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MIXED NORM SPACE OF PLURIHARMONIC FUNCTIONS

HASI WULAN

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ABSTRACT. Let Ω be a bounded symmetric domain in $\mathbb{C}^n$. For $0 < p, q < \infty$ and a normal function φ, we show that the mixed norm space $a^{p,q,\varphi}(\Omega)$ of pluriharmonic functions on Ω is a self-conjugate class.

Let Ω be a bounded symmetric domain in the complex vector space $\mathbb{C}^n$ $(n \geq 1)$, $0 \in \Omega$, with Bergman-Silov boundary b, Γ the group of holomorphic automorphisms of Ω, and Γ_0 its isotropy group. It is known that Ω is circular and star-shaped with respect to 0, and that b is circular. The group Γ_0 is transitive on b, and b has a unique normalized Γ_0-invariant measure σ with $\sigma(b) = 1$.

By $H(\Omega)$ denote the class of all holomorphic functions on Ω. Every $f \in H(\Omega)$ has a series expansion ([1])

$$f(z) = \sum_{k,v} a_{kv} \phi_{kv}(z), \qquad a_{kv} = \lim_{r \to 1} \int_b f(r\xi) \overline{\phi_{kv}(\xi)} \, \mathrm{d}\sigma(\xi), \tag{1}$$

which converges uniformly on every compact of Ω, where $\sum\limits_{k,v} = \sum\limits_{k=0}^{\infty} \sum\limits_{v=1}^{u_k}$. The set of functions $\{\phi_{kv}(z)\}$, $k = 0, 1, \dots$, $v = 1, 2, \dots$, $u_k = C^k_{n+k-1}$ is a complete orthogonal system of homogeneous polynomials on Ω which are orthonormal on b ([2]).

Let $f \in H(\Omega)$ with the expansion (1) and $\beta > 0$, the βth fractional derivative of f is defined by

$$f^{[\beta]}(z) = \sum_{k,v} \frac{\Gamma(k+\beta+1)}{\Gamma(k+1)} a_{kv} \phi_{kv}(z),$$

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where $\Gamma(\cdot)$ denotes the gamma function, and we know that $f^{[\beta]}$ is holomorphic on Ω ([3]).

A positive continuous function φ on $[0,1]$ is called *normal* if there exist a and b $(0 < a < b)$ such that

1. $\dfrac{\varphi(r)}{(1-r)^a}$ is non-increasing and $\lim\limits_{r\to 1} \dfrac{\varphi(r)}{(1-r)^a} = 0$,
2. $\dfrac{\varphi(r)}{(1-r)^b}$ is non-decreasing and $\lim\limits_{r\to 1} \dfrac{\varphi(r)}{(1-r)^b} = \infty$.

A continuous real function u on Ω is called *pluriharmonic* if for every holomorphic mapping γ of the unit disk D into Ω, $u \circ \gamma$ is harmonic on D. Since Ω is simply connected ([9; p. 311]), every pluriharmonic function on Ω is the real part of a holomorphic function ([10; p. 44]). Let u be pluriharmonic on Ω, then $u = \operatorname{Re} f$, where $f = u + \mathrm{i} v$ is holomorphic on Ω, and v is called the *pluriharmonic conjugate* of u.

For a normal function φ and $0 < p, q < \infty$, we introduce the mixed norm space $a^{p,q,\varphi}(\Omega)$ as the set of pluriharmonic functions u on Ω with finite norm

$$\|u\|_{p,q,\varphi} = \left\{ \int_0^1 (1-r)^{-1} \varphi^p(r) M_q^p(r,u)\, \mathrm{d}r \right\}^{1/p}, \tag{2}$$

where $M_q(r,u) = \left\{ \int_b |u(r\xi)|^q \, \mathrm{d}\sigma(\xi) \right\}^{1/q}$.

