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WEAKLY REGULAR LATTICES

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In [1] G. Birkhoff proposed the following problem: to find necessary and sufficient conditions under which there exists a one-to-one correspondence between congruences and ideals of a lattice. More precisely, we have three problems: under which conditions, on a lattice L, each ideal of L is a kernel of

(a) at least one congruence on L;

(b) at most one congruence on L;

(c) just one congruence on L.

Problems (a) and (c) were solved by J. Hashimoto [6]. It is well known that (a) is equivalent to the distributivity of L and (c) is equivalent to the following condition:

(c*) L is a distributive, relatively complemented lattice with zero element.

As far as (b) is concerned, it is not too complicated to prove that each lattice L satisfying (b) has a zero (i.e. a least) element 0, see [5]. Use the terminology of [4]:

An algebra $\mathfrak{A}$ with a constant 0 is *weakly regular* if each two congruence Φ, $\Psi \in Con(\mathfrak{A})$ coincide whenever they have the same congruence class containing 0.

If L is a lattice with 0, the congruence class containing 0 is clearly an ideal of L. Thus (b) is equivalent to characterizing weakly regular lattices.

Sufficient conditions for the weak regularity of lattices were proved in [1]:

Proposition 1. *A lattice L with zero* 0, *where all intervals* $[0, a]$ *are complemented, is weakly regular.*

A full characterization of weakly regular lattices is given in [5]. Recall (Definition 1 in [5]) that for elements a, b, c, d of a lattice L, the pair a, b is weakly projective into the pair c, d (in symbols $\overline{a, b} \to \overline{c, d}$) if

$$c \vee d = [\ldots(\{[(a \vee b) \vee x_1] \wedge x_2\} \vee x_3) \wedge \ldots] \vee x_n$$
$$c \wedge d = [\ldots(\{[(a \wedge b) \vee x_1] \wedge x_2\} \vee x_3) \wedge \ldots] \vee x_n$$

for some $x_1, \ldots, x_n$ of L. A lattice L with zero 0 is *weakly complemented* if to all pairs of elements a, b there exists an element $c \in L$ such that $\overline{a, b} \to \overline{c, 0}$.

Proposition 2. (Theorem 4 in [5]). *L is weakly regular if and only if L has zero* 0 *and every homomorphic immage of L is weakly complemented.*

The disadvantage of this characterization is that we must investigate all homomorphic images. The aim of this paper is to characterize weakly regular lattices in terms of principal congruences. Denote by $\theta(J)$ the least congruence of a lattice L collapsing the set J.

Lemma 1. *A lattice L with zero 0 is weakly regular if and only if for each pair of elements a, b there exists an ideal $J=[0]_{\theta(a,b)}$ such that $\theta(a,b)=\theta(J)$.*

Proof. If L is weakly regular, then clearly the ideal $J=[0]_{\theta(a,b)}$ is the congruence class of just one congruence, namely $\theta(J)$, whence $\theta(a,b)=\theta(J)$.

Conversely, let $\langle x, y\rangle \in \theta \in Con(L)$. By the assumption there exists an ideal $J_{x,y}=[0]_{\theta(x,y)}$ such that

$$\theta(x, y)=\theta(J_{x,y}).$$

Hence

$$\theta(J_{x,y})\subseteq \theta.$$

Denote $J=[0]_\theta$. Clearly $\theta(x,y)\subseteq\theta$ implies $J_{x,y}\subseteq J$, thus

$$\bigvee\{J_{x,y};\ \langle x, y\rangle \in \emptyset\}\subseteq J$$

in the ideal lattice of L. Hence

$$\theta=\bigvee\{\theta(x, y);\ \langle x, y\rangle\in\theta\}=\bigvee\{\theta(J_{x,y});\ \langle x, y\rangle \in\theta\}\subseteq\theta(J),$$

i.e. $\theta=\theta(J)$, proving the weak regularity of L.

Lemma 2. *Let L be a lattice with 0. The following conditions are equivalent:*
(A) *for each a, b of L there exists an ideal $J=[0]_{\theta(a,b)}$ such that $\theta(a,b)=\theta(J)$;*
(B) *for each a, b of L there exists an element $c\in L$ with $\theta(a,b)=\theta(c,0)$.*

Proof. (B) $\Rightarrow$ (A) is trivial if we put $J=(c]$. Prove (A) $\Rightarrow$ (B). Let $\theta(a,b)=\theta(J)$ for the ideal $J=[0]_{\theta(a,b)}$. By Lemma 1, L is weakly regular, thus $\theta(J)$ is equal to the least congruence on L collapsing the subset $\{0\}\times J\subseteq L\times L$ (denote it by $\theta\{\{0\}\times J\}$), i.e.

$$\langle a, b\rangle \in \theta\{\{0\}\times J\}.$$

Hence there follows the existence of a finite $F\subseteq J$ such that

$$\langle a, b\rangle \in\theta\{\{0\}\times F\}.$$

Denote $c=\bigvee F$. Clearly $c\in J$ and $(c]$ is a principal ideal of L contained in J, thus

$$\langle a, b\rangle\in\{\{0\}\times(c]\}.$$

By Lemma 1 we have $\langle a, b\rangle\in\theta((c])$, thus

$$\theta(a, b)\subseteq\theta((c])\subseteq\theta(J)=\theta(a, b),$$

whence $\theta(a,b)=\theta((c])$. Since $(c]$ is a kernel of $\theta(c,0)$, we have $\theta((c])=\theta(c,0)$ proving (B).

