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ON JOINT DISTRIBUTIONS OF OBSERVABLES

ANATOLIJ DVUREČENSKIJ—SYLVIA PULMANNOVÁ

In the paper the joint distributions of an infinite set of observables on a logic are studied. In the special case of the Hilbert — space logic, the conditions of the existence of joint distributions for finite and infinite sets of observables are formulated.

Throughout the paper, the word logic means a $\sigma$-orthomodular lattice and the word state means a $\sigma$-additive probability measure on a logic. Both definitions of the latter notions together with some basic facts and physical interpretation can be found in [2]. The reader may also consult reference [2] for the basics on real observables. We shall deal with generalized observables, $\mathcal{X}$-observables, defined as follows. Let $\mathcal{X}$ be a complete separable metric space and $B(\mathcal{X})$ the $\sigma$-algebra of Borel sets of $\mathcal{X}$. An $\mathcal{X}$-observable is a map $x$ from $B(\mathcal{X})$ to a logic $L$ such that

1. $x(\emptyset) = 1$,
2. $x(E) \perp x(F)$ if $E \cap F = \emptyset$, $E, F \in B(\mathcal{X})$,
3. $x(\bigcup E_i) = \bigvee x(E_i)$ if $E_i \cap E_j = \emptyset$, $i \neq j$, $i, j = 1, 2, \ldots$.

We shall frequently use the following simple observation. If $f: \mathcal{X} \to \mathcal{X}$ is a Borel measurable mapping between two complete separable metric spaces, then $f^* x: E \to x(f^{-1}(E))$, $E \in B(\mathcal{X})$ is an $\mathcal{X}$-observable.

Obviously, we obtain the “traditional” observable if we set $\mathcal{X} = \mathbb{R}^1$ and $L$ is the logic $L(H)$ of all closed subspaces of a separable Hilbert space (real or complex). As known, there is a one-to-one correspondence between $\mathbb{R}^1$-observables and self-adjoint operators on $H$ [3].

Since we assume the space $\mathcal{X}$ to be fixed throughout the paper, we shall write simply an observable instead of an $\mathcal{X}$-observable.

Suppose we are given observables $x_1, x_2, \ldots, x_n: B(\mathcal{X}) \to L$ and a state $m$ on $L$. We say that the collection $x_1, \ldots, x_n$ has a joint distribution in the state $m$ if there is a probability measure $p$ on $B(\mathcal{X})$ such that

$$p(E_1 \times \ldots \times E_n) = m\left(\bigwedge_{i=1}^n x_i(E_i)\right)$$

for any $E_i \in B(\mathcal{X})$, $i = 1, 2, \ldots, n$. Evidently, such a measure $p$ is then unique and we may (and shall) denote by $p_{x_1}^x, \ldots, x_n$ the measure corresponding to the
collection \(x_1, \ldots, x_n\) and the state \(m\). This type of joint distributions was introduced by S. Gudder [4]. It is called the type-1 joint distribution.

We denote by \(\text{Com}(x_1, \ldots, x_n)\) for a collection \(x_1, \ldots, x_n\) of observables the set of all states on \(L\) in which the joint distribution exists. Let us recall a useful criterion for a state to belong to \(\text{Com}(x_1, \ldots, x_n)\) (c. f. [1]). In order to simplify the expressions, let us state first a few conventions. Put \(D = \{0, 1\}\) and denote by \(d_i\) the \(i\)-th coordinate of a point \(d \in D^n\), \(n \in \mathbb{N}\). Write \(E^d = E\) if \(d_i = 1\), and \(E^d = E_i = \mathbb{X} - E\) if \(d_i = 0\), for any \(E \in B(\mathbb{X})\). The criterion reads as follows. The observables \(x_1, \ldots, x_n\) have a joint distribution in a state \(m\) if

\[
\sum_{d \in D^n} m\left(\bigwedge_{i=1}^n x_i(E_i^d)\right) = 1
\]

for any \(E_1, E_2, \ldots, E_n \in B(\mathbb{X})\).

Let us set

\[
a(E_1, E_2, \ldots, E_n) = \bigvee_{d \in D^n} x_i(E_i^d),
\]

\(E_i \in B(\mathbb{X})\). One sees easily that criterion (2) can be reformulated as follows:

\[
m \in \text{Com}(x_1, \ldots, x_n) \text{ if } m(a(E_1, \ldots, E_n)) = 1
\]

for any \(E_1, \ldots, E_n \in B(\mathbb{X})\).

**Proposition 1.** (i) Let \(f_1, \ldots, f_n\) be Borel measurable functions from \(\mathbb{X}\) to \(\mathbb{X}_1\). If \(p_{x_1, \ldots, x_n}\) exists, then \(p_{f_1(x_1), \ldots, f_n(x_n)}\) exists as well and there holds

\[
p_{x_1, \ldots, x_n}(f_1^{-1}(E_1) \times \cdots \times f_n^{-1}(E_n)) = p_{f_1(x_1), \ldots, f_n(x_n)}(E_1 \times \cdots \times E_n)
\]

for any \(E_1, \ldots, E_n \in B(\mathbb{X}_1)\).

