α. Therefore if H' is a nonzero homomorphic image of G_α^*, then H' is isomorphic either to K_1 or to G_α^*. Thus in view of 2.4 and 2.13 we obtain:

3.7. Lemma. $T_c(K_1) \prec T_c(G_\alpha^*)$.

According to 3.4 and 3.7 we have

3.8. Lemma. $T_c(G_\alpha^*) \in \mathscr{A}_2$.

If β is an infinite cardinal with $\beta \neq \alpha$, then G_α and G_β are not isomorphic, thus G_α^* is not isomorphic to G_β^*. We also easily obtain that $T(G_\alpha^*) \neq T(G_\beta^*)$. Now from 3.8 and 3.4 we get

3.9. Theorem. *Let X be an atom in $\mathscr{R}_{c0}$. Then the collection of all elements of $\mathscr{A}_2$ which cover X is a proper collection.*

3.10. Proposition. *Let α and K_1 be as above. Then the interval* $[0^-, T(G_\alpha^*)]$ *of the lattice $\mathscr{R}_c$ is a three-element chain.*

The proof will be omitted.

Let us denote by X_a the join of all atoms of the lattice $\mathscr{R}_{c0}$.

3.11. Proposition. *The radical class X_a fails to be the greatest element of the lattice $\mathscr{R}_{c0}$.*

Proof. If X_i $(i \in I)$ are atoms of the lattice $\mathscr{R}_{c0}$, then in view of 2.14 we have

(1) $$\vee_{i \in I} X_i = \cup_{i \in I} X_i .$$

Also, from 3.4 we infer that if $\{0\} \neq G \in \cup_{i \in I} X_i$, then G is isomorphic to a subgroup of K.

Let $L \neq \{0\}$ be a linearly ordered group and let $K_1 \neq \{0\}$ be a subgroup of K. Then $K_1 \otimes L$ belongs to $\mathscr{R}_{c0}$ and does not belong to $\cup_{i \in I} X_i$. Hence $X_a \neq (\mathscr{C}_a)_0$.

From the above result and from (1) we conclude:

3.12. Corollary. (i) *The lattice $\mathscr{R}_{c0}$ fails to be atomic.* (ii) *The interval* $[0^-, X_a]$ *of the lattice $\mathscr{R}_{c0}$ is isomorphic to the Boolean algebra of all subsets of nonisomorphic types of nonzero subgroups of K.*

4. Dual atoms of the lattice $\mathscr{R}_c$

As we already remarked above, there are no dual atoms in the lattice $\mathscr{R}$. Let us now investigate the existence of dual atoms in the lattices $\mathscr{R}_c$ and $\mathscr{R}_{c0}$.

4.1. Proposition. *The lattice $\mathscr{R}_{c0}$ has no dual atom.*

Proof. By way of contradiction, assume that Y is a dual atom of the lattice $\mathscr{R}_{c0}$. Then there exists $\{0\} \neq G \in (\mathscr{C}_a)_0$ such that G does not belong to Y. Next there exists a nonzero subgroup K_1 of K such that $K_1 \in \operatorname{Hom}\{G\}$. Let α be an infinite cardinal, $\alpha > \operatorname{card} G$. Let G_α be as in Section 3. Put

$$H = G_a \otimes G.$$

Since $G_1 \in \operatorname{Hom}\{H\}$, H fails to be linearly ordered. Hence $\{0\} \neq H \in (\mathscr{C}_a)_0$. In view of 2.13 we have $T(G) = \operatorname{Hom}\{G\}$, hence $H \notin T(G)$. Since Y is a dual atom in $\mathscr{R}_{c0}$, we obtain

$$H \in (\mathscr{C}_a)_0 = Y \vee T(G) = Y \vee \operatorname{Hom}\{G\}.$$

In view of 2.13 and 2.14 we have

$$Y \vee \operatorname{Hom}\{G\} = \operatorname{Hom} Y \cup \operatorname{Hom}\{G\} = Y \cup \operatorname{Hom}\{G\},$$

hence $H \in Y \cup \operatorname{Hom}\{G\}$. As H does not belong to $\operatorname{Hom}\{G\}$ we get $H \in Y$. Since $G \in \operatorname{Hom}\{H\}$ we obtain $G \in Y$, which is a contradiction.