Theorem 1. *Let L be a lattice. The following conditions are equivalent:*
(1) *L has 0 and is weakly regular;*
(2) *L has 0 and for each a, $b \in L$ there exists $c \in L$ such that $\theta(a, b) = \theta(c, 0)$.*
The proof is a trivial consequence of Lemmas 1 and 2.

Remark. By Proposition 1, every relatively complemented lattice L with 0 is weakly regular. Theorem 1 enables us to prove whether a relatively noncomplemented lattice L has this property. An example of such a lattice is in Fig. 1. This

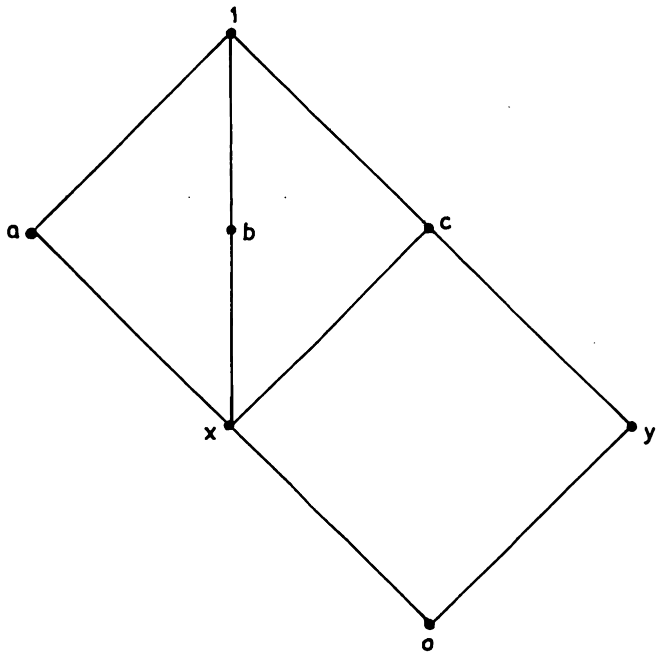

Fig. 1

lattice L is not relatively complemented (x has not a complement in L). On the other hand, there are only two congruences Φ, Ψ different from ω and ι; Φ has the classes $\{0, x\}$, $\{c, y\}$ and the one-element ones, Ψ has two classes: $\{0, y\}$ and $\{a, b, c, x, 1\}$. Clearly $\Phi = \theta(0, x)$, $\Psi = \theta(0, y)$ and, by Theorem 1, L is weakly regular.

Corollary. *Let L be a lattice with 0. The following conditions are equivalent:*
(1) *L is weakly regular;*
(2) *for each a, $b \in L$, $a \geq b$, there exists $c \in L$ and elements $e_0, \ldots, e_m, f_0, \ldots, f_n \in L$* such that

(i) $a = e_0 \geq e_1 \geq \ldots \geq e_m = b$, *where* $\overline{c, 0} \rightarrow \overline{e_{j+1}, e_j}$ $(j = 0, \ldots, m-1)$

(ii) $c = f_0 \geq f_1 \geq \ldots \geq f_n = 0$, *where* $\overline{b, a} \rightarrow \overline{f_{i+1}, f_i}$ $(i = 0, \ldots, n-1)$.

Proof. Since $\theta(a, b) = \theta(a \vee b, a \wedge b)$, it suffices to investigate only $\theta(a, b)$ with $a \geq b$ in (2) of Theorem 1. Thus L is weakly regular if and only if for each $a \geq b$ there exists c such that

$$\langle a, b \rangle \in \theta(c, 0) \quad \text{and} \quad \langle c, 0 \rangle \in \theta(a, b).$$

By the Dilworth theorem [3], we immediately obtain (i) and (ii).

In [2] there are given characterizations of varieties (with 0) whose congruences satisfy (2) of Theorem 1 (one of those characterizations is of the Maľcev type, the other of the Csákány type). Applying these results, we can formulate our next theorem. Let $\mathcal{V}$ be a variety. Let us call $\mathcal{V}$ a lattice variety if there exist binary polynomials $p(x, y)$, $q(x, y)$ such that each $\mathfrak{A} \in \mathcal{V}$ is a lattice with respