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EXTENSION AND REGULARITY OF l-GROUP VALUED MEASURES

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In his paper [4] J. D. Maitland Wright considered measures which take their values in a boundedly σ-complete vector lattice V. He studied the measure extension property of V and proved the main theorem which characterizes this quality of V through the property of the regularity of the V-valued Baire measure on a compact Hausdorff space.

In the first part of this paper we consider the extension theorem for l-group valued measures. We extend the measure μ from the algebra $\mathcal{A}$ to the σ-algebra $\mathcal{C}$ containing $\mathcal{A}$. In the second part the sufficient condition for the regularity of the l-group valued measure μ defined on the σ-algebra $\mathcal{S}$ of Borel sets of the topological space is given.

Let us introduce some notation first. $x \vee y$, $x \wedge y$-will denote lattice operations. $x_n \nearrow x$ ($x_n \searrow x$) will be written iff $x_n \leqq x_{n+1}$ ($x_n \geqq x_{n+1}$) for every n and $\bigvee_{n=1}^{\infty} x_n = x$ ($\bigwedge_{n=1}^{\infty} x_n = x$). A similar notation is used for sequences of sets.

Let X be a nonempty set and $\mathcal{A}$ be an algebra of subsets of X. Let $\mathcal{L}$ be a commutative l-group.

Definition 1. *The set function μ: $\mathcal{A} \to \mathcal{L}$ is a measure iff*

(i) *$\mu(A) \geqq 0$ for every $A \in \mathcal{A}$ (0 is a zero element of $\mathcal{L}$)*

(ii) *μ is finitely additive, i.e. if $A_i \in \mathcal{A}$, $i = 1, 2, \ldots, n$, and $A_i \cap A_j = \emptyset$ for $i \neq j$, then*

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$$

(iii) *μ is continuous from above at $\emptyset$, i.e. if $A_i \in \mathcal{A}$, $i = 1, 2, \ldots, A_i \searrow \emptyset$, then $\mu(A_i) \searrow 0$.*

Observe that the measure μ has the following properties:

(1) $\mu(\emptyset) = 0$

(2) μ is monotone, i.e. if $A, B \in \mathcal{A}$, $A \subset B$, then $\mu(A) \leqq \mu(B)$

(3) μ is subtractive, i.e. if $A, B \in \mathcal{A}$, $A \subset B$, then $\mu(B - A) = \mu(B) - \mu(A)$

(4) μ is countable additive, i.e. if $A_i \in \mathcal{A}$, $i = 1, 2, \ldots$, $A_i \cap A_j = \emptyset$, $i \neq j$, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$, then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$

(5) μ is continuous from below at any set $A \in \mathcal{A}$, i.e. for every sequence $\{A_i\}_{i=1}^{\infty}$, $A_i \in \mathcal{A}$, for which $A_i \nearrow A$ we have $\mu(A) = \vee \mu(A_i)$.

Definition 2. *A l-group $\mathcal{L}$ has a countable type if the following holds: if $\mathcal{M} \subset \mathcal{L}$ and $c = \sup \mathcal{M}$, then there exists a countable chain $\mathcal{K}$, $\mathcal{K} \subset \mathcal{M}$, such that $c = \sup \mathcal{K}$. The l-group $\mathcal{L}$ is regular if there holds: if $a_k^i \in \mathcal{L}$ for $i = 1, 2, \ldots$, $k = 1, 2, \ldots$, are such that $a_k^i \searrow 0$ $(i \nearrow \infty)$ for $k = 1, 2, \ldots$, and $b \in \mathcal{L}$ is such that for every sequence $\{i_1, i_2, i_3, \ldots\}$ of positive integers $b \leqq \bigvee_n \left(\sum_{k=1}^{n} a_k^{i_k}\right)$, then $b \leqq 0$.*

Lemma. *Every l-group is a distributive lattice. Every complete l-group is a commutative group. (See* Birkhoff G. [1])

