Pavol Híc

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A CONSTRUCTION OF GEODETIC BLOCKS

PAVOL HÍC

1. Introduction

Geodetic graphs were first defined by Ore [5] as graphs in which every pair of vertices is connected by a unique shortest path. Since a graph is geodetic iff each of its blocks is geodetic (see Stemple and Watkins [13]), it is sufficient to study the geodetic blocks only. The geodetic blocks of diameter two have been studied by Lee [4], Stemple [11], Zelinka [14]. For geodetic blocks of higher diameters there are available some general constructions only (see for example [1, 2, 6, 7, 8, 9, 12, 15]).

In this paper we present one construction of new geodetic blocks \( \tilde{G} \) or \( \tilde{G}(s) \) from a known geodetic block \( G \) if a geodetic block \( G \) can be decomposed into two edge-disjoint geodetic subgraphs \( G_1, G_2 \) with two special properties. This construction consists of replacing certain vertices by new edges. The construction unifies the construction described by Bosák [1] (\( g(m, s) \)-graphs), Plesník [8] (\( WPd \)-graphs), and Stemple [12] (\( K^s \)-graphs). We used this construction to study a special class of geodetic block which are homeomorphic to \( \mathcal{G}(p + 1, 2p) \)-graphs (see [4]), namely to study \( G(p, 2 + s) \)-graphs.

2. Definitions and preliminary results

We use the general notation and terminology of Harary [3]. The graphs considered are simple undirected graphs. If \( G \) is a graph, then \( V(G) \) and \( E(G) \) denote its vertex set and edge set, respectively. The distance between vertices \( u, v \in V(G) \) is denoted by \( d_{G}(u, v) \). A shortest \( u - v \) path in \( G \) is called a geodesic and it is denoted by \( \Gamma_{G}(u, v) \). Any subpath of a geodesic is also a geodesic. If \( S \) is a path, \( |S| \) will mean the length of \( S \). Clearly, if \( \Gamma_{G}(u, v) \) exists, then \( d_{G}(u, v) = |\Gamma_{G}(u, v)| \). The supremum of all distances in \( G \) is the diameter of \( G \), \( d(G) \). If \( v \in V(G) \), then we put \( V_{G}(v) = \{ u \in V(G) | d_{G}(u, v) = i \} \).

A clique is defined as a maximal complete subgraph \( U_{k} \) of order \( k \geq 3 \), that is, a complete subgraph on at least three vertices which is contained in no larger complete subgraph.
Theorem A (see Stemple [11, Theorem 5.5]). If $G$ is a geodetic block of
diameter two and $U_k, U_j$ are cliques of $G$, then $k = j$.

Now, if $G$ is a geodetic block of diameter two and $G$ contains a clique $U_k$, we call
$k$ the clique size of $G$. If $G$ contains no clique, we let $k = 2$ be the clique size.

Theorem B (see Stemple [11, Theorem 5.11]). Let $G$ have clique size $k \geq 3$
and assume that $G$ contains a clique $H$ with the property that for each vertex
$v_i \in V(H)$, $i = 1, 2, \ldots, k$; there exists a clique $H_i$, where

$$V(H) \cap V(H_i) = \{v_i\} \text{ and } V_{G_i}(v_i) \subseteq V(H) \cup V(H_i).$$

Then $G_1 = G - H$ is geodetic of diameter two with clique size $k - 1$. If $G_1$ contains
cliques (i.e., $k \geq 4$), then each clique in $G_1$ is at distance two from every other
clique.

3. The construction of $\hat{G}$ and $\hat{G}(s)$

By a decomposition of a graph $G$ we mean a set of edge-disjoint subgraphs
$G_1, G_2, \ldots, G_n$ of $G$ which together contain the set of edges of $G$; it is denoted by
$(G_1, G_2, \ldots, G_n)$.

Let $G$ be a geodetic block and $G_1, G_2$ be its geodetic subgraphs (not necessarily
blocks) which form a decomposition $(G_1, G_2)$ of $G$. Then $(G_1, G_2)$ is said to be
a $g$-decomposition of $G$.

We shall say that a $g$-decomposition $(G_1, G_2)$ of a geodetic block $G$ has the
property $P(1)$ if for any two vertices $u, v \in V(G_1) \cup V(G_2)$ every $u - v$
geodesic of $G$ belongs to $G_1$ [to $G_2$] with the exception of $u, v \in V(G_1) \cap V(G_2)$
where either $\Gamma_{G_1}[u, v] = \Gamma_{G_2}[u, v]$ or $\Gamma_{G_1}[u, v] = \Gamma_{G_2}[u, v]$. In other words, $G_1$ and
$G_2$ are geodetically closed in $G$ with the exception of vertices of $V(G_1) \cap V(G_2)$.

Further, we say that $(G_1, G_2)$ has the property $P(2)$ if for any two vertices $u, v \in V(G_1) \cap V(G_2)$ we have

$$|\Gamma_{G_1}[u, v]| - |\Gamma_{G_2}[u, v]| \equiv 1 \pmod{2}.$$

A $g$-decomposition with the properties $P(1)$ and $P(2)$ is called a
$\tilde{g}$-decomposition.

Let $(G_1, G_2)$ be a $\tilde{g}$-decomposition of a geodetic block $G$. From $G$ we construct
a graph $\tilde{G}$ as follows. Let $v$ be any vertex from $V(G_1) \cap V(G_2)$. Then we replace $v$
by two vertices $v^1, v^2$ and join $v^1$ with $v^2$ by an edge. Further we join $v^1$ (or $v^2$)
with each vertex of $V_{G_1}(v)$ ($V_{G_2}(v)$, respectively). We shall denote this construc-
tion by $G \rightarrow \tilde{G}$ and we claim that $\tilde{G}$ is geodetic.

In Fig. 1 we have illustrated the construction of $\tilde{G}$ by taking $K_5$ as $G$ and two
cycles $C_3$ as $G_1, G_2$ ($G_1 = [u, v, x, y, z, u], G_2 = [u, x, z, v, y, u]$). It is obvious that
$(G_1, G_2)$ is a $\tilde{g}$-decomposition of $K_5$ and $\tilde{G}$ is the Petersen graph.
Lemma 1. Let \( x \in V(G_1), y \in V(G_2) \) and \( x, y \notin V(G_1) \cap V(G_2) \). Let \((G_1, G_2)\) be a \( \tilde{g} \)-decomposition of a geodetic block \( G \). Then there exists exactly one vertex \( v \in V(G_1) \cap V(G_2) \) with 

\[
\Gamma_G[x, y] = \Gamma_G[x, v] + \Gamma_G[v, y]
\]

and

\[
\Gamma_G[x, v] \subseteq G_1, \quad \Gamma_G[v, y] \subseteq G_2.
\]

**Proof.** Let \( \Gamma_G[x, y] = [x = v_0, e_1, v_1, \ldots, e_n, v_n = y] \). Let \( v_i, [v_i] \) be the first [last] vertex of \( \Gamma_G[x, y] \) which is in \( V(G_i) \cap V(G_2) \), too. From the property \( P(1) \) we have:

\[
\Gamma_G[v_i, v_j] = \Gamma_{G_l}[v_i, v_j] \subseteq G_1 \quad \text{or} \quad \Gamma_G[v_i, v_j] = \Gamma_{G_2}[v_i, v_j] \subseteq G_2.
\]

Let \( \Gamma_G[v_i, v_j] = \Gamma_{G_l}[v_i, v_j] \); then \( \Gamma_G[x, v_i] + \Gamma_G[v_i, v_j] = \Gamma_G[x, v_j] \subseteq G_1 \) and \( v_i \) is the desired vertex. If \( \Gamma_G[v_i, v_j] = \Gamma_{G_2}[v_i, v_j] \subseteq G_2 \), then we proceed similarly.

Q.E.D.

Corollary 1. Let \((G_1, G_2)\) be a \( \tilde{g} \)-decomposition of a geodetic block \( G \). Let \( x \in V(G_1), y \in V(G_2), x, y \notin V(G_1) \cap V(G_2) \). Let \( \Gamma_G[y, y] \) be a geodesic from \( x \) to \( y \) in the graph \( \tilde{G} \). Then there exists exactly one edge \([v^1, v^2] \subseteq \Gamma_G[x, y]\) with \([v^1, v^2] \notin E(G)\). ([\(v^1, v^2\)] is a new edge corresponding to a vertex \( v \)).

Theorem 1. If \( G \) is a geodetic block, then \( \tilde{G} \) is a geodetic block, too.

**Proof.** It is sufficient to prove that for any two distinct vertices \( u, v \) of \( \tilde{G} \) there exists exactly one geodesic between them. Suppose, on the contrary, that there are two distinct geodesics \( \tilde{F}_1, \tilde{F}_2 \) between \( u \) and \( v \). We can suppose that \( \tilde{F}_1 \) and \( \tilde{F}_2 \) are internally disjoint (otherwise there are internally disjoint subpaths \( P_1 \) of \( \tilde{F}_1 \) and \( P_2 \) of \( \tilde{F}_2 \).
of $\tilde{G}_2$ and we can take $P_1$ and $P_2$ for $\tilde{G}_1$ and $\tilde{G}_2$, respectively). We shall consider the following cases:

Case 1(a). Both $u$ and $v$ belong to $V(G_1) - V(G_2)$. There cannot be two distinct shortest $u - v$ paths, because of the property $P(1)$.

Case 1(b). Both $u$ and $v$ belong to $V(G_2) - V(G_1)$. Then there cannot be two distinct shortest $u - v$ paths, because of the property $P(1)$.

Case 2. One of the vertices $u$ and $v$, say $v$, belongs to $V(G_2) - V(G_1)$. Then there cannot be two distinct shortest $u - v$ paths, because of the property $P(1)$. From the property $P(1)$ it follows that there cannot be two distinct shortest paths between $u$ and $w$. Hence, there cannot be two distinct $u - w$ geodesics.

Case 3. One of the vertices $u$ and $v$, say $v$, belongs to $V(G_1)$ and the other $u$ to $V(G_2)$, $u, v \notin V(G_1) \cap V(G_2)$. By Corollary 1 both $\tilde{G}_1$ and $\tilde{G}_2$ contain exactly one new edge $e_1$ and $e_2$, respectively. From $|\tilde{G}_1| = |\tilde{G}_2|$ it follows for corresponding $u - v$ geodesics $\tilde{G}_1$ and $\tilde{G}_2$ in the graph $G$ that $|\tilde{G}_1| = |\tilde{G}_1| - 1 = |\tilde{G}_2| - 1 = |\tilde{G}_2|$. But this is not possible because $G$ is geodetic.

Case 4. Let $u = u$, $i \in \{1, 2\}$; $v = v$, $j \in \{1, 2\}$. Then by the property $P(2)$ $u^i$, $v^i$, $v^j$, $v^2$ belong to an odd cycle $C = \Gamma_0[u, v] + [v^i, v^j] + \Gamma_0[u, v] + [u^i, u^j]$. Hence the $u^i - v^j$ geodesic is the shorter part of $C$. It is obvious that there cannot be two distinct geodesics.

Q.E.D.

Note 1. The property $P(1)$ cannot be omitted (see Fig. 2). There $G$ is a $K_3$-graph (see [12]) and $G_1, G_2$ are odd cycles which are presented differently. Both vertices $x$ and $y$ belong to $V(G_1)$ but the $x - y$ geodesic does not belong to $G_1$. Then there are two shortest paths between $x$ and $y$ in $\tilde{G}$.

Note 2. The property $P(2)$ cannot be omitted (see Fig. 3). There $G$ is $K_4$ and $|\Gamma_0[v, w]| = 1, |\Gamma_0[v, w]| = 3$. $\tilde{G}$ is not geodetic because there are two distinct shortest paths from $v^2$ to $w^2$. 

Fig. 2

254
Note 3. If $G$ is of diameter $d$, then the diameter of $\tilde{G}$ is $d + 1$ if there exists a pair of vertices $x, y \in V(G_1) \cap V(G_2)$, $x \in V(G_1)$, $y \in V(G_2)$ with $\rho_G(x, y) = d$; otherwise the diameter of $\tilde{G}$ is $d$.

Problem. It would be interesting to find a geodetic block $G$ of diameter $d$ for which $d(\tilde{G}) = d$.

Note 4. The construction cannot be extended for a $\tilde{g}$-decomposition with more than two subgraphs. A counterexample is in Fig. 4(c). Subgraphs $G_1$, $G_2$, $G_3$ are presented differently. There are two distinct shortest paths from $a$ to $u^2$.

Now, let $(G_1, G_2)$ be a $\tilde{g}$-decomposition of a geodetic block $G$. From $G$ we shall construct a graph $\tilde{G}(s)[G \to \tilde{G}(s)]$ which is a generalization of the graph $\tilde{G}$ described above and is obtained as follows: Every vertex $v \in V(G_1) \cap V(G_2)$ is replaced by a path of length $s$, that is $v \to P[v^1, \ldots, v^{s+1}]$ and $v^1$ [or $v^{s+1}$] is joined with each vertex of $V_{G_i}(v)$ [$V_{G_2}(v)$, respectively]. For an illustration, we have a graph $G$ and its graph $\tilde{G}(s)$ in Fig. 4(a) and 4(b), respectively.

Theorem 2. If $G$ is a geodetic block, then $\tilde{G}(s)$ is a geodetic block, too.

Proof. The proof is similar to that for $\tilde{G}$. Let $u, v$ be two distinct vertices of $\tilde{G}(s)$. We shall show that there is exactly one shortest $u-v$ path in $\tilde{G}(s)$. We shall consider the following cases:

Case 1. Both $u$ and $v$ belong to $V(G_1) - V(G_2)$ [or $V(G_2) - V(G_1)$]. Then the assertion is obvious.

Case 2. One of the vertices $u$ and $v$, say $v$, belongs to $V(G_1)$ [$V(G_2)$] and the other $u = w^i$, $i \in \{1, 2, \ldots, s + 1\}$. From the property $P(1)$ it follows that there cannot be two distinct shortest paths between $u$ and $w$ in $G$. Hence, there cannot be two distinct $u-w^i$ geodesics.

Case 3. One of the vertices $u$ and $v$, say $v$, belongs to $V(G_1)$ and the other $u$ to $V(G_2)$; $u, v \in V(G_1) \cap V(G_2)$. From Lemma 1 it follows that there is exactly one new path $P[w^1, w^2, \ldots, w^{s+1}]$ which lies on the $u-v$ geodesic in $\tilde{G}(s)$. Then the existence of two distinct $u-v$ geodesics in $\tilde{G}(s)$ results in the existence of two distinct $u-v$ geodesics in $G$. 

255
Case 4. Let \( u = u' \), \( i \in \{1, 2, ..., s + 1\} \), \( v = v' \), \( j \in \{1, 2, ..., s + 1\} \). Then by the property \( P(2)u', v' \) belong to the odd cycle
\[
C = \Gamma_{G_1}[u', v'] + P[v', ..., v'^{s+1}] + \Gamma_{G_2}[v'^{s+1}, u'^{s+1}] + P[u'^{s+1}, ..., u'].
\]
Hence the \( u' - v' \) geodesic is the shorter part of \( C \).

Q.E.D.

Note 5. If we take \( G \) to be \( K_s \) and \( G_1, G_2 \) are both \( C_s \), then \( \tilde{G}(s) \) is the graph \( WPd \) of Plesník [8], where \( d = s + 1 \), \( s \geq 1 \).

Note 6. If we take \( G \) to be \( K_{m+1} \) and \( G_1 = K_m, G_2 = K_{1,m}, \) then \( \tilde{G}(s) \) is the graph \( g(m, s) \) of Bosáč [1] and taking \( G_2 \) to be a homeomorph of \( K_{1,m} \) successively over 256
each vertex of $K_{m+1}$, then $G(s(v))$ is the graph $K^+_{m+1}$ of Stemple [12], where $s = s(v)$ is a mapping from $V(K_{m+1})$ to the set of nonnegative integers.

Note 7. If $s \geq 2$, then $G(s)$ must contain vertices of degree two, but if $s = 1$, then there exists a geodetic block $G(1)$ without vertices of degree two. It is the Petersen graph in Fig. 1.

**Question.** Is there a geodetic block $G(1)$ without vertices of degree two different from the Petersen graph?

It is obvious that if such a graph exists, then $G$ is without vertices of degree two and for each $v \in V(G_1) \cap V(G_2)$ both $deg_{G_1} v \geq 2$ and $deg_{G_2} v \geq 2$ are true.

4. An application to geodetic graphs of diameter two

Stemple [11] proved that for any geodetic graph $G$ of diameter two, there exist integers $n$ and $m$ satisfying the properties that $G$ contains exactly $n \cdot m + 1$ vertices, and every vertex in $G$ has degree $n$ or $m$. For fixed $n$ and $m$ denote by $\mathcal{G}(m, n)$ the class of all geodetic graphs of diameter two satisfying the above properties. Lee [4, Theorem 1] used orthogonal Latin squares to construct the class $\mathcal{G}(p + 1, 2p)$ for any prime power $p$, $p \geq 3$, (for $p = 2$ such a graph is given in Fig. 4(a)) as follows:

From $p - 1$ orthogonal Latin squares of order $p$ we first construct a $[p^2 \times (p + 1)]$ array $A = (a_{ij})$ of integers, $1 \leq a_{ij} \leq p$ [10, Theorem 1.3].

A graph $G \in \mathcal{G}(p + 1, 2p)$ can be constructed by the following steps:

(i) take vertex disjoint $(p + 1)$-cliques $H_1, H_2, \ldots, H_{p+1}$, and label the vertices of each $H_r$ as $u_i, u_{i,1}, \ldots, u_{i,p}$ for $r = 1, \ldots, p + 1$;

(ii) join every pair $[u_i, u_{i,j}]$ with an edge for $i \neq j$, in this way we make a new clique $H$;

(iii) take new vertices $v_1, v_2, \ldots, v_{p^2}$, not on any $H$, and join $v_i$ and $u_{i,a}$ with an edge for all $t = 1, \ldots, p^2$ and $i = 1, \ldots, p + 1$.

It can be verified that $G$ is geodetic [4, Theorem 1] of diameter two with the following properties:

I. $G$ has clique size $p + 1$.

II. For every vertex $u_{i,0} \in V(H)$, $i = 1, \ldots, p + 1$, there exists a clique $H_i$ where $V(H) \cap V(H_i) = \{u_{i,0}\}$ and $V_c(u_{i,0}) \subseteq V(H) \cup V(H_i)$.

For example, if $p = 3$, then two orthogonal Latin squares of order 3 and its $[9 \times 4]$ array $A$ are:

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

257
The corresponding graph $G \in \mathcal{G}(4, 6)$ is in Fig. 5(a).

From I, II and Theorem B it follows:

**Lemma 2.** Let $p$ be a prime power, $p \geq 2$, and $G \in \mathcal{G}(p + 1, 2p)$. Then $G_1 = G - H$ is a geodetic graph of diameter two with clique size $p$ and $G_1 \in \mathcal{G}(p + 1, 2p - 1)$.

For $p = 2$, the graph $G$ is in Fig. 4(a). The outer triangle is $H$ and $G - H$ is the Petersen graph. The graph in Fig. 5(a) has $p = 3$ and the complete 4-graph with darkened edges is $H$.

**Lemma 3.** Let $G \in \mathcal{G}(p + 1, 2p)$, $G_1 = G - H$, $G_2$ be a subgraph of $G$ consisting of the subgraph $H$ and the edges $\{u_i, o, u_r\}$ for every $i = 1, \ldots, p + 1$, $r = 1, \ldots, p$. Then $(G_1, G_2)$ is a $\mathcal{G}$-decomposition of $G$.

**Proof.** $G_1$ is geodetic, because of Lemma 2. From the definition of $G_2$ it follows that $G_2$ is a geodetic graph, too. Now, we shall prove the property $P(1)$. If $x, y \in V(G_1)$ and $\varrho_{G_1}(x, y) = 1$, then $\varrho_{G}(x, y) = 1$, too. If $x, y \in V(G_1)$ and $\varrho_{G_1}(x, y) = 2$, then $\varrho_{G}(x, y) = 2$, too, since from $\varrho_{G}(x, y) = 1$ it follows that $\varrho_{G_2}(x, y) = 1$. Then, by the definition of $G_2$, at least one of the vertices $x$ and $y$ belongs to $\{u_{i, 0}, \ldots, u_{p+1, 0}\}$ and this is a contradiction to the assumption $x, y \in V(G_1)$. Therefore, the graph $G_1$ is geodetically closed. If $x, y \in V(G_2)$, $x, y \notin V(G_1)$, then $x, y \in \{u_{i, 0}, \ldots, u_{p+1, 0}\}$ and $\varrho_{G_2}(x, y) = \varrho_{G}(x, y) = 1$. Hence, the property $P(1)$ is proved. Now, we shall prove the property $P(2)$. $V(G_1) \cap V(G_2) = \bigcup_{r=1}^{p+1} \{u_{r, 1}, \ldots, u_{r, p}\}$. Let $x, y \in V(G_1) \cap V(G_2)$. We distinguish two cases:

A. $x = u_{r, i}$, $y = u_{r, i}$, $r = 1, \ldots, p + 1$; $j \neq i$, $i \in \{1, \ldots, p\}$; then $\varrho_{G_1}(x, y) = \varrho_{G_2}(u_{r, i}, u_{r, i}) = 1$ and $\varrho_{G}(x, y) = \varrho_{G_2}(u_{r, i}, u_{r, i}) = 2$.

B. $x = u_{m, i}$, $y = u_{r, j}$, $m \neq s$, $m, s \in \{1, \ldots, p + 1\}$; $i, j \in \{1, \ldots, p\}$; then $\varrho_{G_1}(x, y) = \varrho_{G_1}(u_{m, i}, u_{r, j}) = 2$ and $\varrho_{G_2}(x, y) = \varrho_{G_2}(u_{m, i}, u_{r, j}) = 3$. Hence, the property $P(2)$ is proved.

Q.E.D.

**Theorem 3.** For every prime power $p \geq 2$, every integer $s \geq 1$ and every $G \in \mathcal{G}(p + 1, 2p)$, $G(s)$ is a geodetic graph of diameter $2 + s$. (We shall denote it by $G(p, 2 + s)$.)
\( G \in G(4,6) \)

(a)

\text{Fig. 5a}

\( G(3,2+s) \)

(b)

\text{Fig. 5b}
Proof. The geodeticity of \( G(p, 2 + s) \) follows from Lemma 3 and Theorem 2. Using Lemma 1, we evidently have

\[ \varrho_{G_i}(u_r, v_i) = 2 + s, \]

for \( r = 1, \ldots, p + 1; \ i = 1, \ldots, p^2 \). Therefore it is sufficient to prove

\[ \varrho_{G_i}(x, y) \leq 2 + s \]

for any \( x, y \in V(G(p, 2 + s)) \). This is obvious if \( x, y \in V(G_1) \) or \( x, y \in V(G_2) \). If \( x \in V(G_1) - V(G_2) \) and \( y = y^j, j = 1, \ldots, s + 1 \), i.e. the vertex \( y^j \) lies on a new path \( P[y^1, \ldots, y^{s+1}] \) of length \( s \), then \( \varrho_{G_i}(x, y^j) \leq 2 \) and it follows that

\[ \varrho_{G(p, 2+s)}(x, y^j) \leq \varrho_{G_i}(x, y^j) + s \leq 2 + s. \]

Similarly, if \( x \in V(G_2) - V(G_1) \) and \( y = y^j \), then \( \varrho_{G_i}(x, y^{s+1}) \leq 2 \) and hence

\[ \varrho_{G(p, 2+s)}(x, y^j) \leq \varrho_{G_i}(x, y^{s+1}) + s \leq 2 + s. \]

Finally, if \( x = v^i, i = 1, \ldots, s + 1; y = w^j, j = 1, \ldots, s + 1 \); then there exist vertices \( v, w \in V(G_1) \cap V(G_2) \) and corresponding paths \( P_1 = [v^1, \ldots, v^{s+1}] \) and \( P_2 = [w^1, \ldots, w^{s+1}] \) respectively, with \( v^i \in P_1, w^j \in P_2 \). We distinguish two cases:

A. \( \varrho_{G}(v, w) = \varrho_{G(p, 2+s)}(v^1, w^1) = 1 \) and \( \varrho_{G_i}(v, w) = \varrho_{G(p, 2+s)}(v^{s+1}, w^{s+1}) = 2 \). Then \( x, y \) lie on a cycle

\[ C = \Gamma_{G(p, 2+s)}(v^1, w^1) + P_2 + \Gamma_{G(p, 2+s)}(w^{s+1}, v^{s+1}) + P'_1 \]

(where \( P'_1 \) is the path reverse to \( P_1 \)) of length \( 2s + 3 \). Hence,

\[ \varrho_{G(p, 2+s)}(x, y) \leq \lfloor |C|/2 \rfloor \leq 2 + s. \]

B. \( \varrho_{G_i}(v, w) = \varrho_{G(p, 2+s)}(v^1, w^1) = 2 \) and \( \varrho_{G_i}(v, w) = \varrho_{G(p, 2+s)}(v^{s+1}, w^{s+1}) = 3 \). Then \( x, y \) lie on a cycle

\[ C' = \Gamma_{G(p, 2+s)}(v^1, w^1) + P_2 + \Gamma_{G(p, 2+s)}(w^{s+1}, v^{s+1}) + P'_1 \]

of length \( 2s + 5 \). Hence,

\[ \varrho_{G(p, 2+s)}(x, y) \leq \lfloor |C'|/2 \rfloor = 2 + s. \]

Q.E.D.

For illustration, the graph \( G(3, 2 + s) \) is in Fig. 5(b).

REFERENCES


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Katedra matematiky
Vysokej školy pofnoshospodárskej
Mostná 16
949 01 Nitra

ОДНА КОНСТРУКЦИЯ ГЕОДЕЗИЧЕСКИХ ГРАФОВ

Pavol Híc

Резюме

Неориентированный граф называется геодезическим графом, если для каждой пары вершин существует единственная кратчайшая цепь между ними. Автор дает одну конструкцию этих графов. Эта конструкция состоит в натяжении определенного $d$-разложения $(G_1, G_2)$ геодезического графа при каждой из вершин $V(G_1) \cap V(G_2)$ на единицу или больше. Эта конструкция объединяет некоторые известные конструкции геодезических графов.