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ASYMPTOTIC BEHAVIOUR OF A CLASS
OF NONOSCILLATORY SOLUTIONS
OF DIFFERENTIAL EQUATIONS
WITH DEVIATING ARGUMENTS

CH. G. PHILOS

0. Introduction

The present paper is concerned with $n$-th order ($n > 1$) differential equations with deviating arguments, which involve the $r$-derivatives $D_r^i x$ ($i = 0, 1, \ldots, n$) of the unknown function $x$ defined by

$$D_r^0 x = x, \quad D_r^i x = r_i(D_r^{i-1} x)' \quad (i = 1, \ldots, n - 1)$$
and
$$D_r^n x = (D_r^{n-1} x)',$$

where $r_i$ ($i = 1, \ldots, n - 1$) are positive continuous functions on an interval $[t_0, \infty)$. A real-valued function $h$ is said to be $n$-times $r$-differentiable on an interval $[T, \infty)$, $T \geq t_0$ if $D_r^n h$ is defined on $[T, \infty)$, and $h$ is said to be $n$-times continuously $r$-differentiable on $[T, \infty)$ if $D_r^n h$ is continuous on $[T, \infty)$. Note that in the special case where $r_1 = \ldots = r_{n-1} = 1$ the above notion of the $r$-differentiability specializes from the one of the usual differentiability. Recently, there has been an increasing interest in studying the oscillatory and asymptotic behaviour of differential equations involving the $r$-derivatives of the unknown function in place of its usual derivatives.

More precisely, the paper deals with the asymptotic behaviour of nonoscillatory solutions of differential equations with deviating arguments of the form

$$(E) \quad (D_r^n x)(t) + F(t; x < g_0(t)), \quad (D_r^i x)(g_i(t)), \ldots, (D_r^0 x)(g_1(t))) = b(t), \quad t \geq t_0,$$

where: $r_0 = 1$; $l$ is an integer with $0 \leq l \leq n - 1$;

$$g_i = (g_{i1}, \ldots, g_{in_i}) \quad (i = 0, 1, \ldots, l);$$
$g_k$ ($k = 1, \ldots, N_i; \ i = 0, 1, \ldots, l$) are continuous real-valued functions on the interval $[t_0, \infty)$ with

$$\lim_{t \to \infty} g_k(t) = \infty \quad (k = 1, \ldots, N_i; \ i = 0, 1, \ldots, l);$$

$b$ is a continuous real-valued function on $[t_0, \infty)$ and $F$ is a continuous real-valued function defined at least on $[t_0, \infty) \times (R^N \cup R^N)$, where $N = N_0 + N_1 + \ldots + N_l$ and $R_+ = (0, \infty), R = (-\infty, 0)$. Without any further mention, we suppose that: For every $t \geq t_0$, the function $|F(t; \cdot)|$ is increasing on $R^N_+$ and decreasing on $R^N_-$. For real-valued functions defined on subsets of $R^N$ monotonicity is considered with respect to the order in $R^N$ defined by the positive cone $\{Y = (y_1, \ldots, y_N) \in R^N: y_1 \geq 0, \ldots, y_N \geq 0\}$. Sufficient smoothness for the existence of solutions of (E) on a subinterval of $[t_0, \infty)$ will be assumed. In what follows the term “solution” is always used only for such solutions $x(t)$ of (E) which are defined for all large $t$. The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function which is defined on an interval of the form $[T, \infty)$ is called oscillatory if the set of its zeros is unbounded above, and otherwise it is called nonoscillatory.

For our purposes, for any integers $i$ and $\lambda$ with $0 \leq i \leq \lambda \leq n - 1$, we introduce the function $R_{\alpha}$ which is defined on $[t_0, \infty)$ by

$$R_{\alpha}(t) = \left\{ \begin{array}{ll}
1, & \text{if } i = \lambda \\
\int_{t_0}^{t} \frac{1}{r_i(s)} \int_{s}^{s_{i+1}} \frac{1}{r_{i+1}(s_{i+1})} \cdots \int_{s_{i}}^{s_{i-1}} \frac{1}{r_l(s_l)} \, ds_l \cdots ds_{i+2} \, ds_{i+1}, & \text{if } i < \lambda. 
\end{array} \right.$$ 

In particular, for any integer $\lambda$ with $0 \leq \lambda \leq n - 1$, we put

$$R_{\lambda}(t) = R_{\alpha}(t), \quad t \geq t_0. $$

To obtain our results we need the following lemma from [1].

**Lemma 0.1.** Let $\lambda$, $0 < \lambda \leq n - 1$, be an integer such that

$$\int_{t_0}^{\infty} \frac{dt}{r_i(t)} = \infty \quad (i = 1, \ldots, \lambda).$$

Moreover, let $h$ be a function whose $\lambda$-th r-derivative $D_{r_i}^{(\lambda)} h$ exists on an interval $[T, \infty)$, $T \geq t_0$. If $\lim_{t \to \infty} (D_{r_i}^{(\lambda)} h)(t)$ exists in $R^* - \{0\}$, where $R^* = R \cup \{-\infty, \infty\}$ is the extended
real line, then so does $\lim_{t \to \infty} [h(t)/R_k(t)]$ and, moreover,

$$\lim_{t \to \infty} [h(t)/R_k(t)] = \lim_{t \to \infty} (D^{(k)}h)(t).$$

The subject of this paper is the following: Let $m$ be an integer with $l \leq m \leq n - 1$. Provided that

$$(\Sigma[m]) \quad \int_{r(t)}^\infty \frac{dt}{r(t)} = \infty \quad (i = 1, \ldots, m)$$

for the case $m > 0$ and that the function $|b|$ satisfies a smallness condition depending on $m$ (which holds by itself if $b = 0$), we shall find a condition (depending on $m$) for the function $F$, which ensures the existence of at least one (nonoscillatory) solution $x$ of the differential equation (E) with

$$(\star) \quad \lim_{t \to \infty} (D^{(m)}x)(t) = \lim_{t \to \infty} \frac{x(t)}{R_m(t)} = L \in R - \{0\}.$$ 

Next, we shall consider the differential equation (E) with $b = 0$, i.e. the equation

$$(E_0) \quad (D^{(\nu)}x)(t) + F(t; x(g_0(t)), (D^{(i)}x)(g_i(t)), \ldots, (D^{(i)}x)(g_i(t))) = 0,$$

and we will suppose that the functions $r_i$ $(i = 1, \ldots, n - 1)$ are such that

$$(\Sigma) \quad \int_{r_i(t)}^\infty \frac{dt}{r_i(t)} = \infty \quad (i = 1, \ldots, n - 1)$$

and that $F$ has one of the sign properties

$$(I) \quad \begin{cases} F(t; Y) \geq 0 & \text{for every } t \geq t_0 \text{ and } Y \in R^y_n, \\ F(t; Y) \leq 0 & \text{for every } t \geq t_0 \text{ and } Y \in R^y_n, \end{cases}$$

$$(II) \quad \begin{cases} F(t; Y) \leq 0 & \text{for every } t \geq t_0 \text{ and } Y \in R^y_n, \\ F(t; Y) \geq 0 & \text{for every } t \geq t_0 \text{ and } Y \in R^y_n. \end{cases}$$

Then we shall prove that the condition which is sufficient in order that the differential equation $(E_0)$ have at least one (nonoscillatory) solution $x$ satisfying $(\star)$ is also necessary. The results obtained extend previous ones due to the author, Sficas and Staikos [2] and to the author and Staikos [3] concerning the special case where $l = 0$. Notice that the methods used here pattern after that of [2] and [3].

