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ON A THEOREM OF L. LEFTON

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ABSTRACT. In the paper there is proved the theorem of L. Lefton by using the method of Lyapunov-Schmidt. This theorem concerns to the existence of small solutions for ordinary differential equations.

1. Introduction

Recently L. Lefton [1] has investigated small solutions of the boundary value problem

$$\mathcal{L}y = Ly + y^3 = f$$
$$M_1(y) = M_2(y) = 0\,,$$

where L, the linear part of $\mathcal{L}$, is of the form $Ly = y'' + p(x)\cdot y' + q(x)\cdot y$, p, q are integrable on $[a, b]$, f is small and

$$M_1(y) = \alpha_1 y(a) + \alpha_2 y(b) + \alpha_3 y'(a) + \alpha_4 y'(b)\,,$$
$$M_2(y) = \beta_1 y(a) + \beta_2 y(b) + \beta_3 y'(a) + \beta_4 y'(b)\,,$$

α_i and β_i real.

The operator $\mathcal{L}$ is defined on the domain

$$BC = \{y \in C^1[a, b] \mid y' \text{ is absolutely continuous on } [a, b],\ M_1(y) = M_2(y) = 0,\ y'' \in L^1[a, b]\}\,.$$

Lefton assumed that L has an one-dimensional kernel spanned by φ. He pointed out that as a consequence of [2, Lemma 3.2] it follows that if $\varphi^3 \notin \operatorname{Im} L$ (the range of L), then 0 is an isolated solution of $\mathcal{L}y = 0$ and thus $\mathcal{L}y = f$ has a small solution for f small. He studied the case $\varphi^3 \in \operatorname{Im} L$ and proved the following

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THEOREM ([1]). *Suppose* $Lw = \varphi^3$ *with* $w \perp \varphi$, (*i.e.* $\int_a^b w \cdot \varphi \, \mathrm{d}t = 0$), *but* $Ly = w \cdot \varphi^2$ *has no solution in* BC. *Then* $\mathcal{L}y = f$ *has at least one solution for each* $f \in L^1[a, b]$ *small.*

The purpose of this paper is to give a simple proof of this theorem. We shall use only the *Lyapunov-Schmidt method.*

2. Results

Let $a = 0$, $b = 1$ and $Y = BC$, $X = L^1[0, 1]$. Then $L: Y \to X$. We know that $\operatorname{Ker} L = \operatorname{span} \varphi$ and by a proof of Lemma 1.1 in [1] there is $g \in C^0[0, 1]$ such that $h \in \operatorname{Im} L$ if and only if $\int_0^1 h \cdot g \, \mathrm{d}t = 0$. Thus $\operatorname{Im} L$ is a closed subspace of $L^1[0, 1]$. We shall assume $\operatorname{Im} L \neq L^1[0, 1]$. Hence we consider the case $g \neq 0$. Of course, in the Theorem of the Introduction this assumption is satisfied. Our lemmas and theorems will possess structures similar to this theorem and so the condition $g \neq 0$ is necessary. Note that $\operatorname{Im} L \neq L^1[0, 1]$ if and only if M_1 and M_2 are linearly independent boundary value conditions.

We solve the equation $Ly = -y^3 + f$ for $f \in X$ small.

Putting

$$X = X_1 \oplus \operatorname{span} g\,, \qquad X_1 = \operatorname{Im} L$$

$$Y = Y_1 \oplus \operatorname{Ker} L\,, \qquad Y_1 = \operatorname{Ker} P_0$$

$$Q: X \to \operatorname{Im} L\,, \quad P: X \to \operatorname{span} g\,, \qquad Q + P = \mathrm{Id}$$

$$Px = \left(\int_0^1 g(t) \cdot x(t) \, \mathrm{d}t\right) \Big/ \left(\int_0^1 g^2(t) \, \mathrm{d}t\right) \cdot g$$

$$P_0 y = \left(\int_0^1 \varphi(t) \cdot y(t) \, \mathrm{d}t\right) \Big/ \left(\int_0^1 \varphi^2(t) \, \mathrm{d}t\right) \cdot \varphi$$

our equation has the form

$$\begin{aligned} \text{i)} \quad Ly_1 &= -Q(y_1 + c \cdot \varphi)^3 + f_1 \\ \text{ii)} \quad 0 &= -P(y_1 + c \cdot \varphi)^3 + f_2\,, \end{aligned} \tag{2.1}$$

where $y_1 \in Y_1$, $c \in \mathbb{R}$, $f_1 \in X_1$, $f_2 = d \cdot g$, $d \in \mathbb{R}$.

