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CONGRUENCES ON PRODUCTS IN VARIETIES SATISFYING THE CEP

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It is known that the structure of congruences on direct products of similar algebras is very rich. Several properties of congruences on direct products are definable by the Mal'cev conditions; the obtained identities however, are usually extensive and complicated. In order to reduce the Mal'cev conditions for direct decomposability of congruences and of congruence classes the *congruence extension property* was applied in an earlier paper [4]. The aim of this note is to derive reduced identities for varieties formerly investigated by J. Hagemann [9] and by P. Gumm [8] under the same assumption.

First it will be convenient to remind the reader of the definitions. We use the same symbol for an algebra as for its universe, and depend on the context to make clear which is meant. From [7] we adopt the concept of factor congruence: The kernels Π_A, Π_B, of the canonical projections $pr_A\colon A \times B \to A$, $pr_B\colon A \times B \to B$, respectively, are called factor congruences on $A \times B$. In a similar way we introduce: A congruence Ψ on $A \times B$ is called *subfactor* whenever $\Psi \subseteq \Pi_A$ or $\Psi \subseteq \Pi_B$ hold.

Now we can formulate the first property studied in this paper.

Definition 1. *A variety* $\mathbf{V}$ *is said to have directly decomposable subfactor congruences whenever the congruences* $\Theta \wedge \Pi_A$ *and* $\Theta \wedge \Pi_B$ *are uniquely determined by their projections for every* $\Theta \in \mathrm{Con}\ A \times B$; $A, B \in \mathbf{V}$.

We will use the so called *binary scheme*, see [3], to describe the principal congruences:

Proposition 1. *Let* A *be an algebra,* $a, b, c, d \in A$. *The following conditions are equivalent:*

(1) $\langle c, d\rangle \in \Theta(a, b)$;

(2) *there exist a positive integer* n *and binary algebraic functions* $\varphi_1, \ldots, \varphi_n$ *over* A *such that*

$$c = \varphi_1(a, b),$$

$$\varphi_k(b, a) = \varphi_{k+1}(a, b),\ 1 \leq k < n,$$

$$d = \varphi_n(b, a).$$

Proof. See [3; Thm 1].

Then the characterizing identities for direct decomposability of subfactor congruences can be expressed as follows:

Proposition 2. *For a variety* $\mathbf{V}$ *the following conditions are equivalent:*

(1) $\mathbf{V}$ *has directly decomposable subfactor congruences;*

(2) there exist positive integers m, n, *binary polynomials* $p_1, \ldots, p_m, q_1, \ldots, q_m$, *and* $(1+m)$*-ary polynomials* $r_1, \ldots, r_n$ *such that*

$$y = r_i(x, p_1(x, y), \ldots, p_m(x, y)), \ 1 \leq i \leq n,$$

$$x = r_1(a_1, q_1(x, y), \ldots, q_m(x, y))$$

$$r_k(b_k, q_1(x, y), \ldots, q_m(x, y)) = r_{k+1}(a_{k+1}, q_1(x, y), \ldots, q_m(x, y)), \ 1 \leq k < ,$$

$$y = r_n(b_n, q_1(x, y), \ldots, q_m(x, y))$$

hold in $\mathbf{V}$ *where* $\{a_k, b_k\} = \{x, y\}$ *for* $1 \leq k \leqq n$;

(3) there exist positive integers m, n, *binary polynomials* $s_1, \ldots, s_m, t_1, \ldots, t_m$, *and* $(2+m)$*-ary polynomials* $r_1, \ldots, r_n$ *such that*

$$y = r_i(x, x, s_1(x, y), \ldots, s_m(x, y)), \ 1 \leq i \leq n,$$

$$x = r_1(y, x, t_1(x, y), \ldots, t_m(x, y))$$

$$r_k(x, y, t_1(x, y), \ldots, t_m(x, y)) = r_{k+1}(y, x, t_1(x, y), \ldots, t_m(x, y)), \ 1 \leq k < n,$$

$$y = r_n(x, y, t_1(x, y), \ldots, t_m(x, y))$$

hold in $\mathbf{V}$.

Outline of proof. Part (2) is the original characterization from [9].

The binary scheme applied to the subfactor congruence $\Theta(\langle x, y\rangle, \langle x, x\rangle)$ on the square $F_{\mathbf{V}}(x, y) \times F_{\mathbf{V}}(x, y)$ of the free algebra $F_{\mathbf{V}}(x, y)$ yields the identities (3).

Furthermore, the following property of congruences on direct products was studied in [9]:

Definition 2. A variety $\mathbf{V}$ is said to be *factor permutable* whenever every congruence on the product $A \times B \in \mathbf{V}$ permutes with the factor congruences Π_A and Π_B.

Factor permutable varieties have been investigated lately in [8], namely the term „factor permutable variety" is due to H. Peter Gumm. The characterizing identities for factor permutable varieties are exhibited in

Proposition 3. *For a variety* $\mathbf{V}$ *the following conditions are equivalent:*

(1) $\mathbf{V}$ *is factor permutable;*

(2) *there exist positive integers* m, n, *binary polynomials* $q_1, \ldots, q_m$, *ternary polynomials* $s, p_1, \ldots, p_m$, *and* $(1+m)$*-ary polynomials* $r_1, \ldots, r_n$ *such that*

$$s(x, y, z) = r_1(a_1, p_1(x, y, z), \ldots, p_m(x, y, z))$$

$$x = r_1(a_1, q_1(x, y), \ldots, q_m(x, y))$$

$$r_k(b_k, p_1(x, y, z), \ldots, p_m(x, y, z)) = r_{k+1}(a_{k+1}, p_1(x, y, z), \ldots, p_m(x, y, z))$$

$$r_k(b_k, q_1(x, y), \ldots, q_m(x, y)) = r_{k+1}(a_{k+1}, q_1(x, y), \ldots, q_m(x, y)), \quad 1 \leq k < n,$$

$$z = r_n(b_n, p_1(x, y, z), \ldots, p_m(x, y, z))$$

$$y = r_n(b_n, q_1(x, y), \ldots, q_m(x, y))$$

hold in $\mathbf{V}$ *where* $\{a_k, b_k\} = \{x, y\}$ for $1 \leq k \leq n$;

(3) *there exist positive integers* m, n, *a map* l: $\{1, \ldots, n\} \to \{0, 1\}$, *binary polynomials* r_{ij}, *ternary polynomials* s_{ij} $(1 \leq i \leq n,\ 1 \leq j \leq m)$, *and* $(1+m)$*-ary polynomials* $p_1, \ldots, p_n$ *such that*

$$x_0 = p_1(x_{l(1)}, r_{11}(x_0, x_1), \ldots, r_{1m}(x_0, x_1))$$

$$p_k(x_{1-l(k)}, r_{k1}(x_0, x_1), \ldots, r_{km}(x_0, x_1)) = \\ = p_{k+1}(x_{l(k+1)}, r_{k+1,1}(x_0, x_1), \ldots, r_{k+1,m}(x_0, x_1))$$

$$p_k(x_{1-l(k)}, s_{k1}(x_0, x_1, x_2), \ldots, s_{km}(x_0, x_1, x_2)) =$$

$$= p_{k+1}(x_{l(k+1)}, s_{k+1,1}(x_0, x_1, x_2), \ldots, s_{k+1,m}(x_0, x_1, x_2)), \quad 1 \leq k < n.$$

$$x_1 = p_n(x_{1-l(n)}, r_{n1}(x_0, x_1), \ldots, r_{nm}(x_0, x_1))$$

$$x_2 = p_n(x_{1-l(n)}, s_{n1}(x_0, x_1, x_2), \ldots, s_{nm}(x_0, x_1, x_2))$$

hold in $\mathbf{V}$;

(4) *there exist positive integers* m, n, *binary polynomials* $t_1, \ldots, t_m$, *ternary polynomials* $s_1, \ldots, s_m$, *and* $(2+m)$*-ary polynomials* $r_1, \ldots, r_n$ *such that*

$$p(x, y, z) = r_1(x, y, s_1(x, y, z), \ldots, s_m(x, y, z))$$

$$x = r_1(x, y, t_1(x, y), \ldots, t_m(x, y))$$

$$r_k(y, x, s_1(x, y, z), \ldots, s_m(x, y, z)) = r_{k+1}(x, y, s_1(x, y, z), \ldots, s_m(x, y, z))$$

$$r_k(y, x, t_1(x, y), \ldots, t_m(x, y)) = r_{k+1}(x, y, t_1(x, y), \ldots, t_m(x, y)), \quad 1 \leq k < n,$$

$$z = r_n(y, x, s_1(x, y, z), \ldots, s_m(x, y, z))$$

$$y = r_n(y, x, t_1(x, y), \ldots, t_m(x, y))$$

hold in $\mathbf{V}$.

Outline of proof. Conditions (2), (3) are quoted from [9], [8], respectively. To obtain (4) apply the binary scheme to the congruence $\Theta(\langle x, x\rangle, \langle y, y\rangle) \circ \Pi_B = \Pi_B \circ \Theta(\langle x, x\rangle, \langle y, y\rangle)$ on the product $A \times B = F_{\mathbf{V}}(x, y, z) \times F_{\mathbf{V}}(x, y)$.

Now let us turn our attention to varieties satisfying the congruence extension property. We repeat the definition here:

Definition 3. A variety $\mathbf{V}$ is said to have the *congruence extension property* (briefly the CEP) if every congruence on an arbitrary subalgebra of $A \in \mathbf{V}$ is a restriction of some congruence on A.

An important characterization of varieties satisfying the CEP was found by A. Day, [2]. The symbol $\overline{\{a, b, c, d\}}$ denotes the subalgebra generated by elements a, b, c, d.

Proposition 4. *For a variety $\mathbf{V}$ the following conditions are equivalent:*

(1) *$\mathbf{V}$ has the CEP;*

(2) *for any $a, b, c, d \in A \in \mathbf{V}$, $\langle c, d\rangle \in \Theta_A(a, b)$ iff $\langle c, d\rangle \in \Theta_{\overline{\{a, b, c, d\}}}(a, b)$.*

Proposition 4 and binary scheme together yield the following description of principal congruences on varieties satisfying the CEP:

Proposition 5. *Let $\mathbf{V}$ be a variety satisfying the CEP, $a, b, c, d \in A \in \mathbf{V}$. The following conditions are equivalent:*

(á) $\langle c, d\rangle \in \Theta(a, b)$;

(2) *there exist a positive integer n and 6-ary polynomials $p_1, \ldots, p_n$ such that*

$$c = p_1(a, b, a, b, c, d)$$

$$p_k(b, a, a, b, c, d) = p_{k+1}(a, b, a, b, c, d), \ 1 \leqslant k < n,$$

$$d = p_n(b, a, a, b, c, d).$$

Proof. The implication (2) $\Rightarrow$ (1) is evident.

(1) $\Rightarrow$ (2). Applying the binary scheme to (1) we find that

$$c = \varphi_1(a, b)$$

$$\varphi_k(b, a) = \varphi_{k+1}(a, b), \ 1 \leqslant k < n,$$

$$d = \varphi_n(b, a)$$

for some positive integer n and binary algebraic functions $\varphi_1, \ldots, \varphi_n$ over A. By Proposition 4, these functions can be taken over the subalgebra $\overline{\{a, b, c, d\}}$ only. The definition of the algebraic function completes the proof.

We are ready to prove our first result:

Theorem 1. *Let $\mathbf{V}$ be a variety satisfying the CEP. The following conditions are equivalent:*

(1) *$\mathbf{V}$ has directly decomposable subfactor congruences;*

(2) *there exist a positive integer n and 6-ary polynomials $r_1, \ldots, r_n$ such that*

$$y = r_i(x, x, x, x, y, y), \ 1 \leqslant i \leqslant n,$$

$$x = r_1(y, x, y, x, x, y)$$

$$r_k(x, y, y, x, x, y) = r_{k+1}(y, x, y, x, x, y), \ 1 \leq k < n,$$
$$y = r_n(x, y, y, x, x, y)$$

hold in $\mathbf{V}$.

Proof. (1) $\Rightarrow$ (2). Take $A = B = F_{\mathbf{V}}(x, y)$ the free algebra in $\mathbf{V}$ on free generators x and y. Evidently, $\Theta(\langle x, y\rangle, \langle x, x\rangle) = \Theta(\langle x, y\rangle, \langle x, x\rangle) \wedge \Pi_A$, i.e. $\Theta(\langle x, y\rangle, \langle x, x\rangle)$ is a subfactor congruence. By hypothesis, this congruance is directly decomposable hence $\langle\langle y, x\rangle, \langle y, y\rangle\rangle \in \Theta(\langle x, y\rangle, \langle x, x\rangle)$. Applying Proposition 5 to the last statement we get that

$$\langle y, x\rangle = r_1(\langle x, y\rangle, \langle x, x\rangle, \langle x, y\rangle, \langle x, x\rangle, \langle y, x\rangle, \langle y, y\rangle)$$

$$r_k(\langle x, x\rangle, \langle x, y\rangle, \langle x, y\rangle, \langle x, x\rangle, \langle y, x\rangle, \langle y, y\rangle) =$$
$$= r_{k+1}(\langle x, y\rangle, \langle x, x\rangle, \langle x, y\rangle, \langle x, x\rangle, \langle y, x\rangle, \langle y, y\rangle), \ 1 \geq k < n,$$

$$\langle y, y\rangle = r_n(\langle x, x\rangle, \langle x, y\rangle, \langle x, y\rangle, \langle x, x\rangle, \langle y, x\rangle, \langle y, y\rangle)$$

for some positive integer n and 6-ary polynomials $r_1, \ldots, r_n$. Writing this separately in each variable the identities (2) follow.

(2) $\Rightarrow$ (1). This part is straightforward since the assumed identities are exactly those of Proposition 2 (3) when $m = 4$, $s_1 = s_2 = x$, $s_3 = s_4 = y$, $t_1 = y$, $t_2 = t_3 = x$, and $t_4 = y$.

Example 1. It is well known that the variety of distributive lattices or any variety of pseudocomplemented distributive lattices, see [1] or [6], satisfy the CEP. Furthermore, the mentioned varieties have directly decomposable subfactor congruences; this fact follows, e.g., from our Theorem 1 since for $n = 1$, $r_1(x_1, \ldots, x_6) = (x_2 \wedge x_5) \vee (x_2 \wedge x_6) \vee (x_5 \wedge x_6)$ there hold

$$r_1(x, x, x, x, y, y) = (x \wedge y) \vee (x \wedge y) \vee (y \wedge y) = y$$
$$r_1(y, x, y, x, x, y) = (x \wedge x) \vee (x \wedge y) \vee (x \wedge y) = x$$
$$r_1(x, y, y, x, x, y) = (y \wedge x) \vee (y \wedge y) \vee (x \wedge y) = y.$$

Simultaneously, the answer to the natural question whether the CEP follows from the identities exhibited in Theorem 1 (2) is negative:

Counterexample 1. The variety of all lattices evidently satisfies the condition (2) of Theorem 1 but not the CEP.

The reduction of the second Mal'cev condition (see Proposition 3 (4)) is given in

Theorem 2. *Let* $\mathbf{V}$ *be a variety satisfying the CEP. The following conditions are equivalent:*

(1) $\mathbf{V}$ *is factor permutable;*

(2) *there exist a positive integer* n, *a ternary polynomial* p *and 6-ary polynomials* $r_1, \ldots, r_n$ *such that*

$$p(x, y, z) = r_1(x, y, x, y, p(x, y, z), z)$$

$$x = r_1(x, y, x, y, x, y)$$

$$r_k(y, x, x, y, p(x, y, z), z) = r_{k+1}(x, y, x, y, p(x, y, z), z)$$

$$r_k(y, x, x, y, x, y) = r_{k+1}(x, y, x, y, x, y), \ 1 \leq k < n,$$

$$z = r_n(y, x, x, y, p(x, y, z), z)$$

$$y = r_n(y, x, x, y, x, y)$$

hold in $\mathbf{V}$.

Proof. (1)$\Rightarrow$(2). Take $A = F_{\mathbf{V}}(x, y, z)$, $B = F_{\mathbf{V}}(x, y)$, and the principal congruence $\Theta(\langle x, x\rangle, \langle y, y\rangle)$ on $A \times B$. Then $\langle\langle x, x\rangle, \langle z, y\rangle\rangle \in \Theta(\langle x, x\rangle, \langle y, y\rangle) \circ \Pi_B$ since evidently $\langle\langle x, x\rangle, \langle y, y\rangle\rangle \in \Theta(\langle x, x\rangle, \langle y, y\rangle)$ and $\langle\langle y, y\rangle, \langle z, y\rangle\rangle \in \Pi_B$. By hypothesis, we have also $\langle\langle x, x\rangle, \langle z, y\rangle\rangle \in \Pi_B \circ \Theta(\langle x, x\rangle, \langle y, y\rangle)$, i. e. $\langle\langle x, x\rangle, \langle p, q\rangle\rangle \in \Pi_B$ and $\langle\langle p, q\rangle, \langle z, y\rangle\rangle \in \Theta(\langle x, x\rangle, \langle y, y\rangle)$ for suitable $\langle p, q\rangle \in A \times B$. The first statement yields $q = x$; the second one implies $p = p(x, y, z)$ and

$$\langle p(x, y, z), x\rangle = r_1(\langle x, x\rangle, \langle y, y\rangle, \langle x, x\rangle, \langle y, y\rangle, \langle p(x, y, z), x\rangle, \langle z, y\rangle)$$

$$r_k(\langle y, y\rangle, \langle x, x\rangle, \langle x, x\rangle, \langle y, y\rangle, \langle p(x, y, z), x\rangle, \langle z, y\rangle) =$$

$$= r_{k+1}(\langle x, x\rangle, \langle y, y\rangle, \langle x, x\rangle, \langle y, y\rangle, \langle p(x, y, z), x\rangle, \langle z, y\rangle), \ 1 \leq k < n,$$

$$\langle z, y\rangle = r_n(\langle y, y\rangle, \langle x, x\rangle, \langle x, x\rangle, \langle y, y\rangle, \langle p(x, y, z), x\rangle, \langle z, y\rangle)$$

for some positive integer n, a ternary polynomial p, and 6-ary polynomials $r_1, \ldots, r_n$, (see Proposition 5). Writing this componentwisely, the desired identities readily follow.

(2)$\Rightarrow$(1). Apply Proposition 3 with $m = 4$, $s_1 = x$, $s_2 = y$, $s_3 = p(x, y, z)$, $s_4 = z$, $t_1 = x$, $t_2 = y$, $t_3 = x$, $t_4 = y$.

Example 2. Any variety of Abelian groups or any variety of rings in which the identity $(xy)^n = xy$ holds for some $n > 1$ satisfy the CEP, see [1; Thm 2.2 or Thm 3.3]. Trivially, these varieties have also factor permutable congruences. The following concrete identities illustrate part (2) of Theorem 2:

Take $n = 1$, $p(x, y, z) = x - y + z$ and $r_1(x_1, \ldots, x_6) = x_1 - x_3 + x_5$. Then

$$r_1(x, y, x, y, p(x, y, z), z) = x - x + x - y + z = x - y - z = p(x, y, z)$$

$$r_1(x, y, x, y, x, y) = x - x + x = x$$

$$r_1(y, x, x, y, p(x, y, z), z) = y - x + x - y + z = z$$

$$r_1(y, x, x, y, x, y) = y - x + x = y.$$

Similarly as in the first case the identities of Theorem 2 (2) do not suffice for the CEP:

Counterexample 2. The variety of all groups evidently satisfies the condition (2) of Theorem 2 but not the CEP, (see again [1]).

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КОНГРУЭНЦИИ ПРОИЗВЕДЕНИЙ В МНОГООБРАЗНИЯХ ОБЛАДАЮЩИХ СВОЙСТВОМ ЦЭП

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Резюме

В статье найдены сжатые тождества для многообразий алгебр раньше изученных в работах П. Гумма и Й. Хагеманна. Предполагаем свойство расширения конгруэнций (ЦЭП).