Let us denote

$$\mathcal{B} = \left\{A \subset X\colon A = \bigcup_{i=1}^{\infty} A_i,\ \ A_i \in \mathcal{A},\ \ A_i \subset A_{i+1},\ \ i = 1, 2, \ldots\right\}.$$

Let μ be a measure defined on $\mathcal{A}$ with values in $\mathcal{L}$. If $\mathcal{L}$ is a complete l-group, we define a set function $\vartheta\colon \mathcal{B} \to \mathcal{L}$ by

(a) $$\vartheta(A) = \vee \mu(A_i), \quad \text{where} \quad A_i \in \mathcal{A},\ A_i \nearrow A\ .$$

Proposition 1. *The set function ϑ is unambiguously defined.*

Proof. Let $A_n \nearrow A$, $B_n \nearrow A$, A_n, $B_n \in \mathcal{A}$, $n = 1, 2, \ldots$. We have to show that $\vee \mu(A_n) = \vee \mu(B_n)$. But $A_k = \bigcup_{n=1}^{\infty} (A_k \cap B_n)$ and μ is continuous from below at a set A_k.

Hence $\mu(A_k) = \bigvee_n \mu(A_k \cap B_n) \leqq \vee \mu(B_n)$ and $\bigvee_k \mu(A_k) \leqq \bigvee_n \mu(B_n)$. We can reverse the roles of $\{A_n\}$ and $\{B_n\}$ in the argument and show that $\bigvee_k \mu(A_k) = \bigvee_n \mu(B_n)$.

Theorem 1. *Let $\mathcal{L}$ be a complete l-group and ϑ be a function defined on $\mathcal{B}$ by* (**a**). *Then ϑ has the following properties:*

(i) *$\mathcal{A} \subset \mathcal{B}$ and $\vartheta / \mathcal{A} = \mu$*

(ii) *if* $A_n \in \mathscr{B}$, $n = 1, 2, \ldots$, *then* $A_1 \cap A_2 \in \mathscr{B}$ *and* $\bigcup_{i-1}^{\infty} A_i \in \mathscr{B}$

(iii) *if* $A, B \in \mathscr{B}$, *then* $\vartheta(A \cup B) + \vartheta(A \cap B) = \vartheta(a) + \vartheta(B)$

(iv) *if* $A_n \in \mathscr{B}$, $n = 1, 2, .$ *and* $A_n \nearrow A$, *then* $\vartheta(A_1) \leqq \vartheta(A_2)$ *and* $\vartheta(A) = \vee \vartheta(A_n)$.

Proof. (i) is trivial. Clearly (ii) will hold if we prove that if $A_n \nearrow A$, $A_n \in \mathscr{B}$, $n = 1, 2, \ldots$, then $A \in \mathscr{B}$. Let $A_{ni} \nearrow A_n$ if $i \nearrow \infty$, $n = 1, 2, \ldots$. Denote $B_i = \bigcup_{j \leqq i} A_{ji}$. Then B_i is monotone, $B_i \in \mathscr{A}$, and $\bigcup_{i=1}^{\infty} B_i = \bigcup_{n=1}^{\infty} A_n = A$. (iii) holds since for any $A, B \in \mathscr{A}$ we have $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$. $\mathscr{L}$ is a complete l-group and if $a_n \in \mathscr{L}$, $b_n \in \mathscr{L}$, $n = 1, 2, \ldots$, $n = 1, 2, \ldots$, $a_n \nearrow$, $b_n \nearrow$, then $\vee a_n + \vee b_n = \vee (a_n + b_n)$. According to Proposition 1 we prove only the second part of (iv). We use the notation from above. Then $A_{in} \subset B_n \subset A_n$ for $i \leqslant n$, hence $\mu(A_{in}) \leqslant \mu(B_n) \leqslant \vartheta(A_n)$ and $\vartheta(A_i) = \vee_n \mu(A_{in}) \leqq \vee \mu(B_n) \leqq \vee \vartheta(A_n)$ for $i = 1, 2, \ldots$. Thus we have $\vee \vartheta(A_i) \leqq \vee \mu(B_n) \leqq \vee \vartheta(A_n)$ and $\vartheta(A) = \vee \mu(B_n) = \vee \vartheta(A_n)$.