We modify (2.1) on the following form, since by [1, Proposition 1.2] the operator $L\colon Y_1 \to \operatorname{Im} L$ is invertible

$$\begin{aligned} &\text{i)} \quad y_1 = L^{-1}\big(-Q(y_1 + c\cdot\varphi)^3 + f_1\big) \\ &\text{ii)} \quad 0 = -P(y_1 + c\cdot\varphi)^3 + f_2\,, \end{aligned} \tag{2.2}$$

where $y_1 \in C^0[0, 1]$, $P_0 y_1 = 0$, $c \in \mathbb{R}$, $f_1 \in X_1$, $f_2 = d\cdot g$, $d \in \mathbb{R}$. Note that $Z = \{y \in C^0[0, 1],\ P_0 y = 0\}$ is a Banach space with the supremum norm $\|\cdot\|$.

Applying the implicit function theorem we can solve y_1 in (2.2) i) for c, f_1 small and we have $y_1(c, f_1)$. Indeed, consider the operator $G(y_1, c, f_1) = y_1 - L^{-1}\big(-Q(y_1 + c\cdot\varphi)^3 + f_1\big)$ defined on a neighbourhood of $0 \in Z\times\mathbb{R}\times X_1$. Then G is C^1-smooth and the linearization $G_{y_1}(0, 0, 0) = \mathrm{Id}\colon Z \to Z$ is invertible. We put this solution into the equation (2.2) ii) and obtain the bifurcation function

$$F(c, f_1) = P\big(y_1(c, f_1) + c\cdot\varphi\big)^3 .$$

Now we seek small solutions of $F(c, f_1) = f_2$. Since $c \in \mathbb{R}$, $f_1 \in X_1$, $f_2 \in \operatorname{Im} P$ and $\dim \operatorname{Im} P = 1$, we can consider F as a map defined on a neighbourhood of $0 \in \mathbb{R} \times X_1$ into $\mathbb{R}$. We shall study the singularity of $F(c, 0)$ at $c = 0$.

Lemma 2.1. *If $\varphi^3 \notin \operatorname{Im} L$, then $F(c, 0) = a\cdot c^3 + O(c^4)$ with $a \neq 0$.*

P r o o f. By (2.2) i) it follows

$$y_1(c, 0) = L^{-1}\big(-Q(y_1(c, 0) + c\cdot\varphi)^3\big).$$

Further, for c small $y_1(c, 0)$ is small as well, hence

$$\begin{aligned} \|y_1(c, 0)\| &\leqq \|L^{-1}\|\cdot\|Q\|\cdot\big(\|y_1(c, 0)\| + |c|\cdot\|\varphi\|\big)^3 \\ &\leqq \|L^{-1}\|\cdot\|Q\|\cdot 4\cdot\big(\|y_1(c, 0)\|^3 + |c|^3\cdot\|\varphi\|^3\big) \\ &\leqq \|L^{-1}\|\cdot\|Q\|\cdot 4\cdot\|y_1(c, 0)\|^2\cdot\|(y_1(c, 0)\| + O(c^3) \\ &\leqq \big(\|y_1(c, 0)\| + O(c^3)\big)/2 \end{aligned}$$

and this gives $y_1(c, 0) = O(c^3)$.
(We have used the inequality $(a + b)^3 \leqq 4\cdot(a^3 + b^3)$ for $a \geqq 0$, $b \geqq 0$.) Hence

$$F(c, 0) = P\big(y_1(c, 0) + c\cdot\varphi\big)^3 = c^3\cdot P\varphi^3 + O(c^4).$$

Using $P\varphi^3 \neq 0$ we obtain the assertion.

From Lemma 2.1 it follows that if $\varphi^3 \notin \operatorname{Im} L$, then 0 is an isolated solution of $\mathcal{L}y = 0$. Indeed, the equation $\mathcal{L}y = 0$ is equivalent to $F(c, 0) = 0$, $F(c, 0) = a \cdot c^3 + O(c^4)$, $a \neq 0$ and $c = 0$ is an isolated solution of this equation. This result was mentioned in the Introduction of this paper.

