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ON THE STRUCTURE FUNCTION
OF A G-STRUCTURE

IVAN KOLÁŘ—IVETA VADOVIČOVÁ

The first author deduced in [3] that the structure function of a generalized G-structure can be naturally defined in terms of the difference tensor of a semi-holonomic 2-jet. This approach leads to an original direct construction of the structure function of a classical (i.e. first order) G-structure. Since this problem is a matter of considerable interest to geometers, we now develop a complete version of the first order case including a detailed comparison with the classical constructions. Our approach underlines the fact that the structure function vanishes if and only if there exists the holonomic prolongation of the G-structure in question. (This result was recently derived in another way by A. Trautman, [6].) Then we deduce that every flat G-structure has the holonomic prolongation. The converse assertion is not true in general, which gives a clear interpretation of the fact that the vanishing of the structure function is only a necessary condition for flatness. Our consideration is in the category $C^\infty$.

1. Semi-holonomic 2-jets

We first recall the basic facts from the theory of semi-holonomic 2-jets, [1]. Given two manifolds $M, N$, we denote by $J^1(M, N)$ the space of all first order jets of $M$ into $N$ and by $\alpha: J^1(M, N) \to M$ or $\beta: J^1(M, N) \to N$ the source or target projection, respectively. Consider a local map $\varphi$ of a neighbourhood of a point $x \in M$ into $J^1(M, N)$ satisfying $\alpha \circ \varphi = \text{id}$ and

$$\varphi(x) = j_1^\alpha (\beta \circ \varphi).$$

The 1-jet $A = j_1^\alpha \varphi$ of such a map is said to be a semi-holonomic 2-jet of $M$ into $N$ with source $\alpha A = x$ and target $\beta A = \beta \varphi(x)$. The space of all these jets is denoted by $J^2(M, N)$. The canonical coordinates $u^i$ on $R^m$ and $v^p$ on $R^n$ induce the additional coordinates $v^p = \partial v^p / \partial u^i$ on $J^1(R^m, R^n)$. If $v^p = f^p(u^1, \ldots, u^m) = f^p(u)$ and $v^p = f^p(u)$ is the coordinate expression of $\varphi$, then (1) implies $f^p(x) = (\partial f^p / \partial u^i)(x)$. Hence $v^p_i = (\partial v^p / \partial u^i)$ are the additional coordinates on $J^2(R^m, R^n)$. There is a canonical inclusion $J^2(M, N) \subseteq J^2(M, N)$ of holonomic 2-jets into
semi-holonomic ones defined by $j^2f \mapsto j^1(j^1f)$. In coordinates, a holonomic 2-jet is characterized by the property that its second order coordinates are symmetric in the subscripts.

The composition of semi-holonomic 2-jets is defined as follows. Consider $A \in \bar{J}^2(M, N), A = j^1\varphi$ and $B \in \bar{J}^2(N, Q), B = j^1\psi$ satisfying $\beta A = \alpha B$. Then the composition of first order jets $\psi(\beta \varphi(u)) \varphi(u)$ is a local map of $M$ into $J^1(M, Q)$ of the type required in the definition of a semi-holonomic 2-jet and we set

$$B \circ A = j^1[\psi(\beta \varphi(u)) \varphi(u)] \in \bar{J}^2(M, Q).$$

If $A$ and $B$ are holonomic, one gets the usual composition of holonomic 2-jets.

This composition is associative. Hence the set $\tilde{L}_m^2$ of all invertible semi-holonomic 2-jets of $\mathbb{R}^m$ into $\mathbb{R}^m$ with source 0 and target 0 is a Lie group. By (3), its composition law in the coordinates $a^i, a^i_\lambda, \det a^i_\lambda \neq 0$ is expressed by

$$b^i \circ (a^i_\lambda, a^i_\mu) = (b^i a^i_\lambda, b^i_\lambda a^i_\lambda + b^i a^i_\mu).$$

The subset $L^2_{2\mathbb{R}^m}$ of all holonomic 2-jets is a Lie subgroup.

Consider further the first jet prolongation $J^1H^1M$ of the fibred manifold $H^1M \to M$ of all first order frames on $M$. We introduce a map $i: J^1H^1M \to \tilde{H}^2M$ as follows. Denote by $t_u$: $\mathbb{R}^m \to \mathbb{R}^m$ the translation $x \mapsto x + u$. Having $X \in J^1H^1M, X = j^1s, s(x) = j^0\varphi, \varphi: \mathbb{R}^m \to M$, we construct the composition $s(\varphi(u)) j^0(t_u^{-1}) \in J^1(\mathbb{R}^m, M)$ and set

$$i(X) = j^0[\varphi(u)] \circ j^0(t_u^{-1})] \in \tilde{H}^2M.$$

The coordinates on $H^1\mathbb{R}^m$ being $u^i, u^i_\lambda, \det u^i_\lambda \neq 0$, we introduce the additional coordinates on $J^1H^1\mathbb{R}^m$ by $\bar{u}^i_\lambda = \partial u^i_\lambda/\partial u^k$. If $X = (u^i, u^i_\lambda, \bar{u}^i_\lambda) \in J^1H^1\mathbb{R}^m$, then we deduce from (5) that the coordinates of $i(X) \in \tilde{H}^2\mathbb{R}^m$ are $u^i, u^i_\lambda$, and

$$u^i_\lambda = \bar{u}^i_\lambda u^k.$$

As $u^i_\lambda$ is a regular matrix, we have proved
Proposition 1. \(i: J^1H^1M \to \tilde{H}^2M \) is a fibred manifold isomorphism over \(H^1M\).

2. Prolongations of groups

By (4) the restriction of the group composition to the kernel of the jet projection \(\beta_1: \tilde{L}_m^2 \to L_m^1\) is the vector addition, so that we have an exact sequence of groups

\[
0 \to R^n \otimes R^m^* \otimes R^m^* \to \tilde{L}_m^2 \xto{\beta_1} L_m^1 \to 0.
\]

There is a splitting \(\lambda: L_m^1 \to \tilde{L}_m^2 \subset \tilde{L}_m^2\), \(a_i \mapsto (a_i, 0)\). (Geometrically, every \(a = (a_i) \in L_m^1\) determines a linear transformation \(\text{lin } a: \tilde{u}^i = a^i u^i\) and we set \(\lambda(a) = j_0^i (\text{lin } a)\). Hence \(\tilde{L}_m^2\) can be expressed as a semi-direct product of \(L_m^1\) and \(\text{Ker } \beta_1\). For every \(A \in \tilde{L}_m^2\), we have \(a = \beta_1 A \in L_m^1\) and \(A_1 := \lambda(a)^{-1} \circ A \in \text{Ker } \beta_1\). We shall write \(A = (a, A_1)\), which determines a decomposition \(\tilde{L}_m^2 = L_m^1 \times (R^n \otimes R^m^* \otimes R^m^*).\) If \(a = (a_i)\) and \(A_1 = (\delta_i^j, a_{jk})\), then

\[
(8) \quad A = (a_i, 0) \cdot (\delta_i^j, a_{jk}) = (a_i, a_a^i a_{jk}).
\]

For any \(a = (a_i) \in L_m^1\) and \(A = (a_{jk}) \in R^n \otimes R^m^* \otimes R^m^*\), we set

\[
(9) \quad \text{ad}(a)(A) = (a_a^i a^i_a a_{jk}), \nonumber
\]

where \(a_a^i\) means the inverse matrix to \(a^i_a\).

**Lemma 1.** In the decomposition (8) the multiplication in \(\tilde{L}_m^2\) is expressed by

\[
(10) \quad (b, B_1) \cdot (a, A_1) = (ba, \text{ad}(a^{-1})B_1 + A_1). \nonumber
\]

**Proof.** According to (4), \((b_i^l, b^l b_{jk}) \cdot (a_i, a_{jk}) = (b_i^l a_i^k, b^l b_{mn} a^m_a^k + b_m a^{mn} a_{jk}) = (b_i^l a_i^k, b^l a^k (a^m_a^k a_{jk} + a_{jk}))\), QED.

Let \(G\) be any Lie group, whose multiplication will be denoted by a dot. Then the space \(T^1_m G\) of all 1-jets of \(R^n\) into \(G\) with source 0 is also a Lie group with the composition law

\[
(11) \quad (j_0^i \varphi(u)) \cdot (j_0^j \psi(u)) = j_0^i (\varphi(u) \cdot \psi(u)). \nonumber
\]

The target projection \(\beta: T^1_m G \to G\) is a group homomorphism. Let \(\mathfrak{g}\) be the Lie algebra of \(G\).

**Lemma 2.** We have an exact sequence of groups

\[
(12) \quad 0 \to \mathfrak{g} \otimes R^m^* \to T^1_m G \xto{\beta} G \to 0. \nonumber
\]

**Proof.** The kernel of \(\beta\) is the set of all 1-jets of \(R^n\) into \(G\) with source 0 and the target at the unit of \(G\). As a set, this is equal to \(\text{Hom}(R^n, \mathfrak{g}) = \mathfrak{g} \otimes R^m^*\). Using the
basic facts on the Lie groups, one finds easily that the group composition in $\mathfrak{g} \otimes \mathbb{R}^{m*}$ coincides with the vector addition, QED.

In particular, if $G = L_m$ and the additional coordinates on $T_m^1 L_m$ are $a'_i = (\partial a_i / \partial u^k)(0)$, then the multiplication in $T_m^1 L_m$ is given by

$$ (b'_i, b'_k) \cdot (a'_i, a'_k) = (b'_k a'_i, b'_k a'_i + b'_k a'_k). $$

On the other hand, we introduce a map $\nu : T_m^1 L_m \rightarrow \tilde{L}_m^2$ as follows. Having $A \in T_m^1 L_m$, $A = j_0^1 \gamma(u)$, $\gamma(0) = j_0^1 \psi(u)$, we set

$$ \nu(A) = j_0^1 [j_0^1 (t^1_\psi(u)) \cdot \gamma(u) \cdot j_0^1 (t^{-1}_\psi(u))]. $$

If $A$ has some coordinates $a'_i, a'_k$ in $T_m^1 L_m$, then $\nu(A)$ has the same coordinates in $\tilde{L}_m^2$. Comparing (4) and (13), we find that $\nu$ is not a group homomorphism. Nevertheless, (14) and (2) imply

**Lemma 3.** If $G$ is a subgroup in $L_m^1$, then $\nu(T_m^1 G)$ is a subgroup in $\tilde{L}_m^2$.

The latter group will be denoted by $\tilde{G}$ and called the semi-holonomic prolongation of $G$.

**Lemma 4.** We have an exact sequence of groups

$$ 0 \rightarrow \mathfrak{g} \otimes \mathbb{R}^{m*} \rightarrow \tilde{G} \xrightarrow{\beta} G \rightarrow 0 $$

**Proof.** By (4) and (13) the composition laws in $T_m^1 L_m$ and $\tilde{L}_m^2$ coincide on $\text{Ker} \beta$ and $\text{Ker} \beta_1$, so that (15) is a consequence of (12).

Since the coordinates in $T_m^1 L_m$ coincide with those in $\tilde{L}_m^2$, there holds $\lambda(G) \subset \tilde{G}$.

Then Lemma 1 gives

**Proposition 2.** We have $\tilde{G} = G \times (\mathfrak{g} \otimes \mathbb{R}^{m*})$ with composition law (10).

The intersection $G' := \tilde{G} \cap L_m^2$ will be called the (holonomic) prolongation of $G$. Obviously, $(a'_i, 0) \in G'$ for each $a'_i \in G$, so that $\beta_1 : G' \rightarrow G$ is surjective. Let $p(\mathfrak{g}) = (\mathfrak{g} \otimes \mathbb{R}^{m*}) \cap (\mathbb{R}^{m} \otimes \mathbb{R}^{m*} \circ \mathbb{R}^{m*})$ be the Spencer prolongation of $\mathfrak{g}$. By Proposition 2 we obtain immediately

**Proposition 3.** We have $G' = G \times p(\mathfrak{g})$ with composition law (10).

### 3. The structure function

Consider a $G$-structure $P \subset H^1 M$. Hence $J^1 P \subset J^1 H^1 M$. If $X \in J^1 P$ and $A \in T_m^1 G$ are as in (5) and (14), then

$$ i(X) \nu(A) = j_0^1 [s(\varphi(\psi(u))) \cdot \gamma(u) \cdot j_0^1 (t^{-1}_\psi(u))]. $$

Conversely, for any other $\tilde{X} \in J^1 P$, $\tilde{X} = j_0^1 \tilde{s}$, $s(x) = j_0^1 \varphi$, there exists exactly one
A \in T^i_m G \text{ satisfying } i(X) \cdot \nu(A) = i(\tilde{X}). \text{ Indeed, the equation } s \in \mu = \tilde{s} \text{ determines a local map of } M \text{ into } G \text{ and } A = j^i_0 \psi(\tilde{u}(u)). \text{ Thus, we have proved}

**Proposition 4.** \(i(J^1 P)\) is a reduction of \(H^2 M\) to \(G \subset L^2_m\).

For every \(B \in H^2 M\) we can construct its difference tensor \(\Delta(B) \in T_m M \otimes \wedge^2 R^{m*}\), \(x = \beta B\), further, \(B = \beta, b \in H^2 M\) can be interpreted as a linear map \(b : R^m \to T_m M\).

Then \(\Delta(B) = b^{-1} \Delta(B) \in R^m \otimes \wedge^2 R^{m*}\). If in coordinates \(B = (x^i, b^i, b^k)\), then \(\Delta(B) = \tilde{b} i b_{[ijk]}\).

**Definition 1.** The structure function \(\nu(b)\) of a G-structure \(P\) at \(b \in P\) is the set \(\Lambda(i(X))\) for all \(X \in J^1 P, \beta X = b\).

Since \(g \subset R^m \otimes R^{m*}\), there is \(g \otimes R^{m*} \subset R^m \otimes R^{m*} \otimes R^{m*}\) and \(\psi(g \otimes R^{m*}) \subset R^m \otimes \wedge^2 R^{m*}\), where \(\psi\) means the antisymetrization with respect to \(R^{m*} \otimes R^{m*}\). The space \(H^{0,2}(g) = R^m \otimes \wedge^2 R^{m*} / \psi(g \otimes R^{m*})\) is the Spencer cohomology class of bidegree \((0,2)\) of \(g\).

**Proposition 5.** \(\nu(b)\) belongs to \(H^{0,2}(g)\) for every \(b \in P\).

**Proof.** By Proposition 4 any other \(i(Y) \in i(J^1 P)\), \(\beta Y = b\), is of the form \(i(X) \circ A, A \in g \otimes R^{m*}\). If \(A = (\delta_i, a_k)\) and \(i(X) = (x^i, u^i, u^k)\), then \(i(X) \circ A = (x^i, u^i, u^k + u^j a_k)\) and \(\Lambda(i(X) \circ A) = \tilde{u}^i u^i_{[jk]} + a^i_{[jk]}\). QED.

In coordinates, one verifies easily that our structure function coincides with the classical one, see, e.g., [5]. We remark that our method leads to a simple derivation of the classical transformation law of the structure function. The space \(\psi(g \otimes R^{m*})\) being invariant with respect to the action (9) of \(G\), [5], we have an induced action \(\varphi\) of \(G\) on the factor space \(H^{0,2}(g)\).

**Proposition 6.** There holds \(\varphi(g^{-1}) \nu(b) = \nu(bg)\) for all \(g \in G\) and \(b \in P\).

**Proof.** By (6), if \((u^i, u^j, u^k)\) are coordinates of \(i(X)\), then the coordinates of \(X\) are \((u^i, u^j, u^k a^k)\). Take an element \(a^j \in G\) and construct the image \(X'\) of \(X\) by the right translation determined by \(a^j\). Then the coordinates of \(X'\) are \((u^i, u^i a^k, u^k a^l a^m)\) and the second order coordinates of \(i(X')\) are \(u^i_{[lm]} a^k a^m\). Hence \(\Lambda(i(X')) = a^l a^m_{[lm]} a^k a^m\), which proves our assertion.

### 4. Prolongability and flatness

**Definition 2.** A G-structure \(P\) is called prolongable if the intersection of \(i(J^1 P)\) and \(H^2 M\) is non-empty over every \(b \in P\).

If \(P\) is prolongable, then the intersection \(P' := i(J^1 P) \cap H^2 M\) is said to be the (holonomic) prolongation of \(P\).

**Proposition 7.** If \(P\) is prolongable, then \(P'\) is a reduction of \(H^2 M\) to \(G' \subset L^2_m\).

**Proof.** This follows from Proposition 4 and from the fact that the composition of two holonomic 2-jets is holonomic.
Proposition 8. A G-structure $P$ is prolongable if and only if its structure function vanishes.

Proof. By definition, $P$ is prolongable if and only if for every $b \in P$ there exists an $X \in \mathfrak{J}^1 P$, $\beta X = b$, such that $i(X) \in H^2 M$. This is equivalent to $\triangle(i(X)) = 0$, which is the same as $\tau(b) = 0 \in H^{0,2}(g)$.

We recall that a G-structure on $M$ is said to be flat if it is locally isomorphic to the standard flat G-structure $R^m \times G \subset H^1 R^m$. The well-known fact that the structure function of a flat G-structure vanishes can be rederived as follows. If we take a constant section $s: u' \mapsto (u', a'_i)$ of $R^m \times G$, we have $j^s = (x^i, a'_i, 0)$ and $i(j^s) \in H^2 M$. This implies

Proposition 9. Every flat G-structure is prolongable.

The converse assertion is not true in general. This clarifies in a conceptual way the relation between the vanishing of the structure function and the flatness of a G-structure.

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Received March 31, 1983