Theorem 2. *Let the symbols* μ, $\mathscr{A}$, $\mathscr{B}$, ϑ *denote the same as in the Theorem 1 and let* $\mathscr{L}$ *be a complete, regular l-group which has a countable type. Then a function* μ^* *defined on* 2^X *by*

(b) $$\mu^*(C) = \wedge \{\vartheta(B) : C \subset B \in \mathscr{B}\}$$

has the following properties:

(i) $\mu^*/\mathscr{B} = \vartheta$, $\mu^*(C) \geqq 0$ *for all* $C \subset X$

(ii) $\mu^*(C_1 \cup C_2) + \mu^*(C_1 \cap C_2) \leqq \mu^*(C_1) + \mu^*(C_2)$ *for all* C_1, C_2

(iii) *if* $C_1, C_2 \subset X$ *and* $C_1 \subset C_2$, *then* $\mu^*(C_1) \leqq \mu^*(C_2)$

(iv) *if* $C_n \subset X$, $n = 1, 2, \ldots$, $C_n \nearrow C$ $(n \nearrow \infty)$, *then* $\mu^*(C_n) \nearrow \mu^*(C)$.

Proof. (i) is trivial. Let $B_1^n \in \mathscr{B}$, $B_2^n \in \mathscr{B}$, $n = 1, 2, \ldots$ such that $\vartheta(B_1^n) \searrow \mu^*(C_1)$ and $\vartheta(B_2^n) \searrow \mu^*(C_2)$. According to (iii) Theorem 1 $\vartheta(B_1^n) + \vartheta(B_2^n) = \vartheta(B_1^n \cap B_2^n) + \vartheta(B_1^n \cup B_2^n) \geqq \mu^*(C_1 \cap C_2) + \mu^*(C_1 \cup C_2)$, hence $\mu^*(C_1) + \mu^*(C_2) \geqq \mu^*(C_1 \cap C_2) + \mu^*(C_1 \cup C_2)$. (iii) is trivial. Let $C_n \subset X$, $n = 1, 2, \ldots$, $C_n \nearrow C$. $\mathscr{L}$ has a countable type and hence there exist $B_n^i \in \mathscr{B}$, $n = 1, 2, \ldots$, such that for every n $\vartheta(B_n^i) \searrow \mu^*(C_n)$ $(i \nearrow \infty)$. Denote $a_n^i = \vartheta(B_n^i) - \mu^*(C_n)$ and $b = \mu^*(C) - \vee \mu^*(C_n)$. For any sequence $\{i_1, i_2, \ldots\}$ of positive integers we have $b \leqq \mu^*\left(\bigcup_{n=1}^{\infty} B_n^{i_n}\right) - \vee \mu^*(C_n) \leqq \vartheta\left(\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} B_n^{i_n}\right) - \vee \mu^*(C_n) \leqq \vee_k \vartheta\left(\bigcup_{n=1}^{k} B_n^{i_n}\right) - \vee \mu^*(C_n) \leqq \vee_k \left[\vartheta\left(\bigcup_{n=1}^{k} B_n^{i_n}\right) - \mu^*(C_k)\right]$. The difference $\vartheta\left(\bigcup_{n=1}^{k} B_n^{i_n}\right) - \mu^*(C_k)$ can be bounded by $\vartheta(B_1^{i_1} \cup \ldots \cup B_k^{i_k}) \leqq \sum_{j=1}^{K} \vartheta(B_j^{i_j}) - \sum_{i=1}^{k-1} \mu^*(C_i)$. This inequality may be verified using mathematical induction. Hence $b \leqq \bigvee_k \left| \sum_{n=1}^{k} (\vartheta(B_n^{l_n}) - \mu^*(C_n)) \right| = \bigvee_k \left(\sum_{n=1}^{k} a_n^{l_n} \right)$. With respect to regularity of $\mathcal{S}$ we have $b \leqq 0$ and $\mu^*(C) = \vee \mu^*(C_n)$.