Let $\varphi^3 \in \operatorname{Im} L$, i.e., $P\varphi^3 = 0$ and $Lw = \varphi^3$ for some $w \in Y_1$. Putting $y_1 = y_2 - c^3 \cdot w$, $y_2 \in Z$, we have from (2.2)

$$\text{i)}\quad y_2 - c^3 \cdot w = L^{-1}\Big(-Q\big((y_2 - c^3 w)^3 + 3(y_2 - c^3 w)^2 \cdot c \cdot \varphi + 3(y_2 - c^3 w) \cdot c^2 \cdot \varphi^2\big) - c^3 \cdot \varphi^3 + f_1\Big)$$

$$\text{ii)}\quad 0 = -P\big((y_2 - c^3 w)^3 + 3(y_2 - c^3 w)^2 \cdot c \cdot \varphi + 3(y_2 - c^3 w) \cdot c^2 \cdot \varphi^2\big) + f_2\,,$$

i.e.,

$$\begin{aligned} \text{i)}\quad y_2 &= L^{-1}\big(-Q(y_2^3 - 3y_2^2 c^3 w + 3y_2 c^6 w^2 + 3y_2^2 c\varphi - 6y_2 c^4 \varphi w + 3y_2 c^2 \varphi^2 \\ &\qquad - c^9 w^3 + 3c^7 w^2 \varphi - 3c^5 w \varphi^2) + f_1\big) \\ &= L^{-1}\big(-Q(y_2^3 + c \cdot y_2 \cdot h(y_2, c) - 3c^5 w\varphi^2 + O(c^6)) + f_1\big) \\ \text{ii)}\quad 0 &= -P\big(y_2^3 + c \cdot y_2 \cdot h(y_2, c) - 3c^5 w\varphi^2 + O(c^6)\big) + f_2\,, \end{aligned} \tag{2.3}$$

where $h(y_2, c) = -3y_2 c^2 w + 3y_2\varphi - 6c^3\varphi w + 3c\varphi^2$.

Applying the implicit function theorem we can solve y_2 from the first equation (2.3) i) for c, f_1 small and putting this solution $y_2(c, f_1)$ into

$$-P\big(y_2^3 + c \cdot y_2 \cdot h(y_2, c) - 3c^5 w\varphi^2 + O(c^6)\big)$$

we obtain as in the above procedure the bifurcation function

$$G(c, f_1) = -P\big(y_2^3(c, f_1) + c \cdot y_2(c, f_1) \cdot h(y_2(c, f_1), c) - 3c^5 w\varphi^2 + O(c^6)\big)\,.$$

By (2.3) i) it follows that

$$y_2(c, 0) = L^{-1}\Big(-Q\big(y_2^3(c, 0) + c \cdot y_2(c, 0) \cdot h(y_2(c, 0), c) - 3c^5 w\varphi^2 + O(c^6)\big)\Big)\,.$$

Further, for c small $y_2(c,0)$ is small as well and in the same way as in the proof of Lemma 2.1 we have

$$\|y_2(c, 0)\| \leqq \big(\|y_2(c, 0)\| + O(c^5)\big)/2 \qquad \text{for } c \text{ small}.$$

Hence $y_2(c, 0) = O(c^5)$ and we have for $w \cdot \varphi^2 \notin \operatorname{Im} L$, i.e., $Pw \cdot \varphi^2 \neq 0$

$$G(c, 0) = b \cdot c^5 + O(c^6)\,, \quad b \neq 0\,.$$

(We consider G as a map defined on a neighbourhood of $0 \in \mathbb{R} \times X_1$ into $\mathbb{R}$, since $c \in \mathbb{R}$, $f_1 \in X_1$, $G(\cdot, \cdot) \in \operatorname{Im} P$, $\dim \operatorname{Im} P = 1$.)

Summing up we obtain

LEMMA 2.2. *If $\varphi^3 \in \operatorname{Im} L$, $Lw = \varphi^3$, $w \in Y_1$ and $w \cdot \varphi^2 \notin \operatorname{Im} L$, then the bifurcation function has the form*

$$G(c, 0) = b \cdot c^5 + O(c^6), \quad b \neq 0.$$

By Lemma 2.2 for d, f_1 small the equation $G(c, f_1) + d \cdot g = 0$ has always at least one solution near $c = 0$ and hence we obtain the proof of the above theorem from [1].

