Bedřich Pondělíček

On the intersection graph of a commutative distributive groupoid

Mathematica Slovaca, Vol. 29 (1979), No. 1, 57--62

Persistent URL: http://dml.cz/dmlcz/129550

Math. Slovaca 29, 1979, No. 1, 57—62

ON THE INTERSECTION GRAPH OF A COMMUTATIVE DISTRIBUTIVE GROUPOID

BEDŘICH PONDĚLÍČEK

Let F be some family of sets. By the *intersection graph* of F we mean the undirected graph whose set of vertices is F and in which two distinct vertices are joined by an edge if and only if they have a non-empty intersection. Some authors studied the case when F is the family of all proper subalgebras of a given algebra. This study was begun by J. Bosák [1]. It is known for example that *the intersection graph of every semigroup (commutative semigroup) with more than two (three) elements is connected and its diameter does not exceed three (two).* See [2] and [3].

The purpose of this paper is to discuss the connectedness of the intersection graph of a *commutative distributive groupoid.*

A groupoid P is called

— *commutative* if $ab=ba$ for all $a, b \in P$;

— *distributive* if $a.bc=ab.ac$ and $bc.a=ba.ca$ for all $a, b, c \in P$;

— *idempotent* if $aa=a$ for every $a \in P$ and

— *abelian* if $ab.cd=ac.bd$ for all $a, b, c, d \in P$.

The intersection graph of the family of all proper subgroupoids of a groupoid P is denoted by $G(P)$. If $G(P)$ is a connected graph, then by $\delta(P)$ we denote its *diameter.*

By the symbol $[A]$, where A is a non-empty subset of a groupoid P, we denote a subgroupod of P generated by A. The set of all idempotents of a groupoid P is denoted by $E(P)$.

Theorem 1. *The graph $G(P)$ of a distributive groupoid P is non-empty if and only if* card $P \geqq 2$. *Moreover,*

1. $\delta(P) \leqq 3$ *if and only if for any two idempotents a, b of P with $[a, b]=P$ there exists $c \in P$ such that*

$$[a, c] \neq P \neq [c, b].$$

2. $\delta(P) \leqq 2$ *if and only if $[a, b] \neq P$ for any two idempotents a, b of P.*

3. $\delta(P) \leqq 1$ *if and only if P contains just one idempotent.*

4. $\delta(P)=0$ *if and only if P is isomorphic to the semigroup with zero multiplication and containing just two elements.*

Proof. Let P be a distributive groupoid. It follows from Proposition 1.1 of [4] that the set $E(P)$ of all idempotents of P is a subgroupoid of P and

$$(1) \qquad a.bc,\ ab.c \in E(P) \text{ for al } a, b, c \in P.$$

It is clear that $G(P)$ is empty if and only if $P=E(P)$ and card $P=1$.

Now we shall suppose that card $P \geqq 2$.

1. Suppose that $\delta(P) \leqq 3$. Let a, $b \in E(P)$ and $[a, b]=P$. Then $\{a\}$, $\{b\}$ are proper subgroupoids of P and so there exist proper subgroupoids A, B of P such that $a \in A$, $b \in B$ and $A \cap B \neq \emptyset$. We can choose $c \in A \cap B$. Then we have $[a, c] \subset A$ and $[c, b] \subset B$. Hence $[a, c] \neq P \neq [c, b]$.

Now, we assume that for any a, b of $E(P)$ with $[a, b]=P$ there exists $c \in P$ such that $[a, c] \neq P \neq [c, b]$. Let A, B be two proper subgroupoids of P. Then we can choose $u \in A$ and $v \in B$. It follows from (1) that $a=uu.u \in A$ and $b=vv.v \in B$ are idemponts of P. If $[a, b] \neq P$, then we put $C=D=[a, b]$. If $[a, b]=P$, then there exists $c \in P$ such that $C=[a, c] \neq P$ and $D=[c, b] \neq P$. This gives in both cases $A \cap C \neq \emptyset$, $C \cap D \neq \emptyset$, $D \cap B \neq \emptyset$ and so $\delta(P) \leqq 3$.