1. **Sufficient conditions**

To obtain our first result (Theorem 1.1) we shall apply the fixed point technique by using the following Schauder's theorem (Schauder [4]).
The Schauder theorem. Let $E$ be a Banach space and $X$ a nonempty, convex and closed subset of $E$. Moreover, let $S$ be a continuous mapping of $X$ into itself. If $SX$ is relatively compact, then the mapping $S$ has at least one fixed point (i.e. there exists an $x \in X$ with $x = Sx$).

A set $\mathcal{F}$ of real-valued functions defined on the interval $[T, \infty)$ is said to be (cf. [5]) equiconvergent at $\infty$ if all functions in $\mathcal{F}$ are convergent in $\mathbb{R}$ at the point $\infty$ and, moreover, for every $\varepsilon > 0$ there exists a $T' \geq T$ such that, for all functions $f$ in $\mathcal{F}$,

$$|f(t) - \lim_{s \to \infty} f(s)| < \varepsilon \quad \text{for every} \quad t \geq T'.$$

Let now $B([T, \infty))$ be the Banach space of all continuous and bounded real-valued functions on the interval $[T, \infty)$, endowed with the usual sup-norm $\| \|$. We need the following compactness criterion for subsets of $B([T, \infty))$, which is a corollary of the Arzelà—Ascoli theorem. For a proof of this criterion we refer to Staikos [5].

Compactness criterion. Let $\mathcal{F}$ be an equicontinuous and uniformly bounded subset of the Banach space $B([T, \infty))$. If $\mathcal{F}$ is equiconvergent at $\infty$, it is also relatively compact.

Theorem 1.1. Let $m, l \leq m \leq n - 1$, be an integer such that $(\Sigma[m])$ holds when $m > 0$, and:

$(C[m])$ For some nonzero constant $c$

$$\begin{cases}
\int_{T}^{\infty} |F(t; cR_{0,n-1}(g_0(t)), cR_{1,n-1}(g_1(t)), \ldots, cR_{l,n-1}(g_l(t)))| \, dt < \infty \\
&\text{if } m = n - 1
\end{cases}$$

$$\int_{T}^{\infty} \frac{1}{r_{m+1}(s_{m+1})} \cdots \int_{s_{m+1}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |F(s; cR_{0m}(g_0(s)), cR_{1m}(g_1(s)), \ldots, cR_{lm}(g_l(s)))| \, ds \, ds_{n-1} \cdots ds_{m+1} < \infty \quad \text{if } m < n - 1.$$ 

$(B[m])$ There holds

$$\begin{cases}
\int_{T}^{\infty} |b(t)| \, dt < \infty \quad \text{if } m = n - 1
\end{cases}$$

$$\int_{T}^{\infty} \frac{1}{r_{m+1}(s_{m+1})} \cdots \int_{s_{m+1}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |b(s)| \, ds \, ds_{n-1} \cdots ds_{m+1} < \infty \quad \text{if } m < n - 1.$$ 

Then for every real number $L$ with $Lc > 0$ and $\frac{|c|}{2} < |L| < |c|$, there exists a (nonoscillatory) solution $x$ of the differential equation $(E)$ with
\[
(P_m(x)) \begin{cases}
\lim_{t \to \infty} (D_r^{(i)}x)(t) = (\text{sgn} \, L)^i \quad (i = 0, 1, \ldots, m - 1) \quad \text{when } m > 0 \\
\lim_{t \to \infty} (D_r^{(m)}x)(t) = \lim_{t \to \infty} \frac{x(t)}{R_m(t)} = L \\
\lim_{t \to \infty} (D_r^{(j)}x)(t) = 0 \quad (j = m + 1, \ldots, n - 1) \quad \text{when } m < n - 1.
\end{cases}
\]

Proof. The substitution \( u = -x \) transforms (E) into the equation

\[
(D_r^{(m)}u)(t) + \hat{F}(t; u(g_0(t)), (D_r^{(i)}u)(g_i(t)), \ldots, (D_r^{(i)}u)(g_i(t))) = \hat{b}(t),
\]

where \( \hat{F}(t; Y) = -F(t; -Y) \) and \( \hat{b}(t) = -b(t) \). The transformed equation is subject to the assumptions of the theorem with \(-c\) in place of \(c\). Thus we can confine our discussion only to the case of positive \(c\).

Let \( L \) be a (positive) number with \( \frac{c}{2} < L < c \). By condition \((B[m])\) we choose a \( T_0 > t_0 \) such that \( c - L > d \), where

\[
d = \begin{cases}
\int_{T_0}^{\infty} |b(t)| \, dt & \text{if } m = n - 1 \\
\frac{1}{r_m(s_{m+1})} \cdots \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |b(s)| \, ds & \text{if } m < n - 1.
\end{cases}
\]

Next, by taking into account condition \((C[m])\), we consider a \( T \geq T_0 \) so that

\[
g_{ik}(t) \geq T_0 \quad \text{for every } t \geq T \quad (k = 1, \ldots, N_i; i = 0, 1, \ldots, l)
\]

and

\[
\int_{T}^{\infty} |F(t; \sum cR_{m_1-1}(g_0(t)), \sum cR_{m_q-1}(g_1(t)), \ldots, cR_{m_n-1}(g_i(t)))| dt \leq c - L - d \\
\int_{T}^{\infty} \frac{1}{r_{m+1}(s_{m+1})} \cdots \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |F(s; \sum cR_{m_1}(g_0(s)), \sum cR_{m_q}(g_1(s)), \ldots, cR_{m_n}(g_i(s)))| ds \, ds_{n-1} \cdots ds_{m+1} \leq c - L - d
\]

Now, we consider the Banach space \( E \) of all real-valued functions with continuous and bounded \( n \)-th \( r \)-derivative on the interval \([T, \infty)\), endowed with the norm \(||| \cdot |||\) defined by

\[
|||h||| = \begin{cases}
||h||, & \text{if } m = 0 \\
||D_r^{(m)}h|| + \sum_{i=0}^{m-1} \frac{|(D_r^{(i)}h)(T)|}{R_{im}(T)} & \text{if } m > 0.
\end{cases}
\]

We observe that the restriction of the function \( LR_m \) on \([T, \infty)\) belongs in \( E \), and so
we consider the nonempty, convex and closed subset $X$ of $E$, which contains all function $x \in E$ with

$$|||x - LR_m||| \leq c - L,$$

namely

$$\begin{cases}
||x - L|| \leq c - L & \text{if } m = 0 \\
||D^{(m)}_r x - L|| + \sum_{i=0}^{m-1} \left| \frac{(D^{(i)}_r x)(T)}{R_m(T)} - L \right| \leq c - L & \text{if } m > 0.
\end{cases}$$

Next, we shall prove that for any function $x$ in $X$ these holds

$$\left| \frac{(D^{(i)}_r x)(t)}{R_m(t)} - L \right| \leq c - L \quad \text{for all } t \geq T \quad (i = 0, 1, \ldots, m).$$

This is obvious for $m = 0$. Let us suppose that $m > 0$ and let us consider an arbitrary function $x$ in $X$. Then we observe that

$$\left| (D^{(m)}_r x)(t) - L \right| \leq c - L \quad \text{for all } t \geq T.$$

Hence for every $t \geq T$ we obtain

$$\frac{(D^{(m-1)}_r x)(t)}{R_{m-1,m}(t)} - L = \frac{1}{R_{m-1,m}(t)} \left[ (D^{(m-1)}_r x)(T) + \int_T^t \frac{1}{r_m(s)} (D^{(m)}_r x)(s) \, ds - LR_{m-1,m}(t) \right]$$

$$= \frac{1}{R_{m-1,m}(t)} \left[ (D^{(m-1)}_r x)(T) + \int_T^t \frac{1}{r_m(s)} [(D^{(m)}_r x)(s) - L] \, ds + L \int_T^t \frac{1}{r_m(s)} - LR_{m-1,m}(t) \right]$$

$$= \frac{R_{m-1,m}(T)}{R_{m-1,m}(t)} \left[ \frac{(D^{(m-1)}_r x)(T)}{R_{m-1,m}(T)} - L \right] + \frac{1}{R_{m-1,m}(t)} \int_T^t \frac{1}{r_m(s)} [(D^{(m)}_r x)(s) - L] \, ds$$

and hence

$$\left| \frac{(D^{(m-1)}_r x)(t)}{R_{m-1,m}(t)} - L \right| \leq \frac{R_{m-1,m}(T)}{R_{m-1,m}(t)} \left| \frac{(D^{(m-1)}_r x)(T)}{R_{m-1,m}(T)} - L \right| +$$

$$+ \frac{1}{R_{m-1,m}(t)} \int_T^t \frac{1}{r_m(s)} \left| (D^{(m)}_r x)(s) - L \right| \, ds$$

$$\leq \frac{R_{m-1,m}(T)}{R_{m-1,m}(t)} \left| \frac{(D^{(m-1)}_r x)(T)}{R_{m-1,m}(T)} - L \right| + \|D^{(m)}_r x - L\| \left[ \frac{1}{R_{m-1,m}(t)} \int_T^t \frac{1}{r_m(s)} \, ds \right]$$

$$\leq \left| \frac{(D^{(m-1)}_r x)(T)}{R_{m-1,m}(T)} - L \right| + \|D^{(m)}_r x - L\|.$$
Thus, \[
\left| \frac{(D_r^{(m-1)}x)(t)}{R_{m-1,m}(t)} - L \right| \leq c - L \quad \text{for all } t \geq T.
\]

Next, provided that \( m > 1 \), for \( t \geq T \) we get
\[
\frac{(D_r^{(m-2)}x)(t)}{R_{m-2,m}(t)} - L = \frac{1}{R_{m-2,m}(t)} \left\{ (D_r^{(m-2)}x)(T) + \int_T^t \frac{1}{R_{m-1,m}(s)} (D_r^{(m-1)}x)(s) \, ds - LR_{m-2,m}(t) \right\}
\]
\[
= \frac{1}{R_{m-2,m}(t)} \left\{ (D_r^{(m-2)}x)(T) + \int_T^t \frac{R_{m-1,m}(s)}{r_{m-1}(s)} \left[ \frac{(D_r^{(m-1)}x)(s)}{R_{m-1,m}(s)} - L \right] \, ds + L \int_T^t \frac{R_{m-1,m}(s)}{r_{m-1}(s)} \, ds - LR_{m-2,m}(t) \right\}
\]
\[
= \frac{1}{R_{m-2,m}(t)} \left\{ [((D_r^{(m-2)}x)(T) - LR_{m-2,m}(T)] + \int_T^t \frac{R_{m-1,m}(s)}{r_{m-1}(s)} \left[ \frac{(D_r^{(m-1)}x)(s)}{R_{m-1,m}(s)} - L \right] \, ds \right\}
\]
\[
= \frac{R_{m-2,m}(T)}{R_{m-2,m}(t)} \left\{ (D_r^{(m-2)}x)(T) - L \right\} + \frac{1}{R_{m-2,m}(t)} \int_T^t \frac{R_{m-1,m}(s)}{r_{m-1}(s)} \left[ \frac{(D_r^{(m-1)}x)(s)}{R_{m-1,m}(s)} - L \right] \, ds,
\]

and consequently
\[
\left| \frac{(D_r^{(m-2)}x)(t)}{R_{m-2,m}(t)} - L \right| \leq \frac{R_{m-2,m}(T)}{R_{m-2,m}(t)} \left| (D_r^{(m-2)}x)(T) - L \right| + \frac{1}{R_{m-2,m}(t)} \int_T^t \frac{R_{m-1,m}(s)}{r_{m-1}(s)} \left| \frac{(D_r^{(m-1)}x)(s)}{R_{m-1,m}(s)} - L \right| \, ds
\]
\[
\leq \frac{R_{m-2,m}(T)}{R_{m-2,m}(t)} \left| (D_r^{(m-2)}x)(T) - L \right| + \left\| \frac{D_r^{(m-1)}x}{R_{m-1,m}} - L \right\| \right\| \frac{1}{R_{m-2,m}(t)} \int_T^t \frac{R_{m-1,m}(s)}{r_{m-1}(s)} \, ds
\]
\[
\leq \left| \frac{(D_r^{(m-2)}x)(T)}{R_{m-2,m}(T)} - L \right| + \left| \frac{(D_r^{(m-1)}x)(T)}{R_{m-1,m}(T)} - L \right| + \left\| D_r^{(m)}x - L \right\|
\]

Hence
\[
\left| \frac{(D_r^{(m-2)}x)(t)}{R_{m-2,m}(t)} - L \right| \leq c - L \quad \text{for all } t \geq T.
\]

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If $m > 2$, then, following the same procedure, we conclude that for every $t \geq T$

$$\left| \frac{x(t)}{R_{om}(t)} - L \right| \leq \sum_{i=0}^{m-1} \left| \frac{(D_i^{(i)}x)(T)}{R_{om}(T)} - L \right| + \|D_i^{(m)}x - L\| \leq c - L.$$ 

We have thus proved our assertion.

Now, let $x$ be an arbitrary function in $X$. Then for $i = 0, 1, ..., l$ we have

$$0 < (D_i^{(i)}x)(t) \leq cR_{im}(t) \quad \text{for every} \quad t \geq T$$

and consequently

$$0 < (D_i^{(l)}x)^*(t) \leq cR_{im}(t) \quad \text{for all} \quad t \geq T_0,$$

where

$$(D_i^{(l)}x)^*(t) = \begin{cases} (D_i^{(l)}x)(t) & \text{if } t \geq T \\ (D_i^{(l)}x)(T) \frac{R_{im}(t)}{R_{im}(T)} & \text{if } T_0 \leq t \leq T. \end{cases}$$

Thus for all $t \geq T$

$$0 < (D_i^{(l)}x)^*([g_{ik}(t)] \leq cR_{im}[g_{ik}(t)] \quad (k = 1, ..., N_i; \ i = 0, 1, ..., l)$$

and hence, since for any $t \geq t_0$ the function $|F(t; \cdot)|$ is increasing on $R^+$, we have

$$|F(t; x^*(g_0(t)), (D_i^{(l)}x)^*(g_i(t)), ..., (D_i^{(l)}x)^*(g_l(t)))| \leq |F(t; \cdot cR_{om}(g_0(t)), cR_{om}(g_l(t)))|$$

for every $t \geq T$.

Thus, because of $(C[m])$ and $(B[m])$ for any $t \geq T$ there holds

$$\int_t^\infty \left[ F(s; x^*(g_0(s)), (D_i^{(l)}x)^*(g_1(s)), ..., (D_i^{(l)}x)^*(g_l(s))) - b(s) \right] ds < \infty \text{ if } m = n - 1$$

$$\int_t^\infty \frac{1}{r_{m+1}(s_{m+1})} \int_{s_{m+1}}^s \frac{1}{r_{m+1}(s_{m+1})} \int_{s_{m+1}}^{s_{m+1}} [F(s; x^*(g_0(s)), (D_i^{(l)}x)^*(g_1(s)), ..., (D_i^{(l)}x)^*(g_l(s))) - b(s)] ds ds_{m+1} \cdots ds_2 ds_1 < \infty \text{ if } m < n - 1.$$

Next, we define the mapping $S$ as follows:

$$(Sx)(t) = LR_m(t) + \int_T^{s_0} \frac{1}{r_1(s_1)} \int_T^{s_1} \frac{1}{r_2(s_2)} \cdots \int_T^{s_{m-1}} \frac{1}{r_{m-1}(s_{m-1})} \int_{s_{m-1}}^{s_{m-1}} [F(s; x^*(g_0(s)), ..., (D_i^{(l)}x)^*(g_l(s))) - b(s)] ds ds_{m-1} \cdots ds_1 ds_2 ds_1$$

if $m = n - 1$,

$$(Sx)(t) = LR_m(t) + (-1)^{m-1} \int_T^{s_0} \frac{1}{r_1(s_1)} \int_T^{s_1} \frac{1}{r_2(s_2)} \cdots \int_T^{s_{m-1}} \frac{1}{r_{m-1}(s_{m-1})} \int_{s_{m-1}}^{s_{m-1}} [F(s; x^*(g_0(s)), ..., (D_i^{(l)}x)^*(g_l(s))) - b(s)] ds ds_{m-1} \cdots ds_2 ds_1$$

if $m < n - 1$. 

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\[
\ldots, (D^m_t x) \ast (g_l(s)) - b(s)] \, ds \, ds_{n-1} \ldots ds_{m+1} \, ds_m \ldots ds_2 \, ds_1
\]

if \( 0 < m < n - 1 \),

\[
(Sx)(t) = LR_m(t) + (-1)^{n-1} \int_t^s \frac{1}{r_1(s_1)} \int_t^{s_1} \frac{1}{r_2(s_2)} \ldots
\]

\[
\ldots \int_t^{s_{n-2}} \frac{1}{r_{m+1}(s_{m+1})} \int_t^{s_{m+2}} \frac{1}{r_{n-1}(s_{n-1})} \int_t^{s_{n-2}} [F(s; x^*(g_0(s))) - b(s)] \, ds \, ds_{n-1} \ldots ds_2 \, ds_1
\]

if \( m = 0 \).

In order to apply the Schauder theorem it must be verified that \( S \) is a mapping of \( X \) into itself, \( SX \) is relatively compact and \( S \) is continuous.

a) \( SX \subseteq X \).

Indeed, for any function \( x \in X \) and every \( t \geq T \) we obtain

\[
| (D^m_t Sx)(t) - L | =
\]

\[
\begin{cases}
\left| \int_t^s [F(s; x^*(g_0(s))), \ldots, (D^m_t x) \ast (g_l(s))) - b(s)] \, ds \right| & \text{if } m = n - 1 \\
\int_t^s \left| \frac{1}{r_1(s_1)} \ldots \frac{1}{r_{m+1}(s_{m+1})} \right| \ldots \frac{1}{r_{n-1}(s_{n-1})} \int_t^{s_{n-1}} [F(s; x^*(g_0(s))), \ldots, (D^m_t x) \ast (g_l(s))) - b(s)] \, ds \, ds_{n-1} \ldots ds_{m+1} & \text{if } m < n - 1
\end{cases}
\]

Also, if \( m > 0 \), we have

\[
\frac{(D^m_t Sx)(T) - L}{R_m(T)} = 0 \quad (i = 0, 1, \ldots, m - 1).
\]

And for any \( x \in X \), \( ||Sx - LR_m|| \leq c - L \) and consequently \( Sx \in X \).
\( \beta \) SX is relatively compact. 
Obviously, it suffices to prove that the set 
\[ \mathcal{F} = \{ D^{(m)}_r x : x \in X \} \]
is a relatively compact subset of the space \( B([T, \infty)) \). Furthermore, by the 
compactness criterion, \( \mathcal{F} \) is relatively compact if it is uniformly bounded, equiconvergent at \( \infty \) and equicontinuous. Now, by the definition of \( X \) and the fact that 
\( SX \subseteq X \), there holds
\[ \| D^{(m)}_r x \| \leq c \quad \text{for every} \quad x \in X, \]
which means that \( \mathcal{F} \) is a uniformly bounded subset of \( B([T, \infty)) \). Also, for any 
function \( x \in X \) and every \( t \in T \) we obtain
\[
\left| (D^{(m)}_r x)(t) - L \right| \leq \\
\begin{cases} \\
\int_t^\infty \left[ |F(s; x*(g_0(s)), \ldots, (D^{(1)}_r x)*(g_i(s)))| + |b(s)| \right] ds & \text{if} \quad m = n - 1 \\
\int_t^\infty \frac{1}{r_{m+1}(s_{m+1})} \cdots \int_t^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_t^\infty |F(s; x*(g_0(s)), \ldots, (D^{(1)}_r x)*(g_i(s)))| + |b(s)|] ds \ ds_{n-1} \ldots ds_{m+1} & \text{if} \quad m < n - 1 \\
\end{cases}
\]
and hence
\[ \lim_{t \to \infty} (D^{(m)}_r x)(t) = L \quad \text{for every} \quad x \in X, \]
i.e. \( \mathcal{F} \) is equiconvergent at \( \infty \). Finally, in order to prove that \( \mathcal{F} \) is equicontinuous, 
for any function \( x \in X \) and every \( t_1, t_2 \) with \( T \leq t_1 \leq t_2 \) we get
\[
\left| (D^{(m)}_r x)(t_2) - (D^{(m)}_r x)(t_1) \right| = \\
\begin{cases} \\
\int_{t_1}^{t_2} \left[ F(s; x*(g_0(s)), \ldots, (D^{(1)}_r x)*(g_i(s))) - b(s) \right] ds & \text{if} \quad m = n - 1 \\
\int_{t_1}^{t_2} \frac{1}{r_{n-1}(s_{n-1})} \int_t^\infty \left[ F(s; x*(g_0(s)), \ldots, (D^{(1)}_r x)*(g_i(s))) - b(s) \right] ds \ ds_{n-1} & \text{if} \quad m = n - 2 \\
\int_{t_1}^{t_2} \frac{1}{r_{m+1}(s_{m+1})} \cdots \int_{t_1}^{t_2} \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{t_1}^{t_2} \frac{1}{r_{n-1}(s_{n-1})} \int_t^\infty |F(s; x*(g_0(s)), \ldots, (D^{(1)}_r x)*(g_i(s))) - b(s)| ds \ ds_{n-1} \ldots ds_{m+2} ds_{m+1} & \text{if} \quad n > 2 \quad \text{and} \quad m < n - 2 \\
\end{cases}
\]
\[
\int_{t_1}^{t_2} \left| F(s; x^*(g_0(s)), ..., (D^{(i)}x)^*(g_i(s))) \right| + |b(s)| \, ds \quad \text{if } m = n - 1
\]

\[
\int_{t_1}^{t_2} \frac{1}{r_{m-1}(s_{m-1})} \sum_{s_{m-1}}^{s_m} \left| F(s; x^*(g_0(s)), ..., (D^{(i)}x)^*(g_i(s))) \right| + |b(s)| \, ds \, ds_{n-1}
\]

\text{if } m = n - 2

\[
\int_{t_1}^{t_2} \frac{1}{r_{m+1}(s_{m+1})} \sum_{s_{m+1}}^{s_{m+2}} \frac{1}{r_{m+2}(s_{m+2})} \sum_{s_{m+2}}^{s_{m+3}} \frac{1}{r_{n-1}(s_{n-1})} \sum_{s_{n-1}}^{s_n} \left| F(s; x^*(g_0(s)), ..., (D^{(i)}x)^*(g_i(s))) \right| + |b(s)| \, ds \, ds_{n-1}...ds_{m+2} \, ds_{m+1}
\]

\text{if } n > 2 \text{ and } m < n - 2

\[
\int_{t_1}^{t_2} \left| F(t; cR_{0,n-1}(g_0(s)), ..., cR_{I,n-1}(g_I(s))) \right| + |b(s)| \, ds \, ds_{n-1}...ds_{m+2} \, ds_{m+1}
\]

\text{if } m = n - 1

\[
\int_{t_1}^{t_2} \frac{1}{r_{m-1}(s_{m-1})} \sum_{s_{m-1}}^{s_m} \left| F(t; cR_{0,n-2}(g_0(s)), ..., cR_{I,n-2}(g_I(s))) \right| + |b(s)| \, ds \, ds_{n-1}...ds_{m+2} \, ds_{m+1}
\]

\text{if } n > 2 \text{ and } m < n - 2.

\gamma) \text{ The mapping } S \text{ is continuous.}

Let } x \in X \text{ and } (x_\nu)_{\nu \in \mathbb{N}} \text{ be an arbitrary sequence in } X \text{ with}

\[ ||| \, ||| \, - \lim_{\nu \to \infty} x_\nu = x. \]

Then it is easy to verify that for every } \nu \geq T_0

\[ \lim_{\nu \to \infty} (D^{(i)}x_\nu)^*(t) = (D^{(i)}x)^*(t) \quad (i = 0, 1, ..., l). \]

On the other hand, for any } \nu \in \mathbb{N} \text{ and every } t \geq T \text{ we have}

\[ |F(t; x_\nu^*(g_0(t), ..., (D^{(i)}x_\nu)^*(g_i(t)))| \leq \]

\[ \leq |F(t; cR_{0m}(g_0(t)), ..., cR_{Im}(g_I(t)))|. \]

Thus, because of condition \((C[m])\), we can apply the Lebesgue dominated convergence theorem to obtain the pointwise convergence

\[ \lim_{\nu \to \infty} (Sx_\nu)(t) = (Sx)(t), \quad t \geq T. \]

In order to prove that

\[ ||| \, ||| \, - \lim_{\nu \to \infty} Sx_\nu = Sx, \]

we consider any subsequence \((\xi_\nu)_{\nu \in \mathbb{N}}\) of \((Sx_\nu)_{\nu \in \mathbb{N}}\). Then, because of the relative compactness of } SX, \text{ there exist a subsequence } (\nu_\nu)_{\nu \in \mathbb{N}} \text{ of } (\xi_\nu)_{\nu \in \mathbb{N}} \text{ and a } w \in E \text{ so that}

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Since $|||\cdot|||$-convergence implies the pointwise one to the same limit function, we always have $w = Sx$, which proves the continuity of $S$.

Finally, the Schauder theorem ensures the existence of at least one fixed point $x$ of the mapping $S$. Then $x = Sx$ and consequently

$$(D_x^n)x(t) = -F(t; x^*(g_o(s)), ..., (D_x^l)x^*(g_l(s))) + b(t), \quad t \geq T,$$

namely the fixed point $x$ of the mapping $S$ is a solution on $[T, \infty)$ of the differential equation (E). Moreover, $x$ satisfies $P_m(x))$. Indeed, for every $t \geq T$ we obtain

$$|(D_x^m)x(t) - L| \leq \left\{ \begin{array}{ll}
\int_t^\infty \left[ |F(s; x^*(g_o(s)), \ldots, (D_x^m)x^*(g_l(s))| + |b(s)| \right] ds & \text{if } m = n - 1 \\
\int_t^\infty \frac{1}{r_{m+1}(s_{m+1})} \ldots \frac{1}{r_{n-1}(s_{n-1})} \int_{s_1}^{s_n} \left[ |F(s; x^*(g_o(s)), \ldots, (D_x^m)x^*(g_l(s))| + |b(s)| \right] ds ds_{n-1} \ldots ds_{m+1} & \text{if } m < n - 1
\end{array} \right.$$
\(m > 0\). Under condition \((C[m])\), for every real number \(L\) with \(Lc > 0\) and \(\frac{|c|}{2} < |L| < |c|\) there exists a (nonoscillatory) solution \(x\) of the differential equation \((E_0)\) satisfying \((P_m(x))\).

Next, let us consider the case of more general differential equations with deviating arguments of the form

\[(E^*)\]

\[(D_i^n x)(t) + \sum_{\varrho = 1}^{\mu} F_{\varrho}(t; x(\tau_{\varrho 0}(t))) = b(t),\]

where for any \(\varrho = 0, 1, \ldots, \mu\) we have: \(0 \leq \varrho_0 \leq n - 1\);

\[
\begin{align*}
&D_i^0 x(t) = ((D_i^0 x)[\tau_{\varrho 1}(t)], \ldots, (D_i^0 x)[\tau_{\varrho \varrho}(t)]) \\
&\tau_{\varrho q}(t) = (\tau_{\varrho 1 \varrho}(t), \ldots, \tau_{\varrho \varrho \varrho}(t))
\end{align*}
\]

\(\tau_{\varrho q}(k = 1, \ldots, N_\varrho; i = 0, 1, \ldots, \varrho_0)\) are continuous real-valued functions on the interval \([t_0, \infty)\) such that

\[
\lim_{t \to \infty} \tau_{\varrho q}(t) = \infty \quad (k = 1, \ldots, N_\varrho; i = 0, 1, \ldots, \varrho_0)
\]

and \(F_{\varrho}\) is a continuous real-valued function defined at least on the set \([t_0, \infty) \times (R^r \cup R^s)\), where \(n_0 = N_0 + N_1 + \ldots + N_\varrho\). It is supposed that: For any \(\varrho = 1, \ldots, \mu\) and every \(t \geq t_0\), the function \(|F_{\varrho}(t; \cdot)|\) is increasing on \(R^r\) and decreasing on \(R^s\). Following the same technique as in the proof of Theorem 1.1, we can obtain the following more general theorem.

**Theorem 1.1'.** Let \(m\) be an integer with \(\max_{\varrho = 1, \ldots, \mu} \varrho_0 \leq m \leq n - 1\) and such that \((\Sigma[m])\) holds when \(m > 0\) and the condition \((B[m])\) is satisfied. Moreover, suppose that:

\((C^*[m])\) For some nonzero constant \(c\) and every \(\varrho = 1, \ldots, \mu\)

\[
\begin{align*}
&\int_{t_0}^{\infty} \left| F_{\varrho}(t; cR_{0, n-1} (\tau_{\varrho 0}(t)), cR_{1, n-1} (\tau_{\varrho 1}(t)), \ldots, cR_{\varrho, n-1} (\tau_{\varrho \varrho}(t))) \right| dt < \infty \quad \text{if} \quad m = n - 1 \\
&\int_{t_0}^{\infty} \frac{1}{r_{m+1}} \ldots \frac{1}{r_{m+1}} \int_{s_{m-1}}^{\infty} \left| F_{\varrho}(s; cR_{0m} (\tau_{\varrho 0}(s)), cR_{1m} (\tau_{\varrho 1}(s)), \ldots, cR_{\varrho m} (\tau_{\varrho \varrho}(s))) \right| ds \, ds_{m-1} \ldots ds_{m+1} < \infty \quad \text{if} \quad m < n - 1.
\end{align*}
\]

Then for every real number \(L\) with \(Lc > 0\) and \(\frac{|c|}{2} < |L| < |c|\) there exists a (nonoscillatory) solution \(x\) of \((E^*)\) satisfying \((P_m(x))\).
Now, let us consider the linear differential equation with deviating arguments

\[ (D_l) \]

\[ (D^{(n)}_r x)(t) + a_i(t)(D^{(i)}_r x)[\sigma_i(t)] + \ldots + a_1(t)(D^{(1)}_r x)[\sigma_1(t)] + a_0(t)x[\sigma_0(t)] = b(t), \]

where \( a_i \) (\( i = 0, 1, \ldots, l \)) are continuous real-valued functions on the interval \([t_0, \infty)\) without any restriction on their sign and \( \sigma_i \) (\( i = 0, 1, \ldots, l \)) are also continuous real-valued functions on \([t_0, \infty)\) with

\[ \lim_{t \to \infty} \sigma_i(t) = \infty \quad (i = 0, 1, \ldots, l). \]

From Theorem 1.1' we obtain Corollary 1.2 below concerning the linear equation \((D_l)\). In particular, for \( l = n - 1 \) we have Corollary 1.3 below.

**Corollary 1.2.** Let \( m \), \( l \leq m \leq n - 1 \), be an integer such that \((\Sigma[m])\) holds when \( m > 0 \) and the condition \((B[m])\) is satisfied. Moreover, suppose that:

\((Y[m])\) For every \( i = 0, 1, \ldots, l \)

\[ \int_{\infty}^{\infty} |a_i(t)| R_{i,n-1}[\sigma_i(t)] \, dt < \infty \quad \text{if} \quad m = n - 1 \]

\[ \int_{\infty}^{\infty} \frac{1}{r_{m+1}(s_{m+1})} \ldots \frac{1}{r_{n-1}(s_{n-1})} \int_{s_n}^{s_{n-1}} |a_i(s)| R_{i,n}[\sigma_i(s)] \, ds \, ds_{n-1} \ldots ds_{m+1} < \infty \]

\( \quad \text{if} \quad m < n - 1. \)

Then for every real number \( L \) with \( L \neq 0 \) there exists a (nonoscillatory) solution \( x \) of the linear equation \((D_l)\) satisfying \((P_m(x))\).

**Corollary 1.3.** Let \((\Sigma)\) be satisfied and suppose that the conditions \((B[n - 1])\) and \((Y[n - 1])\), i.e. the conditions

\((B[n - 1])\)

\[ \int_{\infty}^{\infty} |b(t)| \, dt < \infty \]

and

\((Y[n - 1])\)

\[ \int_{\infty}^{\infty} |a_i(t)| R_{i,n-1}[\sigma_i(t)] \, dt < \infty \quad (i = 0, 1, \ldots, n - 1), \]

hold. Then for every \( L \neq 0 \) there exists a (nonoscillatory) solution \( x \) of the linear equation \((D_{n-1})\), which satisfies \((P_{n-1}(x))\), i.e.

\((P_{n-1}(x))\)

\[ \lim_{t \to \infty} (D^{(i)}_{r} x)(t) = (\text{sgn} \, L)^{\infty} \quad (i = 0, 1, \ldots, n - 2) \]

\[ \lim_{t \to \infty} (D^{(n-1)}_{r} x)(t) = \lim_{t \to \infty} \frac{x(t)}{R_{n-1}(t)} = L. \]
2. Necessary conditions

Our basic purpose in this section is to prove that, for any integer \( m \) with \( l \leq m \leq n - 1 \), \((C[m])\) is also a necessary condition in order that the differential equation \((E_0)\) have at least one (nonoscillatory) solution \( x \) satisfying \((P_m(x))\), provided that \((\Sigma)\) holds and the function \( F \) has the sign property (I) or (II).

**Theorem 2.1.** Suppose that \((\Sigma)\) holds and the function \( F \) has the sign property (I) or (II). Moreover, let \( m \) be an integer with \( l \leq m < n - 1 \).

Then the condition \((C[m])\) is a necessary condition in order that the differential equation \((E_0)\) have at least one (nonoscillatory) solution \( x \) such that \( \lim_{t \to \infty} (D^i(x))(t) \) exists in \( R - \{0\} \).

**Proof.** Let \( x \) be a solution on an interval \([T_0, \infty), T_0 > t_0\), of the differential equation \((E_0)\) with \( \lim_{t \to \infty} (D^i(x))(t) = L \) for some \( L \in R - \{0\} \). Since the substitution \( u = -x \) transforms \((E_0)\) into an equation of the same form satisfying the assumptions of the theorem, we can restrict ourselves to the case where \( L \) is positive.

If \( m > 0 \), then, in view of condition \((\Sigma)\), we can easily derive that
\[
\lim_{t \to \infty} (D^i(x))(t) = \infty \quad (i = 0, 1, \ldots, m - 1).
\]
Thus the \( r \)-derivatives \( D^i(x) (i = 0, 1, \ldots, l) \) are positive on an interval \([T^*, \infty), T^* \geq T_0\). Hence, if we choose a \( T \geq T^* \) such that
\[
g_{ik}(t) \geq T^* \quad \text{for every} \quad t \geq T \quad (k = 1, \ldots, N_i; \ i = 0, 1, \ldots, l),
\]
then from equation \((E_0)\) it follows that for all \( t \geq T \)
\[
\begin{cases}
(D^{(n)}(x))(t) \leq 0 & \text{if (I) holds} \\
(D^{(n)})(x)(t) \geq 0 & \text{if (II) holds}.
\end{cases}
\]
Namely, \( D^{(n)}(x) \) is of constant sign on \([T, \infty)\). Thus the functions \( D^{(j)}(x) (j = m, \ldots, n - 1) \) are also eventually of constant sign. Without loss of generality, we suppose that \( D^{(j)}(x) (j = m, \ldots, n - 1) \) are of constant sign on the whole interval \([T, \infty)\). Next, provided that \( m < n - 1 \) we can use the assumption \((\Sigma)\) to obtain
\[
\lim_{t \to \infty} (D^{(j)})(x)(t) = 0 \quad (j = m + 1, \ldots, n - 1).
\]
Furthermore, we get
\[
|(D^{(m)}(x))(t) - (D^{(m)})(x)(T)| = \int_T^t \frac{1}{r_{m+1}(s)} |D^{(m+1)}(x)(s)| \, ds, \quad t \geq T,
\]
where \( r_n = 1 \), and hence
Also, if \( m < n - 1 \), then

\[
(D^{(j)} x)(t) \leq \int_t^\infty \frac{1}{r_{j+1}(s)} |(D^{(j+1)} x)(s)| \, ds
\]

for \( t \geq T \) \( (j = m + 1, \ldots, n - 1) \).

Indeed, for every \( w \geq t \)

\[
|(D^{(j)} x)(t)| + |(D^{(j)} x)(w)| \geq \int_t^w \frac{1}{r_{j+1}(s)} |(D^{(j+1)} x)(s)| \, ds,
\]

which gives (2), since \( \lim_{w \to \infty} (D^{(j)} x)(w) = 0 \). Combining (1) and (2), we obtain

\[
\left\{ \begin{array}{ll}
\int_T^\infty |(D^{(n)} x)(t)| \, dt < \infty & \text{if } m = n - 1 \\
\int_T^\infty \frac{1}{r_{m+1}(s_{m+1})} \cdots \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty |(D^{(n)} x)(s)| \, ds \, ds_{n-1} \cdots ds_{m+1} < \infty & \text{if } m < n - 1
\end{array} \right.
\]

Now, by the L'Hospital rule, we can derive for \( i = 0, 1, \ldots, l \)

\[
\lim_{t \to \infty} \frac{(D_i^{(i)} x)(t)}{R_{m}(t)} = \lim_{t \to \infty} (D^{(m)} x)(t) = L > 0
\]

and hence there exists a positive constant \( c \) so that

\[
(D_i^{(i)} x)(t) \leq cR_{m}(t) \quad \text{for all } t \geq T^*_0 \quad (i = 0, 1, \ldots, l).
\]

Thus

\[
(D_i^{(i)} x)[g_{ik}(t)] \leq cR_{m}[g_{ik}(t)] \quad \text{for every } t \geq T \quad (k = 1, \ldots, N_i; \quad i = 0, 1, \ldots, l)
\]

and consequently, for all \( t \geq T \), there holds

\[
|(D^{(n)} x)(t)| = |F(t; x \langle g_0(t) \rangle, (D^{(i)} x)(g_1(t)), \ldots, (D^{(i)} x)(g_l(t)))| \\
\geq |F(t; cR_{0m} \langle g_0(t) \rangle, cR_{1m} \langle g_1(t) \rangle, \ldots, cR_{lm} \langle g_l(t) \rangle)|.
\]

And so, because of (3), we have

\[
\left\{ \begin{array}{ll}
\int_T^\infty |F(t; cR_{0,n-1}(g_0(t)), \ldots, cR_{l,n-1}(g_l(t)))| \, dt < \infty & \text{if } m = n - 1 \\
\int_T^\infty \frac{1}{r_{m+1}(s_{m+1})} \cdots \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty |F(t; cR_{0m}(g_0(s)), \ldots, cR_{lm}(g_l(s)))| \, ds \, ds_{n-1} \cdots ds_{m+1} < \infty & \text{if } m < n - 1
\end{array} \right.
\]

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which means that \((C[w])\) is satisfied.

Now we turn our attention to the case of the linear differential equation with deviating arguments

\[
(D_l)_o \quad (D^{(n)}_l x(t) + a_i(t)(D^{(i)}_l x)[\sigma_i(t)] + \ldots + a_1(t)(D^{(1)}_l x)[\sigma_1(t)] + a_0(t)x[\sigma_0(t)] = 0.
\]

For this equation we have the following corollaries of Theorem 2.1.

**Corollary 2.1.** Let \((\Sigma)\) be satisfied and suppose that:

\((A_1)_i, \quad a_i \geq 0 \quad \text{on} \quad [t_0, \infty) \quad (i = 0, 1, \ldots, l)
\]

or

\((A_2)_i, \quad a_i \geq 0 \quad \text{on} \quad [t_0, \infty) \quad (i = 0, 1, \ldots, l).
\]

Moreover, let \(m\) be an integer with \(1 \leq m \leq n - 1\).

Then \((Y[m])\) is a necessary condition in order that the linear equation \((D_l)_o\)

have at least one (nonoscillatory) solution \(x\) so that \(\lim_{t \to \infty} (D^{(m)}_l x)(t)\) exists in \(\mathbb{R} - \{0\}\).

**Corollary 2.2.** Let \((\Sigma)\) be satisfied and suppose that \((A_1)_{n-1}\) or \((A_2)_{n-1}\) holds, i.e.

\((A_1)_{n-1}, \quad a_i \geq 0 \quad \text{on} \quad [t_0, \infty) \quad (i = 0, 1, \ldots, n - 1)
\]

or

\((A_2)_{n-1}, \quad a_i \geq 0 \quad \text{on} \quad [t_0, \infty) \quad (i = 0, 1, \ldots, n - 1).
\]

Then \((Y[n - 1])\) is a necessary condition in order that the linear equation \((D_{n-1})_o\) have at least one (nonoscillatory) solution \(x\) so that \(\lim_{t \to \infty} (D^{(n-1)}_l x)(t)\) exists in \(\mathbb{R} - \{0\}\).

### 3. A special case

Here we shall confine our discussion to the special case where

\[r_1 = \ldots = r_{n-1} = 1.
\]

In this case the differential equations \((E), (E_0), (D_l)\) and \((D_l)_o\) become

\[(\tilde{E}) \quad x^{(n)}(t) + F(t; x(g_0(t)), x'(g_1(t)), \ldots, x^{(i)}(g_i(t)))) = b(t),
\]

\[(\tilde{E}_0) \quad x^{(n)}(t) + F(t; x(g_0(t)), x'(g_1(t)), \ldots, x^{(i)}(g_i(t)))) = 0,
\]

\[(\tilde{D}_l) \quad x^{(n)}(t) + a_i(t)x^{(i)}[\sigma_i(t)] + \ldots + a_1(t)x'[\sigma_1(t)] + a_0(t)x[\sigma_0(t)] = b(t),
\]
and
\[(D_i(t) \rightArrow[14x541]x^{(n)}(t) + a_i(t)x^{(i)}[\sigma_i(t)] + \ldots + a_i(t)x'[\sigma_i(t)] + a_i(t)x[\sigma_i(t)] = 0, \]
respectively. Moreover, (Σ) is always satisfied and for every integer \(m\) with \(0 < m \leq n - 1\) the assumption (Σ[\(m\)]) holds by itself. Also, for any integers \(i\) and \(\lambda\) with \(0 \leq i \leq \lambda \leq n - 1\) we have
\[R_n(t) = \frac{1}{(\lambda - i)!} (t - t_0)^{\lambda - i}, \quad t \geq t_0.\]

In addition, for a nonnegative continuous function \(p\) on an interval of the form \([T, \infty), \quad T \geq t_0\), and for any integer \(m\) with \(0 \leq m < n - 1\) there holds
\[\int_{\lambda_{m+1}}^{\lambda_m} \frac{1}{r_{m+1}(s_{m+1})} \ldots \frac{1}{r_{n-1}(s_{n-1})} \int_{s_n}^{\infty} p(s) \, ds \, ds_{n-1} \ldots ds_{m+1} < \infty \quad \leftrightarrow \quad \int_{\lambda_0}^{\infty} t^{\lambda_n - m} p(t) \, dt < \infty.\]

Hence, applying Theorems 1.1 and 2.1 and Corollaries 1.2—1.3 and 2.1—2.2 to the special case considered, we derive the following results.