Theorem 3. *Let the symbols and assumptions of Theorem 2 hold. Denote* $\mathcal{C} = \{C \subset X: \mu^*(C) + \mu^*(C^c) = \mu(X)\}$. *Then* $\mathcal{C}$ *is the* σ*-algebra of the subsets of* X *and* $\bar{\mu} = \mu^*/\mathcal{C}$ *is the complete measure (if* $A \in \mathcal{C}$, $\mu(A) = 0$ *and* $B \subset A$, *then* $B \in \mathcal{C}$).

Proof. Observe that $\emptyset$, $X \in \mathcal{C}$ and $\mathcal{C}$ is closed with respect to the complementation. Let $B_1, B_2 \in \mathcal{C}$. Then $\mu^*(B_1 \cup B_2) + \mu^*(B_1^c \cap B_2^c) \leqq \mu^*(B_1) + \mu^*(B_2) - \mu^*(B_1 \cap B_2) + \mu^*(B_1^c) + \mu^*(B_2^c) - \mu^*(B_1^c \cup B_2^c) \leqq \mu(X) + \mu(X) - \mu^*(B_1 \cap B_2) - \mu^*(B_1 \cap B_2)^c \leqq \mu(X)$. We have just proved that $\mathcal{C}$ is closed under formation of finite unions. Let $B_n \in \mathcal{C}$, $n = 1, 2, \ldots$, $B_n \nearrow \bigcup_{n=1} B_n$, then $\mu^*(B_m) + \mu^*(B_m^c) \leqq \mu(X)$, $m = 1, 2, \ldots$, and $\mu^*\left(\left(\bigcup_{m=1}^{\infty} B_m\right)^c\right) \leqq \mu^*(B_m^c)$ for all m and $\mu^*(B_m) + \mu^*\left(\left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \leqq \mu(X)$ for all m. We have $\vee \mu^*(B_m) + \mu^*\left(\left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \leqq \mu(X)$ and $\mathcal{C}$ is a σ-algebra. According to (iv) Theorem 2 μ^* is a measure if we show that μ^* is additive. Let $B_1, B_2 \in \mathcal{C}$, then $\mu^*(B_1 \cup B_2) + \mu^*(B_1 \cap B_2) \leqq \mu^*(B_1) + \mu^*(B_2)$ according to (ii) Theorem 2. Also $\mu^*((B_1 \cup B_2)^c) + \mu^*((B_1 \cap B_2)^c) \leqq \mu^*(B_1^c) + \mu^*(B_2^c)$ and the sum on the right-hand sides of the two last equalities is equal to $2\mu(X)$. On the other hand $\mu^*(B_1 \cup B_2) + \mu^*(B_1 \cup B_2)^c \geqq \mu(X)$ and $\mu^*(B_1 \cap B_2) + \mu^*(B_1 \cap B_2)^c \geqq \mu(X)$, hence there is equality in each of these inequalities and μ^* is additive. μ^* is complete since if $A \in \mathcal{C}$, $\mu^*(A) = 0$ and $B \subset A$, $\mu^*(B) + \mu^*(B^c) \leqq \mu^*(A) + \mu(X) = \mu(X)$ holds and $B \in \mathcal{C}$.

Theorem 4. *If* μ *is a measure on the algebra* $\mathcal{A}$ *with values in the complete, regular l-group* $\mathcal{S}$ *which has a countable type, then* μ *has the unique extension* $\bar{\mu}$ *on the* σ*-algebra* $\mathcal{D}$ *generated by the algebra* $\mathcal{A}$.