2. This can bes proved analogously to the proof in 1.

3. Suppose that $\delta(P) \leqq 1$. Let a, $b \in E(P)$. Then $\{a\}$, $\{b\}$ are proper subgroupoids of P and so $\{a\} \cap \{b\} \neq \emptyset$. Hence we have $a=b$.

Let card $E(P)=1$. If A, B are proper subgroupoids of P, then according to (1), we have $xx.x = yy.y$ for all $x \in A$ and $y \in B$. Hence $A \cap B \neq \emptyset$. Thus $\delta(P) \leqq 1$.

4. This follows from (1).

Corollary 1. *If a distributive groupoid P is uncountable, then its graph $G(P)$ is connected and $\delta(P) \leqq 2$.*

Proof. It is clear that $[a, b] \neq P$ for all a, $b \in P$.

Corollary 2. *If a distributive groupoid P is not idempotent, then its graph $G(P)$ is connected and $\delta(P) \leqq 2$.*

Proof. Evidently $[a, b] \subset E(P)$ for all a, $b \in E(P)$.

Now we shall study commutative distributive groupoids. For the sake of brevity, the commutative idempotent abelian groupoids will be called CIA-groupoids. It is known that every CIA-groupoid is distributive.

A commutative semigroup $S(+)$ is called *uniquely* 2-*divisible* if the mapping $\varphi(x)=x+x$ is a permutation of S. In this case, the inverse permutation φ^{-1} is denoted by $\varphi^{-1}(x)=\frac{1}{2}x$. Throughout $\mathcal{Q}$ will denote the set of all numbers of the form $2^{-n}m$, where m, n are integers and $m \geqq 1$, $n \geqq 0$. Denote $2^{-n}mx = \varphi^{-n}(mx)$ for every $x \in S$, where $1x=x$ and $mx=(m-1)x+x$ for $m \geqq 2$. It is easy to see that

(2) $$\alpha(x+y)=\alpha x+\alpha y,\quad (\alpha+\beta)x=\alpha x+\beta x$$

for all $\alpha, \beta \in \mathcal{Q}$ and $x, y \in S$.

In [4] the following has been proved:

Lemma 1. *A groupoid $P(.)$ is a CIA-groupoid if and only if there exists a uniquely 2-divisible commutative semigroup $S(+)$ such that $P \subset S$ and $xy = \frac{1}{2}(x+y)$ for all $x\ y \in P$.*

$\mathscr{C}$ will denote throughout the set of all integers. Let m be an odd positive integer. By $\mathscr{C}_m(+)$ we denote a cyclic group of the order m generated by e. It is clear that $\mathscr{C}_m(+)$ is uniquely 2-divisible. According to Lemma 1, we obtain that $\mathscr{C}_m(.)$ is a CIA-groupoid, where $xy=\frac{1}{2}(x+y)$ for all $x, y \in \mathscr{C}_m$. For $a, u \in \mathscr{C}_m$ we put

$$\mathscr{C}_m(a, u)=\{a+ku\,;\,k \in \mathscr{C}\}.$$

It is clear that $A=\mathscr{C}_m(a, u)$ for some $a, u \in \mathscr{C}_m$ if and only if A is a class of some congruence on the group $\mathscr{C}_m(+)$.

Lemma 2. *Let A be a subgroupoid of $\mathscr{C}_m(.)$. If $a, b \in A$, then $\mathscr{C}_m(a, b-a) \subset A$.*

Proof. Let $a, b \in A$. Put $u=b-a$. First, we shall prove the following implication:

If $x, x+u \in A$, then $x+2u \in A$.

Let $x, x+u \in A$. Put $\mathscr{H}=\{n \in \mathscr{C}\,;\, n \geqq 2 \text{ and } x+nu \in A\}$. It is clear that $2m \in \mathscr{H}$ and so $\mathscr{H} \neq \emptyset$. We shall show that for any $n \in \mathscr{H}$, where $n>2$, there exists $k \in \mathscr{H}$ such that $k<n$.

If n is odd, then we have $\frac{1}{2}(n+1)<n$ and $x+\frac{1}{2}(n+1)u= = (x+u)\ (x+nu) \in A$. If n is even, then we have $\frac{1}{2}n<n$ and $x+\frac{1}{2}nu= =x(x+nu) \in A$.