(ii) \(p_{x_1, \ldots, x_n}\) exists iff for any choice of real valued Borel measurable functions \(f_1, \ldots, f_n\) the observables \(f_1 \circ x_1, \ldots, f_n \circ x_n\) from \(B(\mathbb{R}^1)\) into \(L\) have the joint distribution in the state \(m\).

**Proof.** Part (i) and the necessary condition of (ii) are evident. Now let for all \(f_1, \ldots, f_n : \mathbb{X} \to \mathbb{R}^1\) the observables \(f_1 \circ x_1, \ldots, f_n \circ x_n\) have the joint distribution in \(m\). Let us put \(f_i = \chi_{E_i}, i = 1, 2, \ldots, n\), and let \(p\) be the joint distribution of \(f_1 \circ x_1, \ldots, f_n \circ x_n\). Then

\[
1 = p\{(0, 1) \times \{0, 1\} \times \cdots \times \{0, 1\}\} = \sum_{d \in D^n} p\{(d_1) \times \cdots \times (d_n)\} =
\]

\[
= \sum_{d \in D^n} m\left(\bigwedge_{i=1}^n x_i(\chi_{E_i}^{-1}(d_i))\right) = \sum_{d \in D^n} m\left(\bigwedge_{i=1}^n x_i(E_i^d)\right),
\]

which implies, by criterion (2), the the validity of (ii).

**Corollary 2.** If \(x_1, \ldots, x_n\) are mutually compatible observables, then they have a joint distribution in any state.
Proof. Suppose that \( f_1, \ldots, f_n \) are Borel functions. If \( x_1, \ldots, x_n \) are mutually compatible observables, then \( f_1 \circ x_1, \ldots, f_n \circ x_n \) are mutually compatible real observables. By a result of Varadarajan [2], they have a joint distribution in any state and the rest follows from Proposition 1 (ii).

Let us recall another standard notion. A sequence \( m_n \) of states converges weakly to a state \( m \) if \( m_n(a) \to m(a) \) for any \( a \in L \).

The carrier of a state \( m \) (if it exists) is an element \( a \in L \) such that \( m(b) = 0 \) iff \( a \perp b \).

A state \( m_0 \) is a superposition of a collection \( M \) of states iff \( m(a) = 0 \) for every \( m \in M \) implies \( m_0(a) = 0 \).

The following lemma is an easy consequence of criterion (4).

**Lemma 3.** Let \( a \) be the carrier of the state \( m \). Then \( m \in \text{Com}(x_1, \ldots, x_n) \) iff \( a \equiv a(E_1, \ldots, E_n) \) for any \( E_1, \ldots, E_n \in \mathcal{B}(\mathcal{X}) \).

**Proposition 4.** Let \( m_i \) be a state with 1 for the carrier. Then \( x_1, \ldots, x_n \) are mutually compatible observables iff \( m_i \in \text{Com}(x_1, \ldots, x_n) \).

**Proof.** If \( x_1, \ldots, x_n \) are mutually compatible, then clearly \( m_i \in \text{Com}(x_1, \ldots, x_n) \). Conversely, let \( m_i \in \text{Com}(x_1, \ldots, x_n) \). Then \( m_i \in \text{Com}(x_i, x_j) \) for any \( 1 \leq i, j \leq n \), and, according to Lemma 3, we have

\[
1 = x_i(E) \wedge x_j(F) \vee x_i(E)^{\perp} \wedge x_j(F) \vee x_i(E) \wedge x_j(F)^{\perp} \wedge x_i(E)^{\perp} \wedge x_j(F)^{\perp}
\]

for any \( E, F \in \mathcal{B}(\mathcal{X}) \), and this identity yields that \( x_i \leftrightarrow x_j \) for any \( i, j = 1, 2, \ldots, n \) (c. f. [2]).

**Proposition 5.** Let \( 0 \neq M \subset \text{Com}(x_1, \ldots, x_n) \). If \( m_0 \) is a superposition of the states of \( M \), then \( m_0 \in \text{Com}(x_1, \ldots, x_n) \).

**Proof.** Let \( m \in M \). Then the equality \( m(a(E_1, \ldots, E_n)^{\perp}) = 0 \) implies \( m_0(a(E_1, \ldots, E_n)^{\perp}) = 0 \) and therefore \( m_0 \in \text{Com}(x_1, \ldots, x_n) \).

The next two propositions result from Proposition 5 with regard to the following observations: (i) a state \( m_a \) with the carrier \( a \) is a superposition of a state \( m_b \) with the carrier \( b \) iff \( a \leq b \), (ii) the state \( m = \sum c_i m_i \), where \( \sum c_i = 1 \), \( 0 \leq c_i \leq 1 \), is a superposition of the states \( \{m_i\}_{i=1}^n \), and every \( m_i \) is a superposition of the state \( m \).

