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STRONG SHIFT EQUIVALENCE IN SEMIGROUPS

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(*Communicated by Tibor Katriňák*)

ABSTRACT. We show the property that shift equivalence equals strong shift equivalence in a semigroup (the Williams conjecture) is related to regularity.

1. Introduction

The purpose of this article is to describe a concept, strong shift equivalence, which has connections both to symbolic dynamics and to the theory of semigroups (it can also be generalized to categories).

DEFINITION 1. In a semigroup *strong shift equivalence* is the transitive closure of the binary relation rs is equivalent to sr.

Shift equivalence of a, b is the relation for some r, s such that $ra = br$, $as = sb$, $sr = a^n$, $rs = b^n$ for some $n \in \mathbb{Z}^+$.

Example 1. In any group, both strong shift equivalence and shift equivalence amount to conjugacy.

One checks that shift equivalence is an equivalence relation and that it is implied by strong shift equivalence. These concepts first appeared in the paper [W] of R. F. Williams, which classified certain dynamical systems up to conjugacy.

Dynamical systems are concerned with the evolution of isolated systems over time, and symbolic dynamics specializes to dynamical systems represented as sequences from a finite set of states.

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DEFINITION 2. A *dynamical system* is a topological space X and a continuous mapping $f: X \to X$.

The *full n-shift* is the set $\underline{n}^{\mathbb{Z}}$ of biinfinite sequences from $1, 2, \dots, n$ indexed on $\mathbb{Z}$, where $\underline{n} = \{1, 2, \dots, n\}$. It is topologized as a product of discrete topologies. The continuous mapping is given by shifting coordinates by 1.

For a $(0,1)$-matrix A, the *subshift* of finite type associated with A is the subsystem of all sequences a_i such that the (a_i, a_{i+1})-entry of A is 1 for all i (it is identified with biinfinite walks in the graph of A).

For a nonnegative matrix A, we associate a subshift by taking the edge graph of its multigraph.

Symbolic dynamics is the theory of subsystems of full shifts as dynamical systems.

Example 2. The shift associated with

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

is the set of all biinfinite sequences of $(1,2)$ such that no subsequence $(2,2)$ occurs (since the $(2,2)$-entry is 0).

The problem of classifying subshifts of finite type was reduced to an algebraic (matrix) problem by the following result.

THEOREM 0. ([W]) *Two subshifts of finite type are conjugate if and only if the corresponding matrices are strong shift equivalent over $\mathbb{Z}^+$ (through rectangular matrices).*

So in symbolic dynamics, the semigroup for which strong shift equivalence is mainly to be described is the union of $M_n(\mathbb{Z}^+)$, where $M_n(\mathbb{Z}^+)$ is included in $M_{n+1}(\mathbb{Z}^+)$ by direct sum with 0. However, more general classes of subshifts (in particular, sofic subshifts, images of one subshift of finite type in another) involve strong shift equivalence over other semigroups [KR1].

DEFINITION 3. The *Williams conjecture* for a semigroup is the assertion that shift equivalence equals strong shift equivalence.

Example 3. For groups, this always holds.

For matrices over $\mathbb{Z}^+$, this is probably the most important unsolved problem in symbolic dynamics in the restricted case when the matrices involved are primitive, that is, some powers of them are positive. In [KR3], it was proved false without this restriction.

2. Results for semigroups

For simplicity, we will usually deal with finite semigroups.

DEFINITION 4. In a semigroup S, the $\mathcal{L}$-($\mathcal{R}$-) *class* of an element x is the set of all elements generating the same left (right) ideal as x. Elements in the same $\mathcal{L}$-($\mathcal{R}$-) class are said to be $\mathcal{L}$-($\mathcal{R}$-) *equivalent*. The relation $\mathcal{H}$ is $\mathcal{L} \cap \mathcal{R}$ and $\mathcal{D}$ is the composition $\mathcal{L} \circ \mathcal{R}$ (which is proved equal to $\mathcal{R} \circ \mathcal{L}$).

DEFINITION 5. An element x of a semigroup S is *regular* if and only if $xyx = x$ for some y in S. A semigroup is regular if and only if all its elements are regular.

E x a m p l e 4. The semigroup of transformations on $\underline{n}$ is regular but the semigroup of binary relations on it is not regular for $n \geq 3$.

If a $\mathcal{D}$-class contains one regular element, then all are regular [CP; Theorem 2.11], and every $\mathcal{L}$ and $\mathcal{R}$-class contains an idempotent. An $\mathcal{H}$-class is a group if and only if it contains an idempotent [CP; Theorem 2.16].

DEFINITION 6. Two elements x, y in a semigroup S are (*Vagner-Thierrin*) *inverse* if and only if $xyx = x$ and $yxy = y$.

DEFINITION 7. In a semigroup S, elements a, b are *semiconjugate* if they belong to the same regular $\mathcal{D}$-class and within that $\mathcal{D}$-class there are elements x, y such that $xy = a$, $yx = b$, where $x \in \mathcal{R}_a$, $x \in \mathcal{L}_b$ and $y \in \mathcal{L}_a$, $y \in \mathcal{R}_b$.

PROPOSITION 1. *Let D be a regular $\mathcal{D}$-class. Two elements of D are semiconjugate if and only if*

(1) *they lie in $\mathcal{H}$-classes which are groups;*
(2) *under arbitrary standard isomorphisms of these groups they map to conjugate elements.*

Semiconjugacy is an equivalence relation. Semiconjugacy implies strong shift equivalence.

P r o o f. Let a, b be semiconjugate by s, r. Let $H_x(L_x, R_x)$ be the $\mathcal{H}$-($\mathcal{L}$-, $\mathcal{R}$-) class containing x. According to [CP, Theorem 2.17] $rH_s = H_{rs}$, $H_rH_s = H_b = R_r \cap L_s$, the symmetrical relation holds, and the $\mathcal{H}$-classes of a, b are groups.

The standard isomorphisms between these two groups are given by $f(x) = rxs_1$ and $g(y) = s_1yr_1$ for any s_1 in the $\mathcal{H}$-class of s [CP; Theorem 2.20], where r_1 is the unique inverse in the $\mathcal{H}$-class of r [CP; Theorem 2.18].

If e_a is the idempotent in the $\mathcal{H}$-class of a, then $r_1 e_a = r_1$ by [CP: Lemma 2.14], and symmetrically for e_b and s_1. Then $r_1 e_a$ lies in R_r, L_{e_a}, so by [CP], $r_1 H_{e_a} = r_1 H_a = H_r$ and symmetrically.

Therefore, if we choose a different element in H_r, then we can write that element as $r_1 a_0$ for some a_0 in H_a. Its inverse in the $\mathcal{H}$-class of s_1 by computation is $a_0^{-1} s_1$. Hence, the new standard isomorphism differs from the first one by conjugation by a_0.

We can choose a standard isomorphism mapping a to b by using r and $s_1 = a^{-1}s$; these lie in the correct $\mathcal{H}$-classes. Since $s_1 r = e_a$, s_1 must be the unique inverse of r in its $\mathcal{H}$-class. Also $raa^{-1}s = rs = b$.

Conversely, let a, b satisfy the conditions (1) and (2) above. We may assume a maps for a suitable r, s_1 onto b and conversely. Let $s = as_1$. Then $sr = a$, $rs = b$. The definition of the standard isomorphisms implies s_1 is in R_a, L_b, and r is in R_b, L_a. Then s is in the same $\mathcal{H}$-class as s_1 by [CP; Theorem 2.18]. It follows that a, b are semiconjugate. The last two statements follow from the first two. □

The next result generalizes a result we stated previously in the case of Boolean matrices [KR2].

PROPOSITION 2. *Let S be a semigroup in which every element has some power which lies in an $\mathcal{H}$-class that is a group (e.g., a finite semigroup).*

Then any element a is shift equivalent to an element ae lying in a group with idempotent e. Two elements a, b of any semigroup S which lie in $\mathcal{H}$-classes which are groups with idempotents e, f are shift equivalent if and only if a and b are semiconjugate.

Proof. Let a^n lie in an $\mathcal{H}$-class H_0 which is a group with idempotent e. Then all powers of a^n lie in H_0. It follows that all powers a^k, k greater than or equal to n, are $\mathcal{L}$-equivalent and $\mathcal{R}$-equivalent using powers of a, and therefore also lie in H_0. Then $r = s = a^n$ gives a shift equivalence from a to ae since a, e commute with a^n.

So a, b are shift equivalent if and only if ae and bf are shift equivalent. A semiconjugacy gives a strong shift equivalence, and therefore a shift equivalence. Conversely, let r, s give a shift equivalence. We may multiply r, s by any power of ae, bf on appropriate sides to get a new shift equivalence (involving higher powers of ae, bf).

Then we may assume that

(*) ae divides r on the right and divides s on the left, and bf divides s on the right and r on the left.

Then the relations that $rs = (ae)^m$, $sr = (bf)^m$ together with $(*)$ imply that r and ae generate the same right ideal (so are $\mathcal{R}$-equivalent) and r and bf are $\mathcal{L}$-equivalent and s and bf are $\mathcal{R}$-equivalent and s and ae are $\mathcal{R}$-equivalent. So all these elements lie in a $\mathcal{D}$-class which is regular since it has $\mathcal{H}$-classes which are groups. Then $(ae)^{1-m}r$ and s give a semiconjugacy from ae to bf. □

DEFINITION 8. An element x of a semigroup S is *prime* if and only if whenever $x = yz$, either y or z has a 2-sided inverse u, e.g., $uy = yu = e$, where e is a 2-sided identity.

PROPOSITION 3. *In any semigroup having a prime irregular element and in which some power of every element lies in an $\mathcal{H}$-class which is a group, the Williams conjecture is false.*

Proof. Such an element a is not strong shift equivalent to anything but conjugates of itself by invertible elements since its conjugates only factor in terms of invertibles.

But it is shift equivalent to some power of itself which lies in an $\mathcal{H}$-class containing an idempotent, and is therefore regular, and cannot be a conjugate of the original. □

DEFINITION 9. A semigroup satisfies the order condition for *right* (*left*) *ideals* if given any right (left) ideals $\mathcal{I}$, $\mathcal{J}$ and element x, it is not possible that both $\mathcal{I}$ is a proper subset of $\mathcal{J}$ and $x\mathcal{I} = \mathcal{J}$ ($\mathcal{I}x = \mathcal{J}$).

Example 5. Any finite semigroup satisfies this, as does the multiplicative semigroup of a finite dimensional algebra.

LEMMA 4. *Under the order condition for left ideals, if $a\mathcal{R}a^2$, then they are $\mathcal{H}$-equivalent, and their $\mathcal{D}$-class is a group.*

Proof. Let $a^2u = a$, so a, a^2 have the same right ideals. The left ideal of a^2 is a subset of the left ideal of a. If we have equality, then $a^2\mathcal{H}a$, and we are done by [CP].

If not, then the left ideal Sa^2 is a proper subset of Sa and $(Sa^2)u = Sa$, which contradicts the order condition. □

PROPOSITION 5. *In any regular semigroup having the order condition for left ideals and the descending chain condition for right ideals, the Williams conjecture is true.*

Proof. A semiconjugacy is a strong shift equivalence by Proposition 1. By Proposition 2, it will suffice to show that any element a is strong shift equivalent to an element lying in an $\mathcal{H}$-class which is a group since for the latter shift equivalence implies semiconjugacy. If a, a^2 generated the same right ideals, they would be in the same $\mathcal{H}$-class by Lemma 4, and this would be a group.

Suppose the right ideals are different. Let a' be an inverse of a. Write $a = ea$, where $e = aa'$ is an idempotent in R_a. Then ae will be an element strong shift equivalent to a which lies in the $\mathcal{R}$-class of a^2. Its right ideal is the right ideal of a^2, which is a proper subset of the previous right ideal. So we get a to be strong shift equivalent to an element with smaller right ideal. By the descending chain condition, this process must terminate, and then a, a^2 lie in the same $\mathcal{R}$-class. □

DEFINITION 10. A *lifted shift equivalence invariant* for a semigroup S consists of an epimorphism S_1 to S and an equivalence relation E on S_1 which (1) contains shift equivalence on S_1, (2) contains the congruence induced by the epimorphism.

THEOREM 6. *Every lifted shift equivalence invariant gives an invariant of strong shift equivalence on a semigroup S and the set of all lifted shift equivalence invariants describe strong shift equivalence in S uniquely. The Williams conjecture holds for free semigroups.*

Proof. Let v be a lifted shift equivalence invariant, for homomorphism $h\colon S_1 \to S$. In S, let a, b be one step strong shift equivalent. Then $a = rs$, $b = sr$. Then S_1 choose r_1, s_1 mapping to r, s let $a_1 = r_1 s_1$, $b_1 = s_1 r_1$. Then in v, a_1 is equivalent to b_1, hence a is equivalent to b in the induced relation. By transitivity, if a, b are strong shift equivalent, then they are equivalent in the induced relation.

Take S_1 to be a free semigroup with generators corresponding to the elements of S. We then form the universal lifted shift equivalence invariant, by taking shift equivalence in S_1. In the free semigroup, $rs = a^n$, $sr = b^n$ for any r, s, a, b and positive integer n imply that a^n and b^n are cyclic rearrangements of one another (r, s must be subwords of them). This implies a, b are cyclic rearrangements of one another.

So in S_1, the Williams conjecture is true. So the smallest equivalence relation generated by shift equivalence in S_1 and the congruence for the mapping into S is contained in strong shift equivalences in S. □

Example 6. For Boolean matrices, the Boolean trace is a strong shift equivalence invariant which is not a shift equivalence invariant.

It is obtained in this way from the homomorphism from matrices over $\mathbb{Z}^+$ into Boolean matrices, using the equivalence class of the ordinary trace on $\mathbb{Z}^+$ under the mapping.

3. Conclusion

Semigroup theoretic properties are quite relevant to the question of truth of the Williams conjecture. However, it is known to be true for the semigroup which is the union of $M_n(\mathbb{Z})$, more generally for the union on $M_n(A)$, where A is an algebraic number ring [BH], which has a more complex type than the above. It is known to be false over $\mathbb{Z}^+$ without the primitivity restriction [KR3]. It is also false for the semiring of Boolean matrices but is true for Boolean matrices of trace 1 [KR2].

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