Jaroslav Drahoš
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EQUIVALENCES IN PRESHEAVES,
FUNCTIONAL SEPARATION OF THEIR INDUCTIVE LIMITS
AND REPRESENTATION BY SECTIONS

JAROSLAV PECHANEC—DRAHOŠ

Introduction

In [1], [2], [3] we have dealt with the question of when the topology $t$ of the inductive limit $(I, t)$ of a presheaf $\mathcal{F} = \{(X_\alpha, t_\alpha) \mid q_{\alpha \beta} \langle A \leq \rangle \}$ from the category of topological spaces is Hausdorff. We have also studied there a slightly more general question of when there is a topology $\tilde{t}$ in $I$, such that $(I, \tilde{t})$ is functionally separated, meaning that for any different $x, y$ from $I$ there is a $\tilde{t}$-continuous function separating them. Several theorems have been found answering the question in an even more general setting. But these suffer from two drawbacks. The one is that we need $\leq$ to be a well order, the other is that we need the $q_{\alpha \beta}$’s to be 1—1. These drawbacks have been taken over by [4], where we try to solve the question of the representation of a presheaf by sections in its covering space by means of the theorems of [1]—[3], and they thus set some unwanted boundaries to the usefulness of the theory.

In this paper we try to get rid of these shortcomings. We seek some ways of extending the validity of the theorems of [1]—[3] to the case when $\leq$ is not a well order on the one hand, and when the $q_{\alpha \beta}$’s are not 1—1 on the other. A separation theorem (Th. 4) is found here, in which neither the $q_{\alpha \beta}$’s need be 1—1 nor $\leq$ needs be a well order. The method used in [1] fails to bring such an outcome, therefore the method of equivalences which leads to Th. 4 brings us to the solution of the representation questions in some cases when the former ones fail to do so. From Th. 4 we get a theorem on the representation of sheaves by sections (Th. 5) which could not be obtained by means of [1]—[3] and could not thus occur in [4], and thus enlargen the set of the representation theorems from [4] and help so our knowledge of the problems to become more complete.

Remark 1. Let us have a presheaf $\mathcal{F} = \{(X_\alpha, t_\alpha) \mid q_{\alpha \beta} \langle A \leq \rangle \}$ from the category of topological spaces meaning that the $q_{\alpha \beta}$’s are continuous. In every $X_\alpha$ we shall define the relation “$xR_\alpha y$ iff there is $\beta \geq \alpha$ with $q_{\alpha \beta} (x) = q_{\alpha \beta} (y)$” — as $\langle A \leq \rangle$ is right directed, it is an equivalence. Let $(Y_\alpha, \nu_\alpha)$ be the quotient space of
(X_a, t_a) by R_a, let p_a:(X_a, t_a)→(Y_a, v_a) be the natural maps. If α, β ∈ A, α ≤ β, then clearly q_{αβ}: X_a→X_β agrees with R_a, R_β; there thus is q_{αβ}:(Y_a, v_a)→(Y_β, v_β) — the quotient of q_{αβ}. Clearly it is continuous and q_{αβ} = q_{αγ}q_{γβ} for α ≤ β ≤ γ, hence \( S = \{(Y_a, v_a) | q_{αβ}(A ≤ \rangle) \) is a presheaf from the category of topological spaces. It shall be called quotient presheaf of S. Clearly the following diagram commutes for any α, β ∈ A, α ≤ β:

\[
\begin{array}{ccc}
(X_a, t_a) & \rightarrow & (X_β, t_β) \\
\downarrow p_a & & \downarrow p_β \\
(Y_a, v_a) & \rightarrow & (Y_β, v_β)
\end{array}
\]

Lemma 2. Let \( S = \{(X_a, t_a) | q_{αβ}(A ≤ \rangle) \) be a presheaf from the category of topological spaces, let \( ˇS = \{(Y_a, v_a) | q_{αβ}(A ≤ \rangle) \) be its quotient presheaf. Then

(a): q_{αβ} is 1—1 for any α, β ∈ A, α ≤ β,

(b): If \( \lim_{→} S = \langle (I, t) | \{ξ_a|a ∈ A\} \rangle \), \( \lim_{→} ˇS = \langle (J, v) | \{η_a|a ∈ A\} \rangle \) then (I, t) is homeomorphic to (J, v).

Proof. It is straightforward and, therefore, left to the reader.

Remark 3. Let S be as in Lemma 2.

(a): For the relation R_α on X_a, defined as “xR_α y if \( \xi_α(x) = \xi_α(y) \)”, we have R'_α = R_α.

(b): If α ∈ A, then for every β ≥ α an equivalence R_αβ can be defined as “xR_αβ y if \( q_{αβ}(x) = q_{αβ}(y) \)”. If M ⊂ X_a, then the R_α — saturation S_α(M) of M is related to the R_αβ — saturations S_αβ(M) of M as follows: S_α(M) = \( \cup \{ S_αβ(M) | β ≥ α \} \). Thus M is R_α — saturated iff it is such for every R_αβ with β ≥ α.

(c): If α ≤ β, M ⊂ X_a, then S_α(M) = \( \xi_α^{-1}ξ_α(M) \), S_αβ(M) = \( q_{αβ}^{-1}q_{αβ}(M) \). S_αβ(M) ⊂ S_αβ(M) ⊂ S_α(M) if β ≤ γ.

(d) If M, N ⊂ X_a, then S_α(M) ∩ S_α(N) ≠ \emptyset iff there is β ≥ α with S_αβ(M) ∩ S_αβ(N) ≠ \emptyset.

(e): (i) If all the q_{αβ}'s are open maps, then S_α(M) is open for any α ∈ A and any open M ⊂ X_a.

(ii) If moreover q_{αβ} maps any t_α — open set onto a t_β — open one for each α, β ∈ A, α ≤ β, then ξ_α(M) is t — open for any α ∈ A and any t_α — open set M.

Proof. (a) follows directly from the construction of \( \lim_{→} S \), (b) from the way of definition of R_α and R_αβ, (c) is easy to check. (d): If x ∈ S_α(M) ∩ S_α(N), then, by (b) there are β, γ ≥ α such that x ∈ S_αβ(M) ∩ S_αβ(N). Take δ ≥ β, γ; by (c) we have x ∈ S_αβ(M) ∩ S_αβ(N). The converse is clear. (e): If M ⊂ X_a is open then for any β ≥ αq_{αβ}(M) is open in q_{αβ}(X_a) (openness of q_{αβ}) and q_{αβ}q_{αβ}(M) is t_α.
— open (continuity of \( q_{\alpha\beta} \)) hence (i) follows from (b). (ii): It is known that \( S_{\alpha} \) is / \n— openiff\( 1(S) \) is / \n— openforevery \( \alpha \in A \). If \( M = t_{\alpha} \) is open, \( \beta \in A \), then there is \( \gamma \geq \beta \), \( \alpha \) and we have \( \xi_{\gamma}^{-1}\xi_{\alpha}(M) = \xi_{\gamma}^{-1}\xi_{\alpha}(M) = \mathcal{F}_{\gamma}(q_{\alpha\beta}(M)) \). As \( N = q_{\alpha\beta}(M) \) is \( t_{\alpha} \) — open, so is \( \mathcal{F}_{\gamma}(N) = \xi_{\gamma}^{-1}\xi_{\alpha}(M) \) by (i). Now, \( \xi_{\gamma}^{-1}\xi_{\alpha}(M) = q_{\gamma\gamma}^{-1}\xi_{\gamma}^{-1}\xi_{\alpha}(M) \) and it is open as \( q_{\gamma\gamma} \) is continuous.

**Theorem 4.** Let \( \mathcal{F} = \{(X_{\alpha}, t_{\alpha})| q_{\alpha\beta}|(A \leq)\} \) be a presheaf from the category of topological spaces, let \( \mathcal{F} = \{(Y_{\alpha}, v_{\alpha})| q_{\alpha\beta}|(A \leq)\} \) be its quotient (see 1), let \( \lim \mathcal{F} = \{(I, t)|\{\xi_{\alpha}|\alpha \in A \}\}, \lim \mathcal{F} = \{(J, v)|\{\eta_{\alpha}|\alpha \in A \}\} \). Suppose that for any \( \alpha \in A \) and any \( x, y \in X_{\alpha} \) with \( q_{\alpha\beta}(x) \neq q_{\alpha\beta}(y) \) for all \( \beta \geq \alpha \) there is an open nbd \( U \) of \( x \) and \( V \) of \( y \) with \( q_{\alpha\beta}^{-1}q_{\alpha\beta}(U) \cap q_{\alpha\beta}^{-1}q_{\alpha\beta}(V) = \emptyset \). Then

(a): If any \( q_{\alpha\beta} \) maps any \( t_{\alpha} \) — open set onto a \( t_{\alpha} \) — open one, then \( (I, t) \) and \( (J, v) \) are Hausdorff.

(b): If all the \( q_{\alpha\beta} \)'s are open maps, \( \leq \) a well order, if there is a countable cofinal subset of \( A \), and if every \( (X_{\alpha}, t_{\alpha}) \) is compact and Hausdorff, then \( (J, v) \) and \( (I, t) \) are functionally separated.

**Proof.** (a): If \( a, b \in I, a \neq b \), then there is \( a \in A \) and \( x, y \in X_{\alpha} \) with \( \xi_{\alpha}(x) = a \), \( \xi_{\alpha}(y) = b \). Then \( q_{\alpha\beta}(x) \neq q_{\alpha\beta}(y) \) for all \( \beta \geq \alpha \), and so there are open \( U, V \subset X_{\alpha} \) with \( q_{\alpha\beta}^{-1}q_{\alpha\beta}(U) \cap q_{\alpha\beta}^{-1}q_{\alpha\beta}(V) = \emptyset \). By 3b, we have \( \mathcal{F}_{\alpha}(U) \cap \mathcal{F}_{\alpha}(V) = \emptyset \), for all \( \beta \geq \alpha \), and 3d yields \( \mathcal{F}_{\alpha}(U) \cap \mathcal{F}_{\alpha}(V) = \emptyset \). By 3c \( \xi_{\alpha}^{-1}\xi_{\alpha}(U) \cap \xi_{\alpha}^{-1}\xi_{\alpha}(V) = \mathcal{F}_{\alpha}(U) \cap \mathcal{F}_{\alpha}(V) = \emptyset \) and so \( \xi_{\alpha}(U) \cap \xi_{\alpha}(V) = \emptyset \). By 3e ii \( \xi_{\alpha}(U), \xi_{\alpha}(V) \) is an open neighborhood of \( a, b \) respectively hence \( (I, t) \) is Hausdorff.

(b): We may assume that the ordinal type of \( A \) is \( \omega_{0} \) as there is a cofinal subset of \( A \) of the ordinal type \( \omega_{0} \). Any \( x, y \in X_{\alpha} \) with \( xR_{\alpha}y \) have open \( nbds U, V \) with \( \mathcal{F}_{\alpha}(U) \cap \mathcal{F}_{\alpha}(V) = \emptyset \); by 3e (i) \( \mathcal{F}_{\alpha}(U), \mathcal{F}_{\alpha}(V) \) are open; plainly they are saturated; thus \( p_{\alpha}\mathcal{F}_{\alpha}(U) = p_{\alpha}(U) \) and \( p_{\alpha}\mathcal{F}_{\alpha}(V) = p_{\alpha}(V) \) are \( v_{\alpha} \) — open disjoint \( nbds \) of \( p_{\alpha}(x), p_{\alpha}(y) \) — here \( p_{\alpha}: X_{\alpha} \to Y_{\alpha} \) are the natural maps (see 1). Thus we have the \( (Y_{\alpha}, v_{\alpha})'s \) Hausdorff, hence compact because such are the \( (X_{\alpha}, t_{\alpha})'s \). By 2a the \( q_{\alpha\beta} \)'s are \( 1—1 \), hence they are homeomorphisms. Thus \( \mathcal{F} \) fulfills the conditions of 1.5.5 or of 1.5.6b of [1] and hence \( (J, v) \) is functionally separated. Now we use 2b, which finishes the proof.

Th. 4 yields the following representation theorem:

**Theorem 5.** Let \( \mathcal{F} = \{(X_{\alpha}, t_{\alpha})| q_{U\alpha}|X\} \) be a sheaf from the category of topological spaces such that every \( (X_{U}, t_{U}) \) is compact. Suppose each \( x \in X \) has a filter base \( A_{X} \) of open \( nbds \) such that

(a): either each \( q_{U\alpha} \) maps any \( t_{U} \) — open set onto a \( t_{V} \) — open one, or all the \( q_{U\alpha} \)'s are open and \( x \) is of a countable local character.

(b): For any \( U \in A_{X} \) and any \( x, y \in X_{U} \) with \( q_{U\alpha}(x) = q_{U\alpha}(y) \) for all \( V \in A_{X} \),
there is an open nbd $U$ of $x$ and $V$ of $y$ such that $\varphi_{U,V}(U) \cap \varphi_{U,V}(V) = \emptyset$ for all $V \in \mathcal{A}_x$, $V \subseteq U$. Then there is a separated closure $\hat{i}$ in the covering space $P$ of $S$ such that for the set $\Gamma(U, \hat{i})$ of all continuous sections over $U$, for the natural map $p_U: X_U \rightarrow \Gamma(U, \hat{i})$ mapping $a \in X_U$ onto $\hat{a} \in \Gamma(U, \hat{i})$, where $\hat{a}(x)$ is the germ of $a$ in the stalk over $x$, and for the topology $b_U(\hat{i})$ projectively defined in $A_U$ by the maps $\{ r_{Ux}(x) | x \in U \}$ where $r_{Ux}(\hat{a}) = \hat{a}(x)$ for $x \in U$, we have for any open $U \subseteq X$:

(a): $p_U: (X_U, t_U) \rightarrow (A_U, b_U(\hat{i}))$ is a homeomorphism,

(b): $\Gamma(U, \hat{i}) = A_U$;

(for a more complete statement see [4, Th. 4.2.2]).

Proof. Clearly $\mathcal{S}_{A_x}$ fulfils the conditions of Th. 4 whence if $\lim S_{A_x} = \langle (I_x, t_x) | (\xi_{Ux} | U \in \mathcal{A}_x) \rangle$, then $(I_x, t_x)$ is a Hausdorff topological space; thus $(A_U, b_U)$ is also one for the topology $b_U$ projectively defined in $A_U$ by the maps $\{ r_{Ux}: A_U \rightarrow (I_x, t_x) | x \in U \}$ (see [4, Th. 4.2.2], where $b_U$ is denoted by $b_U(st_x)$). As $p_U: (X_U, t_U) \rightarrow (A_U, b_U)$ are $1-1$ because $\mathcal{S}$ is a sheaf, and continuous, they are homeomorphisms, $(X_U, t_U)$ being compact. The rest can be proven in the same way as [4, Th. 4.2.2].

REFERENCES


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Доказана теорема, при условиях которой индуктивный предел $I$ предпучка $F = \{X,|x|\langle A \leq \rangle\}$ функционально отделим (для всяких $p, q \in F$, $p \neq q$ существует непрерывная функция $f$ такая, что $f(p) = f(q)$) без предположения, что $\leq$ есть хорошее упорядочение и что $\varphi_\alpha$-простые. Это является обобщением подобной теоремы 1.1.7 из [1], в которой упомянутые предложения должны выполняться, а также обобщает некоторые теоремы о представлении пучков сечениями из [4], позволяя отбросить предположение, что $\varphi_\alpha$-простые и что точки $x \in X$ имеют хорошо упорядоченные фильтер — базисы открытых окрестностей.