This implies that $2 \in \mathscr{H}$ and so $x+2u \in A$.

By the induction we can prove that $a+nu \in A$ for all positive integers n and so $\mathscr{C}_m(a, b-a) \subset A$.

Lemma 3. *A subset A of $\mathscr{C}_m$ is a subgroupoid of $\mathscr{C}_m(.)$ if and only if A is a class of some congruence on $\mathscr{C}_m(+)$.*

Proof. Let A be a class of some congruence on $\mathscr{C}_m(+)$. Then $A=\mathscr{C}_m(a, u)$ for some $a, u \in \mathscr{C}_m$. Let $x, y \in \mathscr{C}_m(a, u)$. Then $x=a+ru$ and $y=a+su$ for some $r, s \in \mathscr{C}$. If $r+s$ is even, then $xy = a+\frac{1}{2}(r+s)u \in \mathscr{C}_m(a, u)$. If $r+s$ is odd, then $xy = a+\frac{1}{2}(r+s+m)u \in \mathscr{C}_m(a, u)$. Thus A is a subgroupoid of $\mathscr{C}_m(.)$.

Let A be a subgroupoid of $\mathscr{C}_m(.)$. Let e be a generator of $\mathscr{C}_m(+)$. By $\mathscr{H}$ we denote the set of all positive integers such that for any n of $\mathscr{H}$ there exists $x \in A$ such that $x + ne \in A$. Since $A \neq \emptyset$, we have $m \in \mathscr{H}$ and so $\mathscr{H} \neq \emptyset$. Put $k = \min \mathscr{H}$. Then there exists $a \in A$ such that $b = a + u \in A$, where $u = ke$. It follows from Lemma 2 that $\mathscr{C}_m(a, u) \subset A$.

Now we shall show that $A \subset \mathscr{C}_m(a, u)$. Let $x \in A$. Since e is a generator of $\mathscr{C}_m(+)$, we have $x = a + le$ for some positive integer l. It is well known that there exist $s, r \in \mathscr{C}$ such that $l = sk + r$ and $0 \leqq r < k$. Then $x = a + su + re$, where $a + su \in \mathscr{C}_m(a, u) \subset A$. If $0 < r$, then $r \in \mathscr{H}$, which is a contradiction. Therefore $r = 0$ and so $x = a + su \in \mathscr{C}_m(a, u)$. Consequently $A = \mathscr{C}_m(a, u)$.

Lemma 4. *Every subgroupoid (factor groupoid) of $\mathscr{C}_m(.)$ is isomorphic to $\mathscr{C}_k(.)$ for some odd positive integer k.*

Theorem 2. *Let m be an odd integer $\geqq 3$.*

1. *If m is prime, then the graph $G(\mathscr{C}_m(.))$ is composed of m isolated vertices.*
2. *If m is at least the second power of a prime number p, then the graph $G(\mathscr{C}_m(.))$ has p components whose diameters are equal to two.*
3. *If m is no power of a prime number, then the graph $G(\mathscr{C}_m(.))$ is connected and $\delta(\mathscr{C}_m(.)) = 3$.*

Proof. 1 and 2. This follows from Lemma 3.

3. Let p and q be two different prime numbers such that $p \mid m$ and $q \mid m$. Let $x, y \in \mathscr{C}_m$. Then $\mathscr{C}_m(x, pe) \neq \mathscr{C}_m \neq \mathscr{C}_m(y, qe)$, $\mathscr{C}_m(x, pe) \cap \mathscr{C}_m(y, qe) \neq \emptyset$ and so by Lemma 3 and Theorem 1 we have $\delta(\mathscr{C}_m(.)) \leqq 3$.

It is clear that there exist $r, s \in \mathscr{C}$ such that $rp - sq = 1$. Put $a = sqe$ and $b = rpe$. It follows from Lemma 2 that $\mathscr{C}_m = \mathscr{C}_m(a, e) \subset [a, b]$ and so according to Theorem 1, we have $\delta(\mathscr{C}_m(.)) = 3$.