**Proposition 6.** Let \( m_i, i = 1, 2, \ldots, m_c \) be states with the carriers \( a_i, i = 1, 2, \ldots \), and \( c \), respectively, and let \( c \leq \bigvee a_i \). If \( m_i \in \text{Com}(x_1, \ldots, x_n) \) for \( i = 1, 2, \ldots \), then \( m_c \in \text{Com}(x_1, \ldots, x_n) \).

**Proposition 7.** Let \( m = \sum c_i m_i \), \( c_i > 0 \), \( \sum c_i = 1 \). Then \( m \in \text{Com}(x_1, \ldots, x_n) \) iff \( m_i \in \text{Com}(x_1, \ldots, x_n) \) for any \( i \).
Theorem 8. \( \text{Com}(x_1, \ldots, x_n) \) is a \( \sigma \)-convex sequentially weakly complete subspace in the space of all states of \( L \).

Proof. By Proposition 7, \( \text{Com}(x_1, \ldots, x_n) \) is \( \sigma \)-convex. If \( \{m_i\} \) is a Cauchy sequence in the weak topology, then due to [5], Th. 2.2, the formula \( m(a) = \lim m_i(a) \) defines a state on \( L \). Therefore there is \( \lim p,(E_1 \times \ldots \times E_n) = p(E_1 \times \ldots \times E_n) = m \left( \bigwedge_{i=1}^{n} x_i(E) \right) \) and, consequently, there is \( \lim p_i(A) = p(A) \) for any \( A \) of the algebra generated by all the rectangle sets. As \( m(a(E_1 \times \ldots \times E_n)) = \lim m_i(a(E_1, \ldots, E_n)) = 1 \), we obtain \( m \in \text{Com}(x_1, \ldots, x_n) \) by criterion (4).

Now let \( L(H) \) be the logic of all closed subspaces of a Hilbert space \( H \) (real or complex) with an inner product \((\cdot,\cdot)\) and let \( x_1, \ldots, x_n : B(\mathcal{X}) \to L(H) \) be observables. For \( M \in L(H) \), let us put \( P^M \) for the projector onto \( M \). There is a one-to-one correspondence between the elements of \( L(H) \) and their projections. If \( \varphi \in H \) is a unit vector, then \( m_\varphi : M \mapsto (P^M \varphi, \varphi) \), \( M \in L(H) \), is a state on \( L(H) \). Moreover, the Gleason theorem asserts that any state on \( L(H) \), \( 3 \leq \dim H \leq \aleph_0 \), is of the form \( m(M) = \text{tr}(TP^M) \), \( M \in L(H) \), where \( T \) is the density operator. In the sequel we suppose that \( 3 \leq \dim H \leq \aleph_0 \).

Theorem 9. The observables \( x_1, \ldots, x_n \) on the logic \( L(H) \) have a joint distribution in a state \( m = m_\varphi \) iff

\[
\left( P^x(E_1) \right) \left( P^x(E_2) \right) \ldots \left( P^x(E_n) \right) \varphi = P^{x_{i_1}(E_{i_1})} \ldots P^{x_{i_n}(E_{i_n})} \varphi
\]

for every \( E_1, \ldots, E_n \in B(\mathcal{X}) \) and every permutation \( (i_1, \ldots, i_n) \) of the set \( \{1, 2, \ldots, n\} \).

The proof of the latter theorem follows from the next lemmas.

Lemma 10. Let \( \varphi \) be an arbitrary element of \( H \). If (6) holds for \( x_1, \ldots, x_n \) and for \( \varphi \), then

\[
\left( P^x(E_1) \right) \left( P^x(E_2) \right) \ldots \left( P^x(E_n) \right) \varphi = P^{x_{i_1}(E_{i_1})} \ldots P^{x_{i_n}(E_{i_n})} \varphi
\]

for every \( E_1, \ldots, E_n \in B(\mathcal{X}) \).

Proof. It is known that the equality \( P^M P^N \varphi = P^N P^M \varphi \) implies \( P^{M \wedge N} \varphi = P^M P^N \varphi \). The rest is an elementary induction.

Lemma 11. If (6) holds for \( x_1, \ldots, x_n \) and \( \|\varphi\| = 1 \), then \( x_1, \ldots, x_n \) have a joint distribution in the state \( m = m_\varphi \).

Proof. By the property (7) we have

\[
\sum_{d \in D^*} m \left( \bigwedge_{i=1}^{n} x_i(E_i^d) \right) = \sum_{d \in D^*} \left( P^x(E_1^d) \ldots P^x(E_n^d) \varphi, \varphi \right) = \sum_{d \in D^*} \left( P^{x_{i_1}(E_{i_1})} \ldots P^{x_{i_n}(E_{i_n})} \varphi, \varphi \right) = (\varphi, \varphi) = 1.
\]
The criterion (1) implies that $x_1, ..., x_n$ have a joint distribution in the state $m = m_T$.