Proof. According to Theorem 3 the system $\mathcal{C}$ from Theorem 3 is the σ-algebra containing $\mathcal{A}$ and hence $\mathcal{C} \supset \mathcal{D}$. $\bar{\mu}$ defined by $\bar{\mu}(A) = \mu^*(A)$ for every $A \in \mathcal{D}$ is the extension of the measure μ. Let there exist a measure q on $\mathcal{D}$ such that $q/\mathcal{A} = \mu$. With respect to the definition μ^*, $q \leqq \mu^*$ on $\mathcal{B}$ (observe that $q = \vartheta$ on $\mathcal{B}$). Let $A_0 \in \mathcal{D}$ be such that $q(A_0) < \mu^*(A_0)$. With respect to the last inequalities we have $q(X) = q(A_0) + q(A_0^c) < \mu^*(A_0) + \mu^*(A_0^c) = \mu(X)$, which is impossible since $q = \mu^*$ on $\mathcal{A}$.

2.

Let X be a topological space with a topology $\mathcal{T}$. It is known (see Halmos P. [2]) that in the locally compact spaces the real measure is regular if every compact set is outer regular. Let us investigate the analogy quality of a l-group valued measure. Let $\mathcal{L}$ be a complete, regular l-group which has a countable type. Let X be a topological space with a topology $\mathcal{T}$, let $\mathcal{B}$ be a σ-algebra of Borel sets in X. Denote by $\mathcal{Z}$ the system of all closed sets in X and μ a measure on $\mathcal{B}$ with values in $\mathcal{L}$.

Definition 3. *The set $E \in \mathcal{B}$ is outer regular, if $\mu(E) = \wedge\{\mu(U): E \subset U \in \mathcal{T}\}$. The set $E \in \mathcal{B}$ is inner regular, if $\mu(E) = \vee\{\mu(Z): E \supset Z \in \mathcal{Z}\}$. The set $E \in \mathcal{B}$ is regular if it is both inner and outer regular. A measure μ is regular if every set $E \in \mathcal{B}$ is regular.*

Let $\mathcal{R}$ be a system of all regular subsets of X.

Proposition 2.

(i) *$\mathcal{Z}$ is a lattice of sets .*

(ii) *if $\mathcal{A}$ is a system of subsets of X, denote by $\mathcal{P}\mathcal{A}$ the system $\{A: A = B - C, C \subset B, C, B \in \mathcal{A}\}$. $\mathcal{P}\mathcal{Z}$ is a semiring.*

(iii) *if $\mathcal{A}$ is a system of subsets of X, let $\mathcal{N}\mathcal{A}$ be a normal system generated by $\mathcal{A}$. Then $\mathcal{N}\mathcal{P} = \mathcal{S}\mathcal{P}$, where $\mathcal{P}$ is a semiring and $\mathcal{S}\mathcal{P}$ is a σ-ring generated by $\mathcal{P}$.* (See Halmos P. [2] §5.6)

Theorem 5. *If every set in $\mathcal{Z}$ is outer regular, then $\mathcal{P}\mathcal{Z} \subset \mathcal{R}$.*

Proof. Let $C, D \in \mathcal{Z}$ and $C \subset D$. A set $D - C$ is inner regular since $\mu(D) - \mu(C) = \mu(D) - \wedge\{\mu(U): C \subset U \in \mathcal{T}\} = \vee\{\mu(D) - \mu(U): C \subset U \in \mathcal{T}\} = \vee\{\mu(D - U); \quad B - C \supset D - U \in \mathcal{Z}\} \leqq \vee\{\mu(Z): \quad D - C \supset Z \in \mathcal{Z}\}$. But $\mu(D - C) = \mu(D) - \mu(C) = \wedge\{\mu(U): \quad D \subset U \in \mathcal{T}\} - \mu(C) = \wedge\{\mu(U - C): D - C \subset U - C \in \mathcal{T}\} \geqq \wedge\{\mu(U): D - C \subset U \in \mathcal{T}\}$ and the set $D - C$ is outer regular.