Lemma 5. *Let P be a commutative distributive groupoid with* card $P \geqq 3$. *If for any elements $a, b \in P$ there holds the following implication:*

$$aP = P = Pb \Rightarrow [a, b] \neq P, \tag{3}$$

then $\delta(P(\leqq 3$.

Proof. According to Corollary 2, we can suppose that a commutative distributive groupoid P is idempotent. Let $a, b \in P$ and $[a, b] = P$. It follows from Proposition 1.5 of [4] that P is a CIA-groupoid. By hypothesis (3) we have the following possibilities:

Case 1. $aP \neq P \neq Pb$. It is clear that xP is a subgroupoid of a distributive groupoid P for any $x \in P$. Then we have $[a, ab] \subset aP \neq P$ and $[ab, b] \subset Pb \neq P$.

Case 2. $aP = P \neq Pb$. Then there exists $u \in P$ such that $b = au$. Any $u \in [a, b]$ can be written in the form $u = x_1 x_2 \dots x_n$, where $x_i \in \{., a, b\}$. Put $z = bv$, where

$v = y_1 y_2 \ldots y_n$ and $y_i = .$ if $x_i = .$, $y_i = a$ if $x_i = b$ and $y_i = b$ if $x_i = a$. According to Lemma 1 there exists a uniquely 2-divisible commutative semigroup $S(+)$ such that $P \subset S$ and $xy = \frac{1}{2}(x+y)$ for all $x, y \in P$. By the induction and by (2) we can show that $u = \alpha a + \beta b$ and $v = \beta a + \alpha b$, where $\alpha, \beta \in \mathcal{Q}$ and $\alpha + \beta = 1$. Then we have $b = 2^{-1}(1+\alpha)a + 2^{-1}\beta b$ and $z = 2^{-1}\beta a + 2^{-1}(1+\alpha)b$. Using (2) we obtain that $az = 2^{-2}(2+\beta)a + 2^{-2}(1+\alpha)b = 2^{-1}\beta a + (2^{-2}(1+\alpha)a + 2^{-2}\beta b) + 2^{-1}\alpha b = 2^{-1}\beta a + 2^{-1}b + 2^{-1}\alpha b = z$. If $[a, z] = P$, then $\operatorname{card} P \leqq 2$, which is a contradiction. Hence $[a, z] \neq P$. Evidently $[z, b] \subset Pb \neq P$.

Case 3. $aP \neq P = Pb$. Analogously.

The rest of the proof of Lemma 5 follows from Theorem 1.

Lemma 6. *Let P be a commutative distributive groupoid. If there exist $a, b \in P$ such that $aP = P = Pb$ and $P = [a, b]$, then P is isomorphic to $\mathscr{C}_m(.)$ for some odd positive integer m.*

Proof. Let $P = [a, b]$ and $aP = P = Pb$. It follows from Proposition 1.1 of [4] and from Proposition 1.5 of [4] that P is a CIA-groupoid. By Lemma 1 there exists a uniquely 2-divisible commutative semigroup $S(+)$ such that $P(.)$ is a subgroupoid of $S(.)$, where $xy = \frac{1}{2}(x+y)$ for all $x, y \in S$.

We first show that there exists an odd integer $k \geqq 3$ such that $ka = kb$. Since $aP = P = Pb$, there exist $u, v \in P$ such that $a = b(b \,.\, au)$ and $b = a(a \,.\, bv)$. It follows from $P = [a, b]$ that $a = \alpha a + \beta b$, $b = \gamma a + \varepsilon b$, where $\alpha, \beta, \gamma, \varepsilon \in \mathcal{Q}$ and $\alpha + \beta = 1 = \gamma + \varepsilon$, $\alpha < \frac{1}{2} < \beta$ and $\varepsilon < \frac{1}{2} < \gamma$. Then $\eta = 1 + \gamma - \alpha = 1 + \beta - \varepsilon \in \mathcal{Q}$ and so, by (2), we have $\eta a = a + \gamma a - \alpha a = \gamma a + \beta b = b + \beta b - \varepsilon b = \eta b$. This implies that $ka = kb$ for some odd integer $k \geqq 3$.