From 4.1 and 3.6 we infer:

4.2. Corollary. *The lattice $\mathscr{R}_{c0}$ is not self-dual.*

4.3. Lemma. *Let K_1 be a subgroup of K, $K_1 \neq \{0\}$. Let $\{Y_i\}_{i \in I}$ be the class of all $Y \in \mathscr{R}_c$ such that K_1 does not belong to Y_i. Put $Y = \vee_{i \in I} Y_i$. Then Y is a dual atom of the lattice $\mathscr{R}_c$.*

Proof. If Z is any class of cyclically ordered groups such that $K_1 \notin Z$, then $K_1 \notin \operatorname{Ext} Z$. From this and from 2.5 we infer that K_1 does not belong to Y. Hence $Y \subset \mathscr{C}_a$.

By way of contradiction, assume that Y fails to be a dual atom of $\mathscr{R}_c$. Then there exists $Z \in \mathscr{R}_c$ such that $Y \subset Z \subset \mathscr{C}_a$. Thus there is $G \in Z \setminus Y$. Since $T(G) \subseteq \subseteq Z$, we obtain

$$Y \vee T(G) \subseteq Z.$$

We distinguish two cases.

(i) $K_1 \notin \operatorname{Hom}\{G\}$. Then $K_1 \notin T(G)$, hence $T(G) \leqq Y$, implying $G \in Y$, which is a contradiction.

(ii) $K_1 \in \operatorname{Hom}\{G\}$. Let $G_1 \in \mathscr{C}_a$. If $K_1 \notin \operatorname{Hom}\{G_1\}$, then $K_1 \notin T(G_1)$, thus $T(G_1) \leqq Y$ and therefore $G_1 \in Y$. Now assume that $K_1 \in \operatorname{Hom}\{G_1\}$. Then G_1 is an extension of a linearly ordered group L_1 by means of K_1. Thus we have $L_1 \in Y$, $K_1 \in T(G)$, hence

$$\{L_1, K_1\} \subseteq Y \vee T(G).$$

Since the class $Y \vee T(G)$ is closed with respect to extensions, we get $G_1 \in Y \vee \vee T(G)$. Therefore $\mathscr{C}_a \subseteq Z$, which is a contradiction.

Let K_1 and Y be as in 4.3. We denote $Y = K_1^\delta$.

4.4. Lemma. *Let K_1 and K_2 be nonzero subgroups of K. Assume that K_1 is not isomorphic to K_2. Then $K_1^\delta \neq K_2^\delta$.*

Proof. We have $K_1 \in K_2^\delta$, $K_1 \notin K_1^\delta$.

Since there exists an infinite set of mutually nonisomorphic subgroups of K, from 4.3 and 4.4 we infer

4.5. Theorem. *The lattice $\mathscr{R}_c$ has infinitely many dual atoms.*

4.6. Lemma. *Let Y be an atom of the lattice $\mathscr{R}_c$. Then $(\mathscr{C}_a)_l \subseteq Y$.*

Proof. We have to verify that $Y_l = (\mathscr{C}_a)_l$. By way of contradiction, assume that $Y_l \neq (\mathscr{C}_a)_l$. In view of 2.9 we have $Y_l \in \mathscr{R}$. Since $(\mathscr{C}_a)_l$ is the greatest element of $\mathscr{R}$ and since $\mathscr{R}$ has no dual atoms, there is $Y_1 \in \mathscr{R}$ such that $Y_l \subset Y_1 \subset (\mathscr{C}_a)_l$. Hence there is a linearly ordered group G_1 which does not belong to Y_1. Also, there is a linearly ordered group G belonging to $Y_1 \backslash Y_l$. Then $G \notin Y$, whence $Y \vee T(G) = \mathscr{C}_a$. Thus

$$G_1 \in Y \vee T(G) = \operatorname{Ext}(Y \cup T(G)) \subseteq \operatorname{Ext}(Y \cup Y_1).$$

Because G_1 is linearly ordered, we have

$$G_1 \in \operatorname{Ext}(Y_l \cup Y_1) = \operatorname{Ext} Y_1 = Y_1,$$

which is a contradiction.

4.7. Proposition. *Let Y be a dual atom of the lattice $\mathscr{R}_c$. Then there is a nonzero subgroup K_1 of K such that $Y = K_1^{\delta}$.*

Proof. In view of 4.6 we have $(\mathscr{C}_a)_l \subseteq Y$. If all nonzero subgroups of K belong to Y, then we have $Y = \mathscr{C}_a$ (since Y is closed with respect to extensions). Thus there is a nonzero subgroup K_1 of K such that K_1 does not belong to Y. According to the definition of K_1^{δ} we have $Y \subseteq K_1^{\delta} \subset \mathscr{C}_a$. Since Y is a dual atom of $\mathscr{R}_c$ and since $K_1^{\delta} \in \mathscr{R}_c$ we obtain that $Y = K_1^{\delta}$.

4.8. Corollary. *Let $Y \in \mathscr{R}_c$. The following conditions are equivalent:*

(i) *Y is a dual atom of the lattice $\mathscr{R}_c$.*

(ii) *There is a nonzero subgroup K_1 of K such that $Y = K_1^{\delta}$.*

4.9. Corollary. *Let $\mathscr{D}$ be the collection of all dual atoms of the lattice $\mathscr{R}_c$. Then $\mathscr{D}$ fails to be a proper collection.*

5. K-radical classes

For $G \in \mathscr{C}_a$ we denote by $c(G)$ the system of all c-convex subgroups of G. The system $c(G)$ is partially ordered by inclusion.

Since $H \subseteq G^0$ is valid for each $H \in c(G)$ with $H \neq G$ and since G^0 is linearly ordered, we have

5.1. Lemma. *$c(G)$ is a linearly ordered set.*

Let $X \in \mathscr{R}_c$. If for each $G \in X$ and each $G_1 \in \mathscr{C}_a$ the relation

$$c(G) \simeq c(G_1) \Rightarrow G_1 \in X$$

is valid, then X will be said to be a K-radical class.

K-radical classes of linearly ordered groups are defined analogously (c-convexity is replaced by convexity). We denote by

$\mathscr{R}_k$ — the class of all K-radical classes of linearly ordered groups;

$\mathscr{R}_{ck}$ — the class of all K-radical classes of cyclically ordered groups.

Both $\mathcal{R}_k$ and $\mathcal{R}_{ck}$ are partially ordered by inclusion. In [8] it was proved that $\mathcal{R}_k$ is a lattice.

From the property of G^0 mentioned above we infer:

5.2. Lemma. *Let $G \in (\mathscr{C}_a)_0$, $G \neq \{0\}$. Then G^0 is a dual atom of the linearly ordered set $c(G)$.*

5.3. Lemma. *Let G be a linearly ordered group such that $c(G)$ has a dual atom. Then there is $G_1 \in (\mathscr{C}_a)_0$ such that $c(G) \simeq c(G_1)$.*

Proof. Let G' be a dual atom of $c(G)$. Let K_1 be a nonzero subgroup of K. Put $G_1 = K_1 \otimes G'$. Then $c(G) \simeq c(G_1)$ and $G_1 \in (\mathscr{C}_a)_0$.

Let $X \in \mathcal{R}_k$. We denote by X^+ the class of all $G_1 \in (\mathscr{C}_a)_0$ such that there is $G \in X$ with $c(G) \simeq c(G_1)$.

5.4. Lemma. *Let $X \in \mathcal{R}_k$ and $\{0\} \neq G_1 \in (\mathscr{C}_a)_l$. The following conditions are equivalent*: (i) $G_1 \in X^+$; (ii) *there is $G \in X$ such that chain $c(G)$ has a dual atom and $c(G) \simeq c(G_1)$.*

Proof. This is a consequence of 5.3.

For $X \in \mathcal{R}_k$ we denote $X^* = X^+ \cup X$. From the definition of X^* we obtain immediately:

5.5. Lemma. *Let $X \in \mathcal{R}_k$, $G \in X^*$ and $G_1 \in \mathscr{C}_a$. If $c(G_1) \simeq c(G)$, then $G_1 \in X^*$.*

5.6. Lemma. *Let $X \in \mathcal{R}_k$. Then X^* is closed with respect to homomorphisms.*

Proof. Let $G \in X^*$ and let G' be a homomorphic image of G. If $G \in X$, then clearly $G' \in X \subseteq X^*$. Assume that G does not belong to X. Then $\{0\} \neq G \in (\mathscr{C}_a)_0$ and there is $H \in X$ such that $c(H)$ has a dual atom G_1 and $G_1 \simeq G^0$. The case $G' = \{0\}$ is trivial; suppose that $G' \neq \{0\}$.

In view of lemma 2.11, $(G')^0$ is a homomorphic image of G^0, hence $(G')^0$ is a homomorphic image of G_1. Thus there exists $H_1 \in c(G_1)$ such that $(G')^0 \simeq G_1/H_1$. In particular, $c((G')^0) \simeq c(G_1/H_1)$. From this we infer that $c(G') \simeq c(H/H_1)$. The class X is closed with respect to homomorphisms, whence $H/H_1 \in X$ and thus $G' \in X^*$.