**Lemma 12.** ([4], Lemma 3.5). Let $\{\varphi_i\}$ be an orthonormal set of vectors in $H$. If a vector $\varphi$ satisfies $\|\varphi\|^2 = \sum_i |(\varphi, \varphi_i)|^2$, then $\varphi = \sum_i (\varphi, \varphi_i)\varphi_i$.

**Lemma 13.** Let $M_1, ..., M_n \in L(H)$ and let $d \in D^n$, $0 \neq \varphi \in \bigwedge_i M_i^d$, where $M_i^d = M$ if $d_i = 1$ and $M_i^d = M^\perp$ if $d_i = 0$. Then

$$P^{M_{i_1}} \ldots P^{M_{i_n}} \varphi = P^{M_{i_1}} \ldots P^{M_{i_n}} \varphi$$

for every permutation $(i_1, ..., i_n)$ of $(1, 2, ..., n)$.

**Proof.** We have $P^{M_{i_1}} \ldots P^{M_{i_n}} \varphi = \varphi$ iff $d_i = 1$, $i = 1, ..., n$; otherwise $P^{M_{i_1}} \ldots P^{M_{i_n}} \varphi = 0$. The proof follows immediately.

**Proof of Theorem 9.** The sufficient condition was proved in Lemma 11.

The necessary condition. Let $E_1, ..., E_n \in B(\mathcal{X})$ be given. By criterion (1) we have

$$\sum_{d \in D^n} \left( \bigwedge_{i=1}^n P^{M_{E_{i_1}}} \varphi, \varphi \right) = 1. \quad (9)$$

Let us set $M(d) = \bigwedge_{i=1}^n x_i(E_{i_1}^d)$, $d \in D^n$. The vectors $\{P^{M(d)} \varphi: d \in D^n\}$ are orthogonal and the equality (9) yields

$$\|\varphi\|^2 = 1 = \sum_{\{d \in D^n: M(d)\varphi \neq 0\}} \left| \left( \varphi, \frac{P^{M(d)} \varphi}{\|P^{M(d)} \varphi\|} \right) \right|^2.$$

We see by Lemma 12 that the vector $\varphi$ is a linear combination of vectors $P^{M(d)} \varphi \in \bigwedge_{i=1}^n x_i(E_{i_1}^d)$ and by Lemma 13 that the condition (6) is satisfied.

We recall that any density operator can be written in the form $T = \sum_{i=1}^\infty c_i P^{(\varphi_i)}$ for some partition of unity $\{c_i: i \in N\}$ and an orthonormal system $\{\varphi_i: i \in N\}$, where $P^{(\varphi_i)}: \varphi \mapsto (\varphi, \varphi_i)\varphi_i$ is the projector on the subspace generated by the vector $\varphi_i$.

**Theorem 14.** Let $x_1, ..., x_n$ be observables on the logic $L(H)$. Let $T: \varphi \mapsto \sum_i c_i (\varphi, \varphi_i)\varphi_i$ be a density operator. Let $m = m_T: M \mapsto \text{tr}(P^M T)$ be the state induced by $T$. Then

(i) $x_1, ..., x_n$ have a joint distribution in the state $m = m_T$ iff the condition (6) holds for any $\varphi_i$.  


(ii) if $x_1, \ldots, x_n$ are bounded real observables and $X_1, \ldots, X_n$ are corresponding self-adjoint operators on $H$, then the following conditions are equivalent

(a) $x_1, \ldots, x_n$ have a joint distribution in the state $m = m_T$,
(b) $P_{x_1(E_1)} \ldots P_{x_n(E_n)} \varphi_i = P_{x_1(E_1)} \ldots P_{x_n(E_n)} \varphi_i$,
(c) $X_1 \ldots X_n \varphi_i = X_{i_1} \ldots X_{i_n} \varphi_i$,
(d) $X_1 \ldots X_n T = X_{i_1} \ldots X_{i_n} T$

for any $i = 1, 2, \ldots$, $E_1, \ldots, E_n \in B(R^1)$ and any permutation $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$.

Proof. The statement (i) follows from Theorem 9 and Proposition 7. Using the properties of spectral measures of $X_1, \ldots, X_n$ and Theorem 9 we obtain (ii).

Theorem 15. Let $x_1, \ldots, x_n$ be observables on the logic $L(H)$. The set $H_0$ of all vectors $\varphi \in H$ for which (6) holds is a closed subspace of $H$ which is compatible with any $x_i(E)$, that is, $P_{x_i(E)} P_{x_i(E)}^* = P_{x_i(E)} P_{x_i(E)}^*$ for any $E \in B(\mathcal{A})$, $i = 1, 2, \ldots, n$. Moreover,

$$H_0 = \bigcap_{(E_1, \ldots, E_n)} a(E_1, \ldots, E_n). \tag{10}$$

The state $m = m_T : M \mapsto \text{tr}(TP^M)$ belongs to $\text{Com}(x_1, \ldots, x_n)$ iff the eigenvectors of $T$ belong to $H_0$.

Proof. It is easy to see that $H_0$ is a closed subspace of $H$. Now we show that

$$P_{x_i(E)} P_{x_i(E)}^* = P_{x_i(E)} P_{x_i(E)}^*,$$

or, equivalently,

$$P_{x_i(E)} \varphi \in H_0,$$

whenever $\varphi \in H_0$, $j = 1, 2, \ldots, n$.

Let $(i_1, \ldots, i_n)$ be a permutation of $(1, 2, \ldots, n)$ and $x_j$ be an observable. Let the integer $j$ be at the $s$-th place, $1 \leq s \leq n$. We have

$$P_{x_i(E_1)} \ldots P_{x_{i_n}(E_n)} \varphi =$$

$$= P_{x_i(E_1)} \ldots P_{x_{i_{s-1}}(E_{i_{s-1}})} [P_{x_{i_s}(E_{i_s})} \ldots P_{x_{i_n}(E_n)} \varphi] =$$

$$= P_{x_i(E_1)} \ldots P_{x_{i_{s-1}}(E_{i_{s-1}})} P_{x_{i_s}(E_{i_s})} \varphi =$$

$$= P_{x_i(E_1)} \ldots P_{x_{i_{s-1}}(E_{i_{s-1}})} P_{x_{i_s}(E_{i_s})} \varphi.$$

Similarly,

$$P_{x_i(E_1)} \ldots P_{x_{i_n}(E_n)} P_{x_i(E_s)} \varphi = P_{x_i(E_1)} \ldots P_{x_{i_{s-1}}(E_{i_{s-1}})} P_{x_{i_s}(E_{i_s})} \varphi,$$

and therefore $P_{x_i(E_i)} \varphi \in H_0$.

160
Now if \( \varphi \in H_0 \), then by Theorem 9 the joint distribution of \( x_1, \ldots, x_n \) in the state \( m_\varphi \) exists, hence \( m_\varphi(a(E_1, \ldots, E_n)) = 1 \), \( E_1, \ldots, E_n \in B(\mathcal{H}) \). From this it follows that \( \varphi \in a(E_1, \ldots, E_n) \) for any \( E_1, \ldots, E_n \in B(\mathcal{H}) \). On the other hand, if \( \varphi \in \bigwedge_{E_1, \ldots, E_n} a(E_1, \ldots, E_n) \), then the joint distribution in the state \( m_\varphi \) exists by the criterion (4), therefore \( \varphi \in H_0 \). The last statement of the theorem follows from Theorem 14.

In the following theorem we shall derive a connection between the considered type of joint distribution and another type of joint distributions - the so-called type 2 joint distribution on the logic \( L(H) \) (see [4]). For the sake of simplicity we shall consider only two observables. Before stating the theorem we have to recall some definitions and results. For details see [4] and [7].

If \( x \) is a real observable on \( L(H) \), we write \( X \) for the corresponding self-adjoint operator on \( H \).

If \( X \) and \( Y \) are linear operators on \( H \) with the domains \( D(X), D(Y) \), respectively, then their sum \( X + Y \) is an operator defined on \( D(X) \cap D(Y) \) such that \( (X + Y) \varphi = X\varphi + Y\varphi \) for any \( \varphi \in D(X) \cap D(Y) \).

We say that the real observables \( x \) and \( y \) on the logic \( L(H) \) have \textit{type 2 joint distribution} in a state \( m \) if the observables \( \alpha x + \beta y \) exist (i.e. \( \alpha X + \beta Y \) are self-adjoint operators) for any \( \alpha, \beta \in \mathbb{R} \) and if there is a measure \( \mu \) on \( B(\mathbb{R}^2) \) such that

\[
\mu\{(\omega_1, \omega_2): \alpha \omega_1 + \beta \omega_2 \in E\} = m((\alpha x + \beta y)(E))
\]  

for any \( E \in B(\mathbb{R}^2) \) (see [4]).

A subspace \( H' \) of \( H \) is \textit{invariant} under an operator \( X \) (or \( H' \) reduces \( X \)) if

\(\begin{align*}
(1) \quad & X(H' \cap D(X)) \subset H', \\
(2) \quad & P_{H'}^X D(X) \subset D(X).
\end{align*}\)

The condition (1) is equivalent to

\[
P_{H'}^X X P_{H'} = X P_{H'}^X
\]

A subspace \( H' \) \textit{orthogonally reduces} the operator \( X \) if both \( H' \) and \( H'^\perp \) are invariant under \( X \) (see [7], p. 349).

\textbf{Proposition 16.} A subspace \( H' \) of \( H \) orthogonally reduces \( X \) iff \( X P_{H'} = P_{H'}^X \) (i.e., iff \( X \) commutes with \( P_{H'}^X \)) ([7], p. 352).

\textbf{Proposition 17.} Any invariant subspace of a self-adjoint operator \( X \) orthogonally reduces \( X \) ([7], p. 355).

\textbf{Proposition 18.} The operator \( X' \) induced by a self-adjoint operator \( X \) in his invariant subspace \( H' \) is self-adjoint ([7], p. 365).
**Proposition 19.** A bounded operator $Y$ commutes with a self-adjoint operator $X$ (i.e. $XY = YX$) iff it commutes with the spectral measure of the operator $X$ ([7], p. 521).

**Theorem 20.** Let $x$ and $y$ be real observables on the logic $L(H)$ and let $X$ and $Y$ be the corresponding self-adjoint operators on $H$. Let the domains $D(X)$ and $D(Y)$ be such that $D(X) \cap D(Y)$ is dense in $H$. Then for any $m \in \text{Com}(x, y)$ there exists the type 2 joint distribution of $x, y$ in the state $m$ and this type 2 joint distribution is identical with the type 1 joint distribution.

**Proof.** As $D(X) \cap D(Y)$ is dense in $H$, the linear combinations $aX + \beta Y$ exist for any $a, \beta \in \mathbb{R}$. According to Proposition 7, it satisfies to consider a state $m = \rho$, where $\rho$ is a unit vector in $H$. Let $H_0 \in L(H)$ be defined by (10). Then $\rho \in H_0$ and the subspace $H_0$ is invariant under the operators $X$ and $Y$ (see Theorem 15 and Propositions 16, 17 and 19). The restrictions of $X$ and $Y$ to $H_0$, denoted by $X_0$ and $Y_0$, are self-adjoint operators on $H_0$ which may be treated as a Hilbert space in its own right (see Proposition 18). We show that the subspace $H_0$ is invariant under the operator $aX + \beta Y(a, \beta \in \mathbb{R})$. Indeed, we have

$$P^{H_0}(aX + \beta Y)P^{H_0} = P^{H_0}aXP^{H_0} + P^{H_0}\beta YP^{H_0} = aXP^{H_0} + \beta YP^{H_0} = (aX + \beta Y)P^{H_0}$$

and so the condition (1) in (12) is fulfilled. From the inclusions

$$P^{H_0}D(X) \subset D(X), \quad P^{H_0}D(Y) \subset D(Y)$$

we obtain

$$P^{H_0}D(X) \cap D(Y) \subset D(X) \cap D(Y)$$

and this means that (2) in (12) is fulfilled. Then restriction $(aX + \beta Y)_0$ of $aX + \beta Y$ to $H_0$ is a self-adjoint operator and, clearly, $(aX + \beta Y)_0 = aX_0 + \beta Y_0$. As $X_0$ and $Y_0$ commute, there is the type 2 joint distribution (identical with the type 1 joint distribution (see [4])) for the observables $x_0$ and $y_0$ in the state $m_\rho$, in other words, there is a measure $\mu$ on $B(R^2)$ such that

$$\mu((\omega_1, \omega_2) : a\omega_1 + \beta \omega_2 \in E) = m_\rho((a\omega_0 + \beta \omega_0)(E)),$$

for any $a, \beta \in \mathbb{R}$ and any $E \in B(R^1)$. However,

$$m_\rho((a\omega_0 + \beta \omega_0)(E)) = (P^{(aX_0 + \beta Y_0)(E)} \varphi, \varphi) = (P^{(aX + \beta Y)(E)} \varphi, \varphi) = m_\rho((aX + \beta Y)(E)).$$

From this we see that the type 2 joint distribution in the state $m_\rho$ exists. On the other hand,

$$\mu(E \times F) = m_\rho(x_0(E) \wedge y_0(F)) = (P^{x_0(E) \wedge y_0(F)} \varphi, \varphi) = \mu(E \times F).$$
= (P^{x(E)} P^{y(E)} \varphi, \varphi) = (P^{x(E)} P^{y(E)} P^{x(E)} P^{y(E)} \varphi, \varphi) = \\
= (P^{x(E)} P^{y(E)} \varphi, \varphi) = (P^{x(E)} \wedge y(E) \varphi, \varphi) = m_o(x(E) \wedge y(F)).

Hence, the two types of joint distributions, being identical for the observables \( x_0 \) and \( y_0 \) on \( L(H_0) \), are identical also for \( x \) and \( y \).

The notion of the joint distribution of observables can be generalized to an arbitrary system of observables \( \{x_s : s \in S\} \). We say that a system of observables \( \{x_s : s \in S\} \) has the joint distribution in a state \( m \) if any its finite subsystem has one.