**Corollary 3.1.** Let \(m\) be an integer with \(l \leq m \leq n - 1\) and suppose that: (C[\(m\)])
\[\int_{\lambda_0}^{\infty} t^{\lambda_n - m} |F(t; \frac{c}{m!} (g_0(t))^m, \frac{c}{(m-1)!} (g_1(t))^{m-1}, \ldots, \frac{c}{(m-l)!} (g_{l}(t))^{m-l})| \, dt < \infty,\]
where
\[\langle g_i(t) \rangle^{m-i} = ([g_0(t)]^m, \ldots, [g_{n-1}(t)]^{m-i}) \quad (i = 0, 1, \ldots, l).\]

\[(B[\lambda]) \quad \int_{\lambda_0}^{\infty} t^{\lambda_n - m} |b(t)| \, dt < \infty.\]

Then for every real number \(L\) with \(Lc > 0\) and \(\frac{|c|}{2} < |L| < |c|\) there exists a (nonoscillatory) solution \(x\) of the differential equation (E) with
\[\left\{ \begin{array}{ll}
\lim_{t \to \infty} x^{(i)}(t) = (\text{sgn} \, L)_\infty & (i = 0, 1, \ldots, m - 1) \quad \text{when} \quad m > 0 \\
\lim_{t \to \infty} x^{(m)}(t) = m! \lim_{t \to \infty} \frac{x(t)}{t^m} \times L \\
\lim_{t \to \infty} x^{(0)}(t) = 0 & (j = m + 1, \ldots, n - 1) \quad \text{when} \quad m < n - 1.
\end{array} \right\}
\]

**Corollary 3.2.** Suppose that the function \(F\) has the sign property (I) or (II) and let \(m\) be an integer with \(l \leq m \leq n - 1\).
Then \( (\mathcal{C}[m]) \) is a necessary condition in order that the differential equation \((\mathcal{E}_0)\) has at least one (nonoscillatory) solution \(x\) so that \(\lim_{t \to \infty} x^{(m)}(t)\) exists in \(\mathbb{R} - \{0\}\).

**Corollary 3.3.** Let \(m\) be an integer with \(l \leq m \leq n - 1\) and suppose that \((\mathcal{B}[m])\) holds. Then we have:

\(\alpha\) If 
\[
(\mathcal{Y}[m]) \quad \int t^{n-1-m} |\sigma_i(t)|^{m-i} |a_i(t)| \, dt < \infty \quad (i = 0, 1, \ldots, l),
\]
then for every number \(L\) with \(L \neq 0\) there exists a (nonoscillatory) solution \(x\) of the linear equation \((\mathcal{D}_i)\) satisfying \((\mathcal{P}_m(x))\).

\(\beta\) If \(a_i \geq 0\) on \([t_0, \infty)\) \((i = 0, 1, \ldots, l)\) or \(a_i \leq 0\) on \([t_0, \infty)\) \((i = 0, 1, \ldots, l)\), then \((\mathcal{Y}[m])\) is a necessary condition in order that the linear equation \((\mathcal{D}_i)\) has at least one (nonoscillatory) solution \(x\) so that \(\lim_{t \to \infty} x^{(m)}(t)\) exists in \(\mathbb{R} - \{0\}\).

**Corollary 3.4.** Suppose that the condition \((\mathcal{B}[n - 1])\), i.e.
\[
(\mathcal{B}[n - 1]) \quad \int |b(t)| \, dt < \infty,
\]
holds. Then we have:

\(\alpha\) If \((\mathcal{Y}[n - 1])\), i.e. the condition 
\[
(\mathcal{Y}[n - 1]) \quad \int |\sigma_i(t)^{n-1-i} |a_i(t)| \, dt < \infty \quad (i = 0, 1, \ldots, n - 1),
\]
is satisfied, then for every \(L \neq 0\) there exists a (nonoscillatory) solution \(x\) of the linear equation \((\mathcal{D}_{n-1})\) which satisfies \((\mathcal{P}_{n-1}(x))\), i.e.
\[
(\mathcal{P}_{n-1}(x)) \begin{cases} 
\lim_{t \to \infty} x^{(i)}(t) = (\text{sgn} \, L)^{\infty} & (i = 0, 1, \ldots, n - 2) \\
\lim_{t \to \infty} x^{(n-1)}(t) = (n - 1)! \lim_{t \to \infty} \frac{x(t)}{t^{n-1}} = L.
\end{cases}
\]

\(\beta\) If \(a_i \geq 0\) on \([t_0, \infty)\) \((i = 0, 1, \ldots, n - 1)\) or \(a_i \leq 0\) on \([t_0, \infty)\) \((i = 0, 1, \ldots, n - 1)\), then \((\mathcal{Y}[n - 1])\) is a necessary condition in order that the linear equation \((\mathcal{D}_{n-1})\), i.e. the equation
\[
(\mathcal{D}_{n-1}) \quad x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}[\sigma_{n-1}(t)] + \ldots + a_i(t)x^{(i)}[\sigma_i(t)] + a_0(t)x[\sigma_0(t)] = 0,
\]
have at least one (nonoscillatory) solution \(x\) so that \(\lim_{t \to \infty} x^{(n-1)}(t)\) exists in \(\mathbb{R} - \{0\}\).
АСИМПТОТИЧЕСКОЕ ПОВЕДЕНИЕ ОДНОГО КЛАССА НЕКОЛЕБЛЮЩИХСЯ РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ОТКЛЮЧАЮЩИМИСЯ АРГУМЕНТАМИ

Ch. G. Philos

Резюме

В статье изучается асимптотическое поведение одного класса неколеблющихся решений дифференциальных уравнений n-го порядка (n > 1) с отклоняющимися аргументами. Эти уравнения содержат r-производные $D^{(r)}_0 x$ (i = 0, 1, ..., n) неизвестной функции x, определяемые следующим образом:

$$D^{(r)}_0 x = x, \quad D^{(r)}_i x = r(D^{(r-1)}_i x)' (i = 1, 2, ..., n - 1) \quad \text{и} \quad D^{(r)}_n x = (D^{(n-1)}_1 x)'.$$

где r, (i = 1, 2, ..., n - 1) — положительные непрерывные функции на интервале $[t_0, \infty)$.

Даются достаточные и необходимые и достаточные условия для существования хотя бы одного (неколеблющегося) решения x такого, что

$$\lim_{t \to \infty} (D^{(m)}_r x)(t)$$

существует в R - {0}, где m — целое, $1 \leq m \leq n - 1$.

Полученные результаты обобщают результаты, данные в [2] и [3].

REFERENCES


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