5.7. Lemma. *Let $X \in \mathcal{R}_k$. Then X^* is closed with respect to transfinite extensions.*

Proof. Let $G \in \mathscr{C}_a$. Assume that $G_\alpha \in c(G)$ for each $\alpha < \beta$,

$$\{0\} \subseteq G_0 \subseteq G_1 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots, \quad \cup_{\alpha<\beta} G_\alpha = G$$

and that $G_\alpha / \cup_{\gamma<\alpha} G_\gamma \in X^*$ is valid for each $\alpha < \beta$. We have to verify that G belongs to X^*. We distinguish two cases.

At first suppose that G is linearly ordered. Then all G_α are linearly ordered and all $H_\alpha = G_\alpha / \cup_{\gamma<\alpha} G_\gamma$ are linearly ordered as well. Hence all H_α belong to X; since X is a radical class, we obtain $G \in X \subseteq X^*$.

Now suppose that G is not linearly ordered. Then in view of 2.8 there is $\alpha(0) < \beta$ such that $G_{\alpha(0)} = G$ and $G_\alpha < G$ for each $\alpha < \alpha(0)$. Thus $G_\alpha \subseteq (G_{\alpha(0)})^0$

for each $\alpha < \alpha(0)$. We have $H_{\alpha(0)} \neq \{0\}$ and $H_{\alpha(0)} \in X^*$. Thus there is $H'_{\alpha(0)} \in X$ such that $c(H_{\alpha(0)}) \simeq c(H'_{\alpha(0)})$. Denote

$$D = \cup_{\alpha < \alpha(0)} G_\alpha .$$

For each $\alpha < \alpha(0)$, G_α is linearly ordered and thus D is linearly ordered as well. Moreover, if $\alpha < \alpha(0)$, then $H_\alpha \in X$. Therefore $D \in X$. Denote $G' = D \circ H'_{\alpha(0)}$ (the symbol $\circ$ stands for the operation of the lexicographic product). G' is an extension of D by means of $H'_{\alpha(0)}$; since $\{D, H_{\alpha(0)}\} \subseteq X$, we obtain $G' \in X$. We have $c(G') \simeq c(G)$. Thus according to 5.4, $G \in X^*$.

5.8. Theorem. *Let $X \in \mathcal{R}_k$. Then X^* belongs to $\mathcal{R}_{ck}$. If $Y \in \mathcal{R}_{ck}$ and $X \subseteq Y$, then $X^* \subseteq Y$.*

Proof. The first assertion follows from 5.5, 5.6 and 5.7. The second assertion is obvious.

The following lemma is easy to verify.

5.9. Lemma. *Let $Y \in \mathcal{R}_{ck}$. Let $f(Y)$ be the set of all linearly ordered groups G having the property that there exists $G_1 \in Y$ with $c(G) \simeq c(G_1)$. Then $f(Y) \in \mathcal{R}_k$ and $(f(Y))^* = Y$.*

5.10. Corollary. $\mathcal{R}_{ck} = \{X^*: X \in \mathcal{R}_k\}$.

If X_1 and X_2 are elements of $\mathcal{R}_k$, then clearly

$$X_1 \subseteq X_2 \Rightarrow X_1^* \subseteq X_2^*.$$

In view of 5.9 we also have

$$X_1^* \subseteq X_2^* \Rightarrow X_1 \subseteq X_2 .$$

Thus according to 5.8 we obtain:

5.11. Theorem. *The mapping $X \to X^*$ is an isomorphism of the lattice $\mathcal{R}_k$ onto the partially ordered collection $\mathcal{R}_{ck}$; hence $\mathcal{R}_{ck}$ is a lattice.*

If $X \in \mathcal{R}_k$, then X need not belong to $\mathcal{R}_{ck}$; hence $\mathcal{R}_k$ is not a subcollection of $\mathcal{R}_{ck}$.

5.12. Theorem. *The lattice $\mathcal{R}_{ck}$ has no atoms and no dual atoms.*

Proof. This is a consequence of 5.11 and of [8], Theorems 4.4 and 4.14.

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РАДИКАЛЬНЫЕ КЛАССЫ ЦИКЛИЧЕСКИ УПОРЯДОЧЕННЫХ ГРУПП

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Резюме

С. Г. ЧЕХАТА и Р. ВИГАНДТ ввели понятие радикального класса линейно упорядоченных групп. В настоящей статье это понятие расширяется для случая циклически упорядоченных групп. Исследуются свойства решетки всех радикальных классов абелевых циклически уподядоченных групп.