Now let a system of observables \( \{x_s : s \in S\} \) on a separable logic \( L \) be given. We recall that a logic \( L \) is \textit{separable} if any subsystem of mutually orthogonal elements of \( L \) is at most countable. For any finite set \( \emptyset \neq F = \{s_1, \ldots, s_n\} \in S \) we put

\[
a(F; E_1, \ldots, E_n) = \bigvee_{d \in D^n} \bigwedge_{i=1}^n x_i(E_{d_i}^i),
\]

where \( D = \{0, 1\} \), \( d = (d_1, \ldots, d_n) \in D^n \), \( E_{d_i}^i = E \) if \( d_i = 1 \) and \( E_{d_i}^i = E^c \) if \( d_i = 0 \),

\[
a_o(F) = \bigwedge_{E_1, \ldots, E_n} a(F; E_1, \ldots, E_n)
\]

(16)

(17)

\textbf{Lemma 21.} Let \( L \) be a separable logic and let \( \{x_s : s \in S\} \) be a given system of observables. Then for finite sets \( F_1, F_2 \subset S, \emptyset \neq F_1 \subset F_2 \) we have

\[
a_o(F_2) \leq a_o(F_1).
\]

(18)

\textbf{Proof.} Let \( F_1 = \{s_1, \ldots, s_n\} \), \( F_2 = \{s_1, \ldots, s_n, s_{n+1}, \ldots, s_m\} \). Then \( a(F_2; E_1, \ldots, E_m, 0, \ldots, 0) = a(F_1; E_1, \ldots, E_n) \) and therefore

\[
a_o(F_2) = \bigwedge_{E_1, \ldots, E_m} a(F_2; E_1, \ldots, E_m) \leq \bigwedge_{E_1, \ldots, E_n} a(F_2; E_1, \ldots, E_n, 0, \ldots, 0) = \\
= \bigwedge_{E_1, \ldots, E_n} a(F_1; E_1, \ldots, E_n) = a_o(F_1).
\]

By Zierler [6] there is in a separable logic to any system of elements \( \{a_s\}_a \) a countable subsystem \( \{a_s\}_a \), such that

\[
\bigvee_{s} a_s = \big\vee_{i=1}^{\infty} a_s(\bigwedge_{i=1}^{\infty} a_s = \bigwedge_{i=1}^{\infty} a_s).
\]

\textbf{Theorem 22.} A system of observables \( \{x_s : s \in S\} \) on a separable logic has the joint distribution in a state \( m \) iff \( m(a_o) = 1 \).

\textbf{Proof.} If the given system of observables has the joint distribution in the state \( m \), then for any finite subset \( F = \{s_1, \ldots, s_n\} \subset S \) we have \( m \in \text{Com}_o(x_{s_1}, \ldots, x_{s_n}) \). Therefore Theorem 2.7 in [1] implies that \( m(a_o(F)) = 1 \). Due to the separability of
There is a sequence of finite subsets \( \{F_n\}_n \) of \( S \) such that \( a_0 = \bigwedge_n a_0(F_n) \). Now we show that

\[
m\left( \bigwedge_{i=1}^{n} a_0(F_i) \right) = 1
\]

for any \( n \). Indeed, if we put \( F_n^* = \bigcup_{i=1}^{n} F_i \), then \( a_0(F_n^*) \leq a_0(F_i), \ i = 1, \ldots, n; \ n = 1, 2, \ldots \). As \( m(a_0(F_n^*)) = 1 \), we obtain \( m(a_0) = \lim_{n} m\left( \bigwedge_{i=1}^{n} a_0(F_i) \right) = 1 \). Since \( a_0 \leq a_0(F), \ F \subset S \), the converse implication follows from Theorem 2.7 in [1].

The following theorem is a generalization of Theorem 2.11 in [1] for the case of an infinite set of observables.

**Theorem 23.** Let \( L \) be a separable logic and let there be, for any \( a \neq 0 \), a state \( m_a \) with the carrier \( a \). If the system of observables \( \{x_s : s \in S\} \) has the joint distribution in the state \( m \), then \( a_0 \neq 0 \) and \( x_s(E) \leftrightarrow a_0 \) for any \( E \in B(\mathcal{X}) \) and \( s \in S \). Moreover, all the observables \( x_{a_0} = x_s \wedge a_0 \) are compatible on the logic \( L_{[0, \infty]} = \{b \in L : b \leq a_0\} \).

For the element \( a_0 \) we have

\[
a_0 = \bigvee \{a \in L : \text{there is the joint distribution of } \{x_s : s \in S\} \text{ in } m_a\}.
\]  

(19)

**Proof.** Since \( m(a_0) = 1 \), it is clear that \( a_0 \neq 0 \). There is a sequence of finite subsets \( \{F_n\}_n \) of \( S \) such that \( a_0 = \bigwedge_n a_0(F_n) \). Suppose \( s \in S \). We shall show that \( x_s(E) \leftrightarrow a_0 \) for any \( E \in B(\mathcal{X}) \). For the indexed set \( F'_n = F_n \cup \{s\} \) we have \( a_0(F'_n) \leq a_0(F_n) \) (Lemma 16). Therefore \( a_0 \supseteq \bigwedge_n a_0(F'_n) \), but on the other hand \( a_0 = \bigwedge_{F \subset S} a_0(F) \). Hence, \( a_0 = \bigwedge_n a_0(F_n) \). Theorem 2.11 in [1] implies that \( x_s(E) \leftrightarrow a_0(F_n) \) for any \( E \in B(\mathcal{X}) \) and any \( n \). Therefore \( x_s(E) \leftrightarrow \bigwedge_n a_0(F_n) = a_0 \) (see [2], Lemma 6.10).

