Tomasz Natkaniec

Products of quasi-continuous functions

Mathematica Slovaca, Vol. 40 (1990), No. 4, 401--405

Persistent URL: http://dml.cz/dmlcz/129385

Math. Slovaca 40, 1990, No. 4, 401—405

PRODUCTS OF QUASI-CONTINUOUS FUNCTIONS

TOMASZ NATKANIEC

ABSTRACT. Functions which can be factored into a product of quasi-continuous functions are characterized.

Let us establish some of the terminology to be used. $\mathbb{R}$ denotes the real line. For $f\colon \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$ we define the set $[f < a]$ as $\{x \in \mathbb{R}: f(x) < a\}$. Analogously we define the sets $[f > a]$ and $[f = a]$.

A real function f defined on $\mathbb{R}$ is said to be quasi-continuous (cliquish) at a point $x_0 \in \mathbb{R}$ iff for every $\varepsilon > 0$ and for any neighbourhood U of the point x_0 there exists an open set V such that $\emptyset \neq V \subset U$ and $|f(x) - f(x_0)| < \varepsilon$ for each $x \in V$ (respectively $|f(x_1) - f(x_2)| < \varepsilon$ for $x_1, x_2 \in V$) [1].

A function f is quasi-continuous (cliquish) on an interval $J \subset \mathbb{R}$ iff f is quasi-continuous (cliquish) at every point of J. Of course, every point at which f is continuous is a quasi-continuity point of f and every quasi-continuity point of f is a cliquish point of this function.

It is easy to see that a sum or a product of two cliquish functions are cliquish. Thus a sum and a product of quasi-continuous functions are cliquish (but not necessarily quasi-continuous). Z. Grande proved in [2] (Th. 5) that every cliquish function f is a sum of four quasi-continuous functions. Observe that if f is Lebesgue measurable (or with the Baire property, or of the Baire class α), then the factors from the proof of Grande can be taken from the adequate class. In fact Grande proved his theorem for cliquish functions defined on $\mathbb{R}$ but from that result it follows easily that every cliquish function defined on an open interval I is also sum of four quasi-continuous functions (defined on I). Now we may notice the following fact.

Fact. If g and h are quasi-continuous on an open interval I and $a, b \in \mathbb{R}$, then there exists a function s continuous on I such that $g + s$ and $h - s$ are quasi-continuous functions whose cluster sets at the left and the right end points of I contain a and b, respectively.

AMS Subject Classification (1985): Primary 26A16

Key words: Quasi-continuity, Cliquish, Lebesque measurable, Baire property, Baire class

Of course, if g and h are Lebesgue measurable (or with the Baire property, or of the Baire class α), then $g+s$ and $h-s$ are of the adequate class too.

Consequently, Grande's theorem holds for cliquish functions defined on any interval of an arbitrary type. In the same paper ([2]) Grande remarked that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 1/q & \text{if } x \text{ is rational, } x = p/q,\ q > 0 \text{ and } (p, q) = 1, \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

cannot be the product of a finite number of quasi-continuous functions.

The purpose of this article is to characterize those functions which can be factored into a product of quasi-continuous functions. The result is as follows.

Theorem. *A function $h\colon \mathbb{R} \to \mathbb{R}$ is a product of quasi-continuous functions iff:*

(i) *h is cliquish, and*

(ii) *each of the sets $[h = 0]$, $[h > 0]$ and $[h < 0]$ is the union of an open set and a nowhere-dense set. Moreover, if h is Lebesgue measurable (respectively with the Baire property or of Baire class α), then the factors can be taken to be Lebesgue measurable (respectively with the Baire property or of the Baire class α).*

We begin with the following lemma.

Lemma. *If $I \subset R$ is an interval and $f\colon I \to \mathbb{R}$ is a cliquish function which is always positive or always negative, then f is the product of four quasi-continuous functions $f_1, \ldots, f_4\colon I \to \mathbb{R}$. Moreover, if f is Lebesgue measurable (or with the Baire property, or of the Baire class α), then $f_1, \ldots, f_4$ can be taken from the adequate class.*

Proof. Let $g(x) = |f(x)|$ for $x \in I$. Then $\ln(g)$ is a cliquish function and there exist quasi-continuous functions $g_1, \ldots, g_4\colon I \to \mathbb{R}$ such that $\ln(g) = g_1 + g_2 + g_3 + g_4$ [2]. Now the functions $f_1 = \operatorname{sgn}(f)\cdot\exp(g_1)$, $f_i = \exp(g_i)$ for $i = 2, 3, 4$, are quasi-continuous and $f = f_1 \circ \ldots \circ f_4$.

The second part of this lemma follows from the remark that if f is of the adequate class, then $\ln(g)$ is of this same class and $g_1, \ldots, g_4$ belong to this class too.

Proof of Theorem. First, let us assume that h is a product of n quasi-continuous functions $h = f_1 \circ \ldots \circ f_n$. h, as a product of cliquish functions, is cliquish.

Notice that for a quasi-continuous function f each of the sets $[f = 0]$, $[f < 0]$ and $[f > 0]$ is the union of an open set $G(f, 0)$, $G(f, -)$ and $G(f, +)$ (respectively) and a nowhere-dense set $A(f, 0)$, $A(f, -)$ and $A(f, +)$. Here we can assume that the set $A(f, 0)$ is disjoint with $G(f, 0)$ and analogously for $A(f, -)$ and $A(f, +)$. Thus $[h = 0] = \bigcup_{i=1}^{n} G(f_i, 0) \cup \bigcup_{i=1}^{n} A(f_i, 0)$ and this set is the union of an open set $\bigcup\{G(f_i, 0)\ i = 1, 2, \ldots, n\}$ and nowhere-dense set $\bigcup\{A(f_i, 0)\colon i = 1, 2, \ldots, n\}$. Since $[h < 0] = [f_1 \cdot \ldots \cdot f_{n-1} < 0] \cap [f_n > 0] \cup [f_1 \circ \ldots \circ f_{n-1} > 0] \cap \cap [f_n < 0]$, we obtain (by induction) that this set is the union of an open set and a nowhere-dense set. Similarly we can prove that the set $[h > 0]$ is the union of an open set and a nowhere-dense set. Let us then assume that h satisfies the conditions (i) and (ii). The set $A = (\overline{G(h, 0)} \setminus G(h, 0)) \cup (\overline{G(h, +)} \setminus G(h, +)) \cup \cup (\overline{G(h, -)} \setminus G(h, -)) = \mathbb{R} \setminus (G(h, 0) \cup G(h, +) \cup G(h, -)) = A(h, 0) \cup \cup A(h, +) \cup A(h, -)$ is closed and nowhere-dense and there exists a sequence $(K_n)_n$ of non-empty open intervals such that:

(1) $\bar{K}_n \cap A = \emptyset$, i.e. $\bar{K}_n \subset G(h, 0)$, or $\bar{K}_n \subset G(h, +)$, or $\bar{K}_n \subset G(h, -)$,

(2) if $\bar{K}_n \cap \bar{K}_m \neq \emptyset$, then $n = m$, $n, m \in \mathbb{N}$,

(3) for every $x \in A$ and for each neighbourhood U of x the set $\{n\colon \bar{K}_n \subset U\}$ is infinite,

(4) if $x \notin A$, then there exists a neighbourhood U of x such that the set $\{n\colon U \cap \bar{K}_n \neq \emptyset\}$ has at most one element.

Fix $n \in \mathbb{N}$. We can choose a sequence $(K_{n,m})_{m \leqslant n}$ of open subintervals of K_n such that:

(5) $\bar{K}_{n,m} \subset K_n$ for $m \leqslant n$,

(6) if $\bar{K}_{n,m} \cap \bar{K}_{n,t} \neq \emptyset$, then $m = t$.

Notice that the set $A \cup \overline{G(h, 0)} \cup \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{n} \bar{K}_{n,m}$ is closed. Let $(I_n)_n$ be a sequence of all components of the complement of this set. Then $I_n \subset G(h, -)$ or $I_n \subset G(h, +)$ for every component I_n. Let $t_{n,1}, \ldots, t_{n,4}\colon I_n \to \mathbb{R}$ be quasi-continuous functions such that $h | I_n = t_{n,1} \circ \ldots \circ t_{n,4}$.

Similarly, if the interval $\bar{K}_{n,m}$ is not contained in the set $G(h, 0)$, then $\bar{K}_{n,m} \subset G(h, -)$ or $\bar{K}_{n,m} \subset G(h, +)$ and there exist quasi-continuous functions $k_{n,m,1}, \ldots, k_{n,m,4}\colon \bar{K}_{n,m} \to \mathbb{R}$ such that $h | \bar{K}_{n,m}$ is the product of $k_{n,m,1}, \ldots, k_{n,m,4}$.