Let n be a positive integer such that $2^{n-1} < k < 2^n$. Define a mapping f of $\mathscr{C}_k$ into S by

$$f(re) = 2^{-n} ra + 2^{-n}(2^n - r)b,$$

where $r = 1, 2, \ldots, k$ and e is a generator of the cyclic group $\mathscr{C}_k(+)$. It is easy to show that f is a homomorphism of $\mathscr{C}_k(.)$ into $S(.)$.

If we put $w = ke$ and $z = (2^n - k)e$, then $f(w) = b$ and $f(z) = a$. Let A be a subgroupoid of $\mathscr{C}_k(.)$ generated by w, z. Then $f(A) = [a, b] = P$ and the restriction $g = f/A$ is a homomorphism of A onto P. This implies that P is isomorphic to some factor groupoid of A. It follows from Lemma 4 that P is isomorphic to the groupoid $\mathscr{C}_m(.)$ for some odd positive integer m.

Theorem 3. *Let P be a commutative distributive groupoid with* card $P \geqq 3$. *If P is not isomorphic to the groupoid $\mathscr{C}_m(.)$, where m is a power of an odd prime number, then the graph G(P) is connected and $1 \leqq \delta(P) \leqq 3$. Moreover,*

1. *If $\delta(P)=3$, then P is idempotent and is generated by two elements.*

2. *If $\delta(P)=2$, then P is not generated by two idempotents and contains two idempotents at least.*

3. *If $\delta(P)=1$, then P contains just one idempotent.*

The proof follows from Theorem 1, Theorem 2, Lemma 5 and Lemma 6.

Note. A commutative distributive groupoid P with card $P=2$ is either a semigroup with the zero multiplication and $\delta(P)=0$ or a semilattice and its graph $G(P)$ is composed of two isolated vertices.

REFERENCES

[1] BOSÁK, J.: The graphs of semigroups. In: Theory of Graphs and its Applications. Proc. of the Symposium held in Smolenice in June 1963, Praha 1964, 119—125.

[2] PONDĚLÍČEK, B.: Průměr grafu pologrupy. Čas. Pěst. Mat., 92, 1967, 206—211.

[3] ZELINKA, B.: Průměr grafu systému vlastních podpologrup komutativní pologrupy. Mat.-fyz. Čas., 15, 1965, 143—145.

[4] JEŽEK, J, KEPKA, T.: Semigroup representations of commutative idempotent abelian groupoids. Comment. Math. Univ. Carolinae, 16, 1975, 487—500.

Received May 19, 1977

Elektrotechnická fakulta ČVUT
290 35 Poděbrady

О ГРАФЕ ПЕРЕСЕЧЕНИЙ КОММУТАТИВНОГО ДИСТРИБУТИВНОГО ГРУППОИДА

Бедржих Понделичек

Резюме

Коммутативный группод P называется дистрибутивным, если $a.bc = ab.ac$ для $a, b, c \in P$. Пусть $G(P)$ — граф, вершинами которого являются все собственные подгруппоиды группоида P и в котором две вершины соединены ребром тогда и только тогда, если соответствующие подгруппоиды имеют непустое пересечение. В статье изучается связность графа пересечений $G(P)$ коммутативного дистрибутивного группоида P.

Пусть $C_m(+)$ — аддитивная группа вычетов по модулю m. Если m — нечетное число, то символом $C_m(.)$ мы обозначим группоид C_m, где $x.y = (x+y)/2$ для всех $x, y \in C_m$. В этой работе доказана следующая теорема:

Если коммутативный дистрибутивний группоид P содержит хотя бы три элемента и отличается от группоида $C_m(.)$, где m является степенью нечетного простого числа, то граф $G(P)$ связный и для его диаметра $\delta(P)$ имеем $1 \leqq \delta(P) \leqq 3$.

При этом:

1. *Если $\delta(P)=3$, то всякий элемент группоида P является идемпотентом и группоид P порожден двумя элементами.*
2. *Если $\delta(P)=2$, то группоид P не является порожденным двумя идемпотентами и содержит хотя бы два идемпотента.*
3. *Если $\delta(P)=1$, то группоид P содержит только один идемпотент.*