Lefton also discussed the case when $w \cdot \varphi^2 \in \operatorname{Im} L$, i.e., $Lv = w \cdot \varphi^2$, $v \in Y_1$. But we can repeat the above procedure. We have transformed (2.2) into (2.3) putting $y_1 = y_2 - c^3 \cdot w$. Now we put in (2.3) $y_2 = y_3 + 3 \cdot c^5 v$, $y_3 \in Z$ and it is easy to see that (2.3) has the form

$$\begin{aligned} &\text{i)} \quad y_3 = L^{-1}\big(-Q\big(y_3^3 + c \cdot y_3 \cdot g(y_3, c) + 3c^7(w^2 \cdot \varphi + 3v \cdot \varphi^2)\big) + O(c^8) + f_1\big) \\ &\text{ii)} \quad 0 \;= -P\big(y_3^3 + c \cdot y_3 \cdot g(y_3, c) + 3c^7(w^2 \cdot \varphi + 3v \cdot \varphi^2) + O(c^8)\big) + f_2, \end{aligned} \tag{2.4}$$

where $g(y_3, c)$ has a similar form as the mapping $h(y_2, c)$.

We can solve (2.4) i) in $y_3 = y_3(c, f_1)$ for c, f_1 small by the implicit function theorem and again we obtain the bifurcation function

$$\begin{aligned} H(c, f_1) = P\big(y_3^3(c, f_1) + c \cdot y_3(c, f_1) \cdot g\big(y_3(c, f_1), c\big) \\ + 3c^7(w^2\varphi + 3v\varphi^2) + O(c^8)\big). \end{aligned}$$

In the same way as in the proof of Lemma 2.1 it follows from (2.4) i) that

$$y_3(c, 0) = O(c^7) \qquad \text{for } c \text{ small}.$$

Hence

$$H(c, 0) = 3 \cdot c^7 \cdot P(w^2\varphi + 3v\varphi^2) + O(c^8) \qquad \text{for } c \text{ small}.$$

LEMMA 2.3. *If $w\varphi^2 = Lv$, $v \in Y_1$ and $w^2\varphi + 3v\varphi^2 \notin \operatorname{Im} L$, then the bifurcation function H has the form*

$$H(c, 0) = d \cdot c^7 + O(c^8), \quad d \neq 0.$$

We consider H as a map defined on a neighbourhood of $0 \in \mathbb{R} \times X_1$ into $\mathbb{R}$.

Applying Lemma 2.3 we can solve $H(c, f_1) = d \cdot g$ for f_1, d small near $c = 0$. Hence we have

THEOREM 2.4. *Under the conditions of Lemma 2.3 the equation $\mathcal{L}y = f$ has at least one small solution for each f small.*

Now, if $w^2\varphi + 3v\varphi^2 \in \operatorname{Im} L$, then we can proceed in the above procedure. Of course, our method has sense only if this procedure stops after a finite number of steps and this holds only if $F(c, 0)$ is not flat at $c = 0$, i.e., $\frac{\partial^i}{\partial^i c}F(0, 0) \neq 0$ for some i. It seems that the example from [1] presents the case when $F(c, 0)$ is flat at $c = 0$.

We also see that $F(c, 0)$ had the forms

$$F(c, 0) = a \cdot c^i + O(c^{i+1}), \quad a \neq 0,$$

where $i = 3$ or $i = 5$ or $i = 7$. This property did not hold by chance, but it follows from the following fact: The map $\mathcal{L}$ is equivariant by the group $\mathbb{Z}_2$, since $\mathcal{L}(-y) = -\mathcal{L}y$ and we can easily derive that $F(c, 0)$ has this property as well, thus

$$F(-c, 0) = -F(c, 0)$$

for c small. Hence there generally holds

$$F(c, 0) = a \cdot c^{2i+1} + O(c^{2i+2}) \qquad a \neq 0$$

when F is not flat and in this case the equation $\mathcal{L}y = f$ has at least one small solution for each f small.

Finally, we can consider similarly the problem

$$Ly \pm y^{2n+1} = f$$

$$M_1(y) = M_2(y) = 0.$$

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