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PRODUCTS OF QUASI-CONTINUOUS FUNCTIONS

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ABSTRACT. Functions which can be factored into a product of quasi-continuous functions are characterized.

Let us establish some of the terminology to be used. \( \mathbb{R} \) denotes the real line. For \( f : \mathbb{R} \to \mathbb{R} \) and \( a \in \mathbb{R} \) we define the set \([f < a]\) as \( \{ x \in \mathbb{R} : f(x) < a \} \). Analogously we define the sets \([f > a]\) and \([f = a]\).

A real function \( f \) defined on \( \mathbb{R} \) is said to be quasi-continuous (cliquish) at a point \( x_0 \in \mathbb{R} \) iff for every \( \varepsilon > 0 \) and for any neighbourhood \( U \) of the point \( x_0 \) there exists an open set \( V \) such that \( 0 \neq V \subset U \) and \( |f(x) - f(x_0)| < \varepsilon \) for each \( x \in V \) (respectively \( |f(x_1) - f(x_2)| < \varepsilon \) for \( x_1, x_2 \in V \) [1].

A function \( f \) is quasi-continuous (cliquish) on an interval \( J \subset \mathbb{R} \) iff \( f \) is quasi-continuous (cliquish) at every point of \( J \). Of course, every point at which \( f \) is continuous is a quasi-continuity point of \( f \) and every quasi-continuity point of \( f \) is a cliquish point of this function.

It is easy to see that a sum or a product of two cliquish functions are cliquish. Thus a sum and a product of quasi-continuous functions are cliquish (but not necessarily quasi-continuous). Z. Grande proved in [2] (Th. 5) that every cliquish function \( f \) is a sum of four quasi-continuous functions. Observe that if \( f \) is Lebesgue measurable (or with the Baire property, or of the Baire class \( a \)), then the factors from the proof of Grande can be taken from the adequate class.

In fact Grande proved his theorem for cliquish functions defined on \( \mathbb{R} \) but from that result it follows easily that every cliquish function defined on an open interval \( I \) is also sum of four quasi-continuous functions (defined on \( I \)). Now we may notice the following fact.

**Fact.** If \( g \) and \( h \) are quasi-continuous on an open interval \( I \) and \( a, b \in \mathbb{R} \), then there exists a function \( s \) continuous on \( I \) such that \( g + s \) and \( h - s \) are quasi-continuous functions whose cluster sets at the left and the right end points of \( I \) contain \( a \) and \( b \), respectively.
Of course, if \( g \) and \( h \) are Lebesgue measurable (or with the Baire property, or of the Baire class \( a \)), then \( g + s \) and \( h - s \) are of the adequate class too.

Consequently, Grande’s theorem holds for cliquish functions defined on any interval of an arbitrary type. In the same paper ([2]) Grande remarked that the function \( f: \mathbb{R} \to \mathbb{R} \) defined as

\[
f(x) = \begin{cases} 
1/q & \text{if } x \text{ is rational, } x = p/q, q > 0 \text{ and } (p, q) = 1, \\
0 & \text{if } x \text{ is irrational}
\end{cases}
\]

cannot be the product of a finite number of quasi-continuous functions.

The purpose of this article is to characterize those functions which can be factored into a product of quasi-continuous functions. The result is as follows.

**Theorem.** A function \( h: \mathbb{R} \to \mathbb{R} \) is a product of quasi-continuous functions iff:

(i) \( h \) is cliquish, and

(ii) each of the sets \([h = 0], [h > 0] \text{ and } [h < 0]\) is the union of an open set and a nowhere-dense set. Moreover, if \( h \) is Lebesgue measurable (respectively with the Baire property or of Baire class \( a \)), then the factors can be taken to be Lebesgue measurable (respectively with the Baire property or of the Baire class \( a \)).

We begin with the following lemma.