Now it is easy to show that \( x_{a_0} : E \mapsto x_s(E) \wedge a_0, \ E \in B(\mathcal{X}) \) is an observable on the logic \( L_{[0, \infty]} \). We claim that \( \{x_{a_0} : s \in S\} \) are compatible observables on \( L_{[0, \infty]} \). Since \( m_{a_0}(a_0) = 1 \), we have (Theorem 22) that \( \{x_s : s \in S\} \) has the joint distribution in the state \( m_{a_0} \). Therefore, for any \( s, t \in S \)

\[
m_{a_0}(x_s(E) \wedge x_t(F) \vee x_s(E)^\perp \wedge x_t(F) \vee x_s(E)^\perp \wedge x_t(F)^\perp \wedge x_s(E)^\perp \wedge x_t(F)^\perp)) = 1
\]

(20)

for any \( E, F \in B(\mathcal{X}) \).
As $x_i(E) \land x_i(F) \leftrightarrow a_0$ for $E, F \in B(\mathcal{B})$, we have

$$m_{a_0}(x_i(E) \land x_i(F)) = m_{a_0}(x_i(E) \land x_i(F) \land a_0) + m_{a_0}(x_i(E) \land x_i(F) \land a_0^\perp) =$$

$$= m_{a_0}(x_i(E) \land x_i(F) \land a_0).$$

The latter fact and (20) imply that

$$m_{a_0}(x_{i_0}(E) \land x_{i_0}(F) \lor x_{i_0}(E)^\perp \land x_{i_0}(F) \lor x_{i_0}(E)^\perp \lor x_{i_0}(F)^\perp \lor x_{i_0}(E)^\perp \lor x_{i_0}(F)^\perp) = 1$$

for any $E, F \in B(\mathcal{B})$, where $\tilde{m}_{a_0} = m_{a_0}/L_{[0, a_0]}$. By the criterion (4) and Proposition 4, $x_{i_0} \leftrightarrow x_{i_0}$.

The equality (19) follows from the observation that the system $\{x_s: s \in S\}$ has the joint distribution in the state $m_\circ$ iff $m_\circ(a_0) = 1$ or, equivalently, iff $a \leq a_0$.

For the case of the logic $L(H)$ the following theorem holds.

**Theorem 24.** Let $L(H)$ be the logic of all closed subspaces of a separable Hilbert space $H$ and let $\{x_s: s \in S\}$ be a system of observables. Then it has the joint distribution in a state $m = \sum c_j m_{\Phi_j}$, $c_j \geq 0$, $\sum c_j = 1$, $\{\Phi_j\}$, an orthogonal system of vectors, iff

$$P_{x_{i_0}^{(B_1)}} \cdots P_{x_{i_n}^{(B_1)}} \Phi_j = P_{x_{i_1}^{(B_1)}} \cdots P_{x_{i_n}^{(B_1)}} \Phi_j$$

(21)

for any permutation $(i_1, \ldots, i_n)$ of $(1, 2, \ldots, n)$; $E_1, \ldots, E_n \in B(\mathcal{B})$ and any finite subsystem of observables and any vector $\Phi_j$. If we put $H_0 = \{\Phi: \Phi \in H, \Phi \text{ fulfills } (21)\}$, then $H_0 \in L(H)$. Moreover, $H_0$ is the element defined by (17) and it reduces the observables $\{x_s: s \in S\}$.

Finally, let us note that an analogical division into three categories of compatibilities as in [1] may be done for the system of observables $\{x_s: s \in S\}$: Let for the system $\{x_s: s \in S\}$ the element $a_0$ be defined by (17) ($L$ is a separable logic). We may say that the system $\{x_s: s \in S\}$ of observables is (i) compatible if $a_0 = 1$, (ii) partially compatible if $0 \neq a_0 \neq 1$, (iii) totally incompatible if $a_0 = 0$. Further investigation may then proceed similarly to that of [1].
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ЗАМЕЧАНИЕ О СОВМЕСТНОМ РАСПРЕДЕЛЕНИИ ВЕРОЯТНОСТИ НАБЛЮДАЕМЫХ

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Резюме

В этой статье исследуются критерии для существования совместного распределения вероятности наблюдаемых на логике и их следствия. Полученные результаты обобщаются для случая бесконечной системы наблюдаемых.