Theorem 6. *The system $\mathcal{R}$ is closed with respect to finite disjoint unions and with respect to the complementation.*

Proof. $\mu(A \cup B) = \mu(A) + \mu(B) = \vee\{\mu(A_k): \quad A \supset A_k \in \mathcal{Z}, \quad k = 1, 2, \dots\} + \vee\{\mu(B_k): B \supset B_k \in \mathcal{Z}, \quad k = 1, 2, \dots\} = \vee\{\mu(A_k) + \mu(B_k): \quad A \supset A_k \in \mathcal{Z}, B \supset B_k \in \mathcal{Z}, k = 1, 2, \dots\} = \vee\{\mu(A_k \cup B_k): A \cup B \supset A_k \cup B_k \in \mathcal{Z}, k = 1, 2, \dots\} \leqq \vee\{\mu(Z): A \cup B \supset Z \in \mathcal{Z}\}$, the reverse inequality is trivial. Let us prove the outer regularity of $A \cup B$. $\mu(A \cup B) = \mu(A) + \mu(B) = \wedge\{\mu(A_k): \quad A \subset A_k \in \mathcal{T}, k = 1, 2, \dots\} + \wedge\{\mu(B_k): \quad B \subset B_k \in \mathcal{T}, \quad k = 1, 2, \dots\} \geqq \wedge\{\mu(A_k \cup B_k): A \cup B \subset A_k \cup B_k \in \mathcal{T}\} \geqq \wedge\{\mu(U): A \cup B \subset U \in \mathcal{T}\}$. At last if $A \in \mathcal{R}$, then A^c is inner regular since $\mu(A^c) = \mu(X) - \mu(A) = \mu(X) - \wedge\{\mu(U): \quad A \subset U \in \mathcal{T}\} = \vee\{\mu(X - U): A \subset U \in \mathcal{T}\} = \vee\{\mu(X - U): A^c \supset X - U, X - U \in \mathcal{Z}\} = \vee\{\mu(Z): A^c \supset Z \in \mathcal{Z}\}$, and outer regular in the dual way.

Theorem 7. *If every set in $\mathcal{Z}$ is outer regular then a system $\mathcal{R}$ is a normal system containing $\mathcal{P}\mathcal{Z}$.*

Proof. According to Theorem 5 and 6 we have to prove (1) and (2):

(1) if $A_i \in \mathcal{R}$, $A_i \searrow \bigcap_{i=1}^{\infty} A_i$, then $\bigcap_{i=1}^{\infty} A_i$ is outer regular

(2) if $A_i \in \mathcal{R}$, $A_i \nearrow \bigcup_{i=1}^{\infty} A_i$, then $\bigcup_{i=1}^{\infty} A_i$ is outer regular.

(1) We have to show that $\mu\left(\bigcap_{i=1}^{\infty} A_i\right) \geqq \wedge \left\{\mu(B) : \bigcap_{i=1}^{\infty} A_i \subset B \in \mathcal{T}\right\}$, since the reverse inequality is trivial. $\mathcal{S}$ has a countable type and for any $i = 1, 2, \ldots$ we have $\mu(A_i) = \bigwedge_k \{\mu(B_{ik}) : \quad A_i \subset B_{ik} \in \mathcal{T}, \quad k = 1, 2, \ldots\}$ and $\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \bigwedge_i \bigwedge_k \{\mu(B_{ik}) :$ $A_i \subset B_{ik} \in \mathcal{T}, \quad i, k = 1, 2, \ldots\} = \bigwedge_k \bigwedge_i \{\mu(B_{ik}) : \quad A_i \subset B_{ik} \in \mathcal{T}, \quad i, k = 1, 2, \ldots\}$ but $\bigwedge_i \{\mu(B_{ik}) : A_i \subset B_{ik} \in \mathcal{T}\} \geqq \wedge \{\mu(B) : \bigcap_{i=1}^{\infty} A_i \subset B \in \mathcal{T}\}$ for any k.