**Lemma.** If \( I \subset \mathbb{R} \) is an interval and \( f: I \to \mathbb{R} \) is a cliquish function which is always positive or always negative, then \( f \) is the product of four quasi-continuous functions \( f_1, \ldots, f_4: I \to \mathbb{R} \). Moreover, if \( f \) is Lebesgue measurable (or with the Baire property, or of the Baire class \( a \)), then \( f_1, \ldots, f_4 \) can be taken from the adequate class.

**Proof.** Let \( g(x) = |f(x)| \) for \( x \in I \). Then \( \ln(g) \) is a cliquish function and there exist quasi-continuous functions \( g_1, \ldots, g_4: I \to \mathbb{R} \) such that \( \ln(g) = g_1 + g_2 + g_3 + g_4 \). Now the functions \( f_i = \text{sgn}(f) \cdot \exp(g_i), f_i = \exp(g_i) \) for \( i = 2, 3, 4 \), are quasi-continuous and \( f = f_1 \circ \ldots \circ f_4 \).

The second part of this lemma follows from the remark that if \( f \) is of the adequate class, then \( \ln(g) \) is of this same class and \( g_1, \ldots, g_4 \) belong to this class too.

**Proof of Theorem.** First, let us assume that \( h \) is a product of \( n \) quasi-continuous functions \( h = f_1 \circ \ldots \circ f_n \). \( h \), as a product of cliquish functions, is cliquish.

Notice that for a quasi-continuous function \( f \) each of the sets \([f = 0], [f < 0] \) and \([f > 0]\) is the union of an open set \( G(f, 0), G(f, -) \) and \( G(f, +) \) (respectively) and a nowhere-dense set \( A(f, 0), A(f, -) \) and \( A(f, +) \). Here we can assume that the set \( A(f, 0) \) is disjoint with \( G(f, 0) \) and analogously for \( A(f, -) \) and \( A(f, +) \). Thus \([h = 0] = \bigcup_{i=1}^n G(f_i, 0) \cup \bigcup_{i=1}^n A(f_i, 0) \) and this set is the union of
an open set $\bigcup \{G(f, 0) \mid i = 1, 2, \ldots, n\}$ and nowhere-dense set $\bigcup \{A(f, 0) \mid i = 1, 2, \ldots, n\}$. Since $[h < 0] = [f_1 \circ \cdots \circ f_{n-1} < 0] \cap [f_n > 0] \cup [f_1 \circ \cdots \circ f_{n-1} > 0] \cap \cdots \cap [f_n < 0]$, we obtain (by induction) that this set is the union of an open set and a nowhere-dense set. Similarly we can prove that the set $[h > 0]$ is the union of an open set and a nowhere-dense set. Let us then assume that $h$ satisfies the conditions (i) and (ii). The set $A = (G(h, 0) \setminus G(h, 0)) \cup (G(h, +) \setminus G(h, +)) \cup (G(h, -) \setminus G(h, -)) = \mathbb{R} \setminus (G(h, 0) \cup G(h, +) \cup G(h, -)) = A(h, 0) \cup A(h, +) \cup A(h, -)$ is closed and nowhere-dense and there exists a sequence $(K_n)_n$ of non-empty open intervals such that:

1. $K_n \cap A = \emptyset$, i.e. $K_n \subset G(h, 0)$, or $K_n \subset G(h, +)$, or $K_n \subset G(h, -)$,
2. if $K_n \cap K_m \neq \emptyset$, then $n = m, n, m \in \mathbb{N}$,
3. for every $x \in A$ and for each neighbourhood $U$ of $x$ the set $\{n \mid K_n \subset U\}$ is infinite,
4. if $x \notin A$, then there exists a neighbourhood $U$ of $x$ such that the set $\{n \mid U \cap K_n \neq \emptyset\}$ has at most one element.

Fix $n \in \mathbb{N}$. We can choose a sequence $(K_{n,m})_{m \leq n}$ of open subintervals of $K_n$ such that:

5. $K_{n,m} \subset K_n$ for $m \leq n$,
6. if $K_{n,m} \cap K_{n,t} \neq \emptyset$, then $m = t$.