Let $(w_n)_n$ be a sequence of all rationals different than zero. Now we can define quasi-continuous functions $f_1, \ldots, f_8\colon \mathbb{R} \to \mathbb{R}$ in the following way.

$$f_1(x) = \begin{cases} h(x) & \text{for } x \in A, \\ w_m & \text{for } x \in \bar{K}_{n,2m},\quad n \geqslant 2m, \\ 0 & \text{for } x \in G(h, 0) \setminus \bigcup_{n,m} \bar{K}_{n,2m}, \\ k_{n,2m+1,1}(x) & \text{for } x \in \bar{K}_{n,2m+1} \setminus G(h, 0),\quad n \geqslant 2m+1, \\ t_{n,1}(x) & \text{for } x \in I_n,\quad n \in \mathbb{N} \end{cases}$$

$$f_2(x) = \begin{cases} 1 & \text{for } x \in A \cup \bigcup_{n,m} \bar{K}_{n,2m+1}, \\ (1/w_m)\, k_{n,2m,1}(x) & \text{for } x \in \bar{K}_{n,2m} \setminus G(h, 0), \quad n \geqslant 2m, \\ 0 & \text{for } x \in G(h, 0) \setminus \bigcup_{n,m} \bar{K}_{n,2m+1}, \\ 1 & \text{for } x \in \bigcup_n I_n \end{cases}$$

$$\begin{array}{l} f_j(x) = \\ j = 3,4,5 \end{array} \begin{cases} 1 & \text{for } x \in A \cup \bigcup_{n,m} \bar{K}_{n,2m} \cup G(h, 0), \\ k_{n,2m+1,j-1}(x) & \text{for } x \in \bar{K}_{n,2m+1} \setminus G(h, 0), \quad n \geqslant 2m+1, \\ t_{n,j-1}(x) & \text{for } x \in I_n, \quad n \in \mathbb{N} \end{cases}$$

$$\begin{array}{l} f_j(x) = \\ j = 6,7,8 \end{array} \begin{cases} 1 & \text{for } x \in A \cup \bigcup_{n,m} \bar{K}_{n,2m+1} \cup G(h, 0), \\ k_{m,2m,j-4}(x) & \text{for } x \in \bar{K}_{n,2m} \setminus G(h,0), \quad n \geqslant 2m, \\ 1 & \text{for } x \in \bigcup_n I_n. \end{cases}$$

Clearly $h = f_1 \circ \ldots \circ f_8$. Let us now show that f_1 is quasi-continuous. The proof that f_i, $i = 2, \ldots, 8$, are quasi-continuous is similar. It is enough to verify that f_1 is quasi-continuous at every point $x_0 \in A$. Fix $\varepsilon > 0$ and a neighbourhood U of x_0. Let w_m be a rational number from the interval $(h(x_0) - \varepsilon, h(x_0) + \varepsilon)$. Then there exists $n \geqslant 2m$ such that $\bar{K}_{n,2m} \subset U$ and $f_1(x) = w_m$ for $x \in K_{n,2m}$. Thus f_1 is quasi-continuous at the point x_0.

If h is a function of the Baire class α (Lebesgue measurable, with the Baire property), then the functions $f_1, \ldots, f_8$ belong to the adequate class. We shall verify this fact for f_1 in the case when h is a Baire 1 function. Then every function $t_{n,i}$ is of the Baire class 1 on I_n ($n \in \mathbb{N}$, $i = 1, 2, 3, 4$) and every function $k_{n,m,i}$ is of the Baire class 1 on $\bar{K}_{n,m}$ ($n \in \mathbb{N}$, $m \leqslant n$, $i = 1, 2, 3, 4$). For an open set $G \subset \mathbb{R}$ each of the sets $h^{-1}(G) \cap A$, $\bigcup \{\bar{K}_{n,2m} \colon w_m \in G\}$, $\bigcup k^{-1}_{n,2m+1,1}(G) \cap (K_{n,2m+1} \setminus G(h,0))$ and $\bigcup_n t^{-1}_{n,1}(G) \cap I_n$ is a F_σ set. Moreover, the set $\bigcup_{n,m} \bar{K}_{n,2m} \cap G(h,0)$ is closed in $G(h, 0)$, thus the set $G(h, 0) \setminus \bigcup_{n,m} \bar{K}_{n,2m}$ is open. It follows that $f_1^{-1}(G)$ is a F_σ set, so f_1 is of the Baire class 1. This finishes the proof of the Theorem.

REFERENCES

[1] BLEDSOE, W. W.: Neighbourly functions, Proc. Amer. Math. Soc. 3, 1952, 114—115.
[2] GRANDE, Z.: Sur les functions cliquish. Časop. Pěst. Mat. 110, 1985, 225—236.

Received June 22, 1989

Instytut Matematyki WSP
ul. Chodkiewicza 30
85 064 Bydgoszcz
Poland