(2) We have to show that $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \geqq \wedge \{\mu(B) : \bigcup_{i=1}^{\infty} A_i \subset B \in \mathcal{T}\}$. But $A_i \in \mathcal{R}$ and $\mu(A_i) = \bigwedge_k \left\{\mu(B_{ik}) : A_i \subset B_{ik} \in \mathcal{T}, k = 1, 2, \ldots\right\}$. Let $\{k_1, k_2, \ldots\}$ be any sequence of positive integers. Denote $a_i^k = \mu(B_{ik}) - \mu(A_i)$. Then $a_i^k \searrow 0$ $(k \nearrow \infty)$, for $i = 1, 2, \ldots$. Denote $a = \wedge \left\{\mu(B) : \bigcup_{i=1}^{\infty} A_i \subset B \in \mathcal{T}\right\} - \vee \mu(A_i)$, then $a \leqq \mu\left(\bigcup_{i=1}^{\infty} B_{ik_i}\right) - \vee \mu(A_i) =$ $\mu\left(\bigcup_{l=1}^{\infty} \bigcup_{i=1}^{l} B_{ik_i}\right) - \vee \mu(A_l) \leqq \bigvee_l \left[\mu\left(\bigcup_{i=1}^{l} B_{ik_i}\right) - \mu(A_l)\right] \quad \leqq \quad \bigvee_l \left[\sum_{i=1}^{l} (\mu(B_{ik_i}) - \mu(A_i))\right] =$ $\bigvee_l \left(\sum_{i=1}^{l} a_i^{k_i}\right)$ and according to the regularity of $\mathcal{S}$ we have $a \leqq 0$. We used the inequality

$$\mu\left(\bigcup_{i=1}^{n} B_{ik_i}\right) \leqq \sum_{i=1}^{n} \mu(B_{ik_i}) - \sum_{i=1}^{n-1} \mu(A_i)$$

which holds since $B_{ik_i} \supset A_i$.

Corollary. *If every set in $\mathcal{Y}$ is outer regular and if $\mathcal{L}$ is a complete, regular l-group which has a countable type and μ is a measure on $\mathcal{B}$ with values in $\mathcal{L}$, then μ is regular in the sense of the Definition 3.*

Proof. According to Proposition 2 and Theorems 5, 6, 7 we have $\mathcal{B} = \mathcal{S}\mathcal{Y} = \mathcal{S}\mathcal{P}\mathcal{Z} = \mathcal{N}\mathcal{P}\mathcal{Z} \subset \mathcal{R}$.

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ПРОДОЛЖЕНИЕ И РЕГУЛЯРНОСТЬ МЕР СО ЗНАЧЕНИЯМИ Б $л$-ГРУППЕ

Петр Болауф

Резюме

В настоящей заметке под мерой будем понимать отображение μ определенное на алгебре $\mathcal{A}$ подмножеств множества X со значениями в $л$-группе $\mathcal{S}$, выполняющее следующие условия: $\mu(A) \geqq 0$ для всяких $A \in \mathcal{A}$, μ конечно-аддитивная и полунепрерывная снизу. Целью заметки является формулировка условий накладываемых на $\mathcal{S}$ достаточных для продолжения меры с алгебры $\mathcal{A}$ на σ-алгебры $\mathcal{B}$ содержащую $\mathcal{A}$.

Во вторий части изучается проблема регулярности меры как она формулирована например в книге Халмоша [2]. В обеих частях центральную роль играют условия счетново типа и регулярности накладываемые на $л$-группу $\mathcal{S}$.