Notice that the set $A \cup G(h, 0) \cup \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{n} K_{n,m}$ is closed. Let $(I_n)_n$ be a sequence of all components of the complement of this set. Then $I_n \subset G(h, -)$ or $I_n \subset G(h, +)$ for every component $I_n$. Let $t_{n,1}, \ldots, t_{n,4} : I_n \to \mathbb{R}$ be quasi-continuous functions such that $h|I_n = t_{n,1} \circ \cdots \circ t_{n,4}$.

Similarly, if the interval $K_{n,m}$ is not contained in the set $G(h, 0)$, then $K_{n,m} \subset G(h, -)$ or $K_{n,m} \subset G(h, +)$ and there exist quasi-continuous functions $k_{n,m,1}, \ldots, k_{n,m,4} : K_{n,m} \to \mathbb{R}$ such that $h|K_{n,m}$ is the product of $k_{n,m,1}, \ldots, k_{n,m,4}$.

Let $(w_n)$ be a sequence of all rationals different than zero. Now we can define quasi-continuous functions $f_1, \ldots, f_8 : \mathbb{R} \to \mathbb{R}$ in the following way.

$$f_1(x) = \begin{cases} h(x) & \text{for } x \in A, \\ w_m & \text{for } x \in K_{n,2m}, \quad n \geq 2m, \\ 0 & \text{for } x \in G(h, 0) \setminus \bigcup_{n,m} K_{n,2m}, \\ k_{n,2m+1,1}(x) & \text{for } x \in K_{n,2m+1} \setminus G(h, 0), \quad n \geq 2m + 1, \\ t_{n,1}(x) & \text{for } x \in I_n, \quad n \in \mathbb{N} \end{cases}$$
Clearly $h = f_1 \circ \ldots \circ f_8$. Let us now show that $f_1$ is quasi-continuous. The proof that $f_i$, $i = 2, \ldots, 8$, are quasi-continuous is similar. It is enough to verify that $f_1$ is quasi-continuous at every point $x_0 \in A$. Fix $\varepsilon > 0$ and a neighbourhood $U$ of $x_0$. Let $w_m$ be a rational number from the interval $(h(x_0) - \varepsilon, h(x_0) + \varepsilon)$. Then there exists $n \geq 2m$ such that $K_{n,2m} \subset U$ and $f_1(x) = w_m$ for $x \in K_{n,2m}$. Thus $f_1$ is quasi-continuous at the point $x_0$.

If $h$ is a function of the Baire class $\alpha$ (Lebesgue measurable, with the Baire property), then the functions $f_1, \ldots, f_8$ belong to the adequate class. We shall verify this fact for $f_1$ in the case when $h$ is a Baire 1 function. Then every function $t_{n,i}$ is of the Baire class 1 on $I_n$ ($n \in \mathbb{N}$, $i = 1, 2, 3, 4$) and every function $k_{n,m,i}$ is of the Baire class 1 on $K_{n,m}$ ($n \in \mathbb{N}$, $m \leq n$, $i = 1, 2, 3, 4$). For an open set $G \subset \mathbb{R}$ each of the sets $h^{-1}(G) \cap A$, $\bigcup \{ K_{n,2m} : w_m \in G \}$, $\bigcup k_{n,2m+1,1}^{-1}(G) \cap (K_{n,2m+1} \setminus G(h,0))$ and $\bigcup t_{n,1}^{-1}(G) \cap I_n$ is a $F_{\alpha}$ set. Moreover, the set $\bigcup K_{n,2m} \cap G(h,0)$ is closed in $G(h,0)$, thus the set $G(h,0) \setminus \bigcup K_{n,2m}$ is open. It follows that $f_1^{-1}(G)$ is a $F_{\alpha}$ set, so $f_1$ is of the Baire class 1. This finishes the proof of the Theorem.
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