For the unit ball B of $\mathbb{C}^n$ and the special case $\varphi(r) = (1-r)^\alpha$, Stoll [4] and Shi [8] proved that $a^{p,q,\varphi}(B) = a^{p,q,\alpha}(B)$ is a self-conjugate class for $0 < p, q < \infty$ and $\alpha > 0$, that is, if $u \in a^{p,q,\alpha}(B)$, then the pluriharmonic conjugate $v \in a^{p,q,\alpha}(B)$. For a bounded symmetric domain Ω, Shi [5] and Xiao [6] proved that $a^{p,q,\alpha}(\Omega)$ is a self-conjugate class for $0 < p, q < \infty$ and $\alpha > 0$. In this article, we generalize all of these results to a general normal function φ on bounded symmetric domains in $\mathbb{C}^n$. Here some new techniques have been used.

THEOREM. *Let $f(z) = u + \mathrm{i} v$ be holomorphic in Ω with $f(0)$ real, and $0 < p' \le p < \infty$, $0 < q' \le q < \infty$, $\beta = n\left(\frac{1}{q'} - \frac{1}{q}\right)$. Then for normal functions φ and $\psi(r) = (1-r)^\beta \varphi(r)$, we have*

$$\|f\|_{p,q,\psi} \le C \|u\|_{p',q',\varphi}. \tag{3}$$

Here and later, C always denotes a positive constant, not necessarily the same one at each occurrence which is independent of f.

COROLLARY. *Let $f(z) = u + \mathrm{i} v$ be holomorphic in Ω with $f(0)$ real, and $0 < p, q < \infty$. Then for each normal function φ we have*

$$\|f\|_{p,q,\varphi} \le C \|u\|_{p,q,\varphi}.$$

From the corollary above, we easily obtain that the space $a^{p,q,\varphi}(\Omega)$ is a self-conjugate class for $0 < p, q < \infty$ and any normal function φ.

To prove the main theorem, we need the following lemmas.

Lemma 1. ([7]) *Let $1 \le k < \infty$, $\lambda > 0$, $\mu > 0$, $h\colon (0,1) \to [0,\infty]$ measurable. Then*

$$\int_0^1 (1-r)^{k\mu-1}\left(\int_0^r (r-t)^{\lambda-1}h(t)\,\mathrm{d}t\right)^k \mathrm{d}r \le C\int_0^1 (1-r)^{k\mu+k\lambda-1}h^k(r)\,\mathrm{d}r\,.$$

Lemma 2. ([5]) *Let $0 < p, q < \infty$, $0 < r < 1$. Then*

$$r^p M_q^p(r,f) \le C\int_0^r (r-t)^{p-1}M_q^p\big(t,f^{[1]}\big)\,\mathrm{d}t\,, \qquad p < q\,; \tag{4}$$

$$r^q M_q^q(r,f) \le C\int_0^r (r-t)^{q-1}M_q^q\big(t,f^{[1]}\big)\,\mathrm{d}t\,, \qquad 0 < q < 1\,; \tag{5}$$

$$r M_q(r,f) \le C\int_0^r M_q\big(t,f^{[1]}\big)\,\mathrm{d}t\,, \qquad 1 \le q < \infty\,. \tag{6}$$

Lemma 3. *Let $f = u + \mathrm{i}v$ be holomorphic in Ω with $f(0)$ real, and $0 < q < \infty$. Then for $1/3 \le r < 1$, we have*

$$M_q\big(r,f^{[1]}\big) \le C(1-r)^{-1}M_q\big(\tfrac{1+r}{2},u\big)\,. \tag{7}$$

Proof. In [5], J. H. Shi proved that (7) is valid for $1 \le q < \infty$. For $0 < q < 1$ and $1/3 \le r < 1$, by [7], we obtain

$$M_q^q\big(r,f_\xi^{[1]} - f(0)\big) \le C(1-r)^{-q-1}\int_{(3r-1)/2}^{(r+1)/2} M_q^q(t,u_\xi)\,\mathrm{d}t\,, \tag{8}$$

where $f_\xi(w) = f(w\xi)$, $\xi \in b$, $w \in D$. Applying the formula in [5]

$$\frac{1}{2\pi}\int_b \mathrm{d}\sigma(\xi)\int_0^{2\pi} g(\xi\,\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta = \int_b g(\xi)\,\mathrm{d}\sigma(\xi)\,, \qquad g \in L^1(b)\,,$$

and (8) we have

$$M_q^q\big(r,f^{[1]} - f(0)\big) \le C(1-r)^{-q}M_q^q\big(\tfrac{1+r}{2},u\big)\,.$$

From $|f(0)|^q = |u(0)|^q \le CM_q^q\big(\frac{1+r}{2},u\big)$, we get

$$M_q^q\big(r,f^{[1]}\big) \le C(1-r)^{-q}M_q^q\big(\tfrac{1+r}{2},u\big)\,.$$

□

LEMMA 4. *Let $f \in H(\Omega)$ and φ be normal, $0 < p, q < \infty$. Then*

$$\int_0^1 (1-r)^{-1}\varphi^p(r)M_q^p(r,f)\,\mathrm{d}r \le C\int_0^1 (1-r)^{p-1}\varphi^p(r)M_q^p(r,f^{[1]})\,\mathrm{d}r\,. \tag{9}$$

Proof. Replacing r by r^{p+1} in the left-hand integral of (9), we have

$$\begin{aligned}\|f\|^p_{p,q,\varphi} &= \int_0^1 (1-r^{p+1})^{pb-1}\varphi^p(r^{p+1})(1-r^{p+1})^{-pb}M_q^p(r^{p+1},f)(p+1)r^p\,\mathrm{d}r\\ &\le C\int_0^1 (1-r)^{-1}\varphi^p(r)r^pM_q^p(r,f)\,\mathrm{d}r\,.\end{aligned} \tag{10 }$$

Case 1. $p < q$. By Lemma 1 and (4), we have

$$\begin{aligned}&\int_0^1 (1-r)^{-1}\varphi^p(r)r^pM_q^p(r,f)\,\mathrm{d}r\\ \le\, &C\int_0^1 (1-r)^{-1}\varphi^p(r)\,\mathrm{d}r\int_0^r (r-t)^{p-1}M_q^p(t,f^{[1]})\,\mathrm{d}t\\ \le\, &C\int_0^1 (1-r)^{ap-1}\,\mathrm{d}r\int_0^r (r-t)^{p-1}\varphi^p(t)(1-t)^{-ap}M_q^p(t,f^{[1]})\,\mathrm{d}t\\ \le\, &C\int_0^1 (1-r)^{p-1}\varphi^p(r)M_q^p(r,f^{[1]})\,\mathrm{d}r\,.\end{aligned} \tag{11}$$

Case 2. $p \ge q$ and $q \ge 1$. By Lemma 1 and (6), we have

$$\begin{aligned}&\int_0^1 (1-r)^{-1}\varphi^p(r)r^pM_q^p(r,f)\,\mathrm{d}r\\ \le\, &C\int_0^1 (1-r)^{-1}\varphi^p(r)\left(\int_0^r M_q(t,f^{[1]})\,\mathrm{d}t\right)^p\mathrm{d}r\end{aligned}$$

$$\le C \int_0^1 (1-r)^{ap-1} \left(\int_0^r \varphi(t)(1-t)^{-a} M_q\big(t, f^{[1]}\big)\, \mathrm{d}t \right)^p \mathrm{d}r$$

$$\le C \int_0^1 (1-r)^{p-1} \varphi^p(r) M_q^p\big(r, f^{[1]}\big)\, \mathrm{d}r\,. \tag{12}$$

Case 3. $p \ge q$ and $0 < q < 1$. By (5) of Lemma 2 and Lemma 1, we have

$$\begin{aligned}
&\int_0^1 (1-r)^{-1} \varphi^p(r) r^p M_q^p(r, f)\, \mathrm{d}r \\
&\le C \int_0^1 (1-r)^{-1} \varphi^p(r) \left(\int_0^r (r-t)^{q-1} M_q^q\big(t, f^{[1]}\big)\, \mathrm{d}t \right)^{p/q} \mathrm{d}r \\
&\le C \int_0^1 (1-r)^{ap-1} \left(\int_0^r (r-t)^{q-1} \varphi^q(t)(1-t)^{-aq} M_q^q\big(t, f^{[1]}\big)\, \mathrm{d}t \right)^{p/q} \mathrm{d}r \\
&\le C \int_0^1 (1-r)^{p-1} \varphi^p(r) M_q^p\big(r, f^{[1]}\big)\, \mathrm{d}r\,.
\end{aligned} \tag{13}$$

Combining (10), (11), (12) and (13), Lemma 4 is proved. □

P r o o f o f T h e o r e m . By Lemma 4, we can obtain

$$\begin{aligned}
\|f\|_{p,q,\psi}^p &= \int_0^1 (1-r)^{-1} \psi^p(r) M_q^p(r, f)\, \mathrm{d}r \\
&\le C \int_0^1 (1-r)^{p-1} \psi^p(r) M_q^p\big(r, f^{[1]}\big)\, \mathrm{d}r \\
&= C \left(\int_0^{1/9} + \int_{1/9}^1 \right) =: C(I_1 + I_2)\,.
\end{aligned} \tag{14}$$

Therefore,

$$
\begin{aligned}
I_1 &= \int_0^{1/9} (1-r)^{p-1+b'p}\psi^p(r)(1-r)^{-b'p}M_q^p\big(r, f^{[1]}\big)\,\mathrm{d}r \\
&\le C\psi^p(1/9)(1-1/9)^{-b'p}M_q^p\big(1/9, f^{[1]}\big) \\
&\le C\int_{1/9}^{1} (1-r)^{p-1+b'p}\psi^p(r)(1-r)^{-b'p}M_q^p\big(r, f^{[1]}\big)\,\mathrm{d}r \\
&= CI_2\,.
\end{aligned} \tag{15}
$$

By [11; Lemma 2], we easily obtain

$$
M_q(r^2, f) \le C(1-r)^{-n(1/q'-1/q)}M_{q'}(r, f)\,, \qquad 0 < q' \le q < \infty\,,
$$

it follows that

$$
\begin{aligned}
I_2 &= \int_{1/9}^{1} (1-r)^{p-1}\psi^p(r)M_q^p\big(r, f^{[1]}\big)\,\mathrm{d}r \\
&= 2\int_{1/3}^{1} (1-r^2)^{p-1}\psi^p(r^2)M_q^p\big(r^2, f^{[1]}\big)r\,\mathrm{d}r \\
&\le C\int_{1/3}^{1} (1-r)^{p-1-p\beta}\psi^p(r)M_{q'}^p\big(r, f^{[1]}\big)\,\mathrm{d}r\,.
\end{aligned}
$$

By Lemma 3 and $\psi(r) = (1-r)^\beta\varphi(r)$, we immediately obtain

$$
\begin{aligned}
I_2 &\le C\int_{1/3}^{1} (1-r)^{-1}\varphi^p(r)M_{q'}^p\big(\tfrac{1+r}{2}, u\big)\,\mathrm{d}r \\
&\le C\int_{1/3}^{1} (1-r)^{bp-1}\varphi^p\big(\tfrac{1+r}{2}\big)\big(1-\tfrac{1+r}{2}\big)^{-bp}M_{q'}^p\big(\tfrac{1+r}{2}, u\big)\,\mathrm{d}r \\
&\le C\int_0^{1} (1-r)^{-1}\varphi^p(r)M_{q'}^p(r, u)\,\mathrm{d}r \\
&\le C\sup_{0\le r<1}\big(\varphi(r)M_{q'}(r, u)\big)^{p-p'}\|u\|_{p',q',\varphi}^{p'}\,.
\end{aligned} \tag{16}
$$

On the other hand,

$$\begin{aligned}\|u\|_{p',q',\varphi}^{p'} &= \int_0^1 (1-r)^{-1}\varphi^{p'}(r)M_{q'}^{p'}(r,u)\,\mathrm{d}r \\ &\geq \int_r^1 (1-t)^{bp'-1}\varphi^{p'}(t)(1-t)^{-bp'}M_{q'}^{p'}(t,u)\,\mathrm{d}t \\ &\geq C\big(\varphi(r)M_{q'}(r,u)\big)^{p'}.\end{aligned}$$

Therefore,

$$\sup_{0\leq r<1}\varphi(r)M_{q'}(r,u) \leq \|u\|_{p',q',\varphi}. \tag{17}$$

From (14), (15), (16) and (17), we obtain that (3) holds. This completes the proof of Theorem. □

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Department of Mathematics
Inner Mongolia Normal University
Huhhot 010022
P. R. CHINA

E-mail: wulan@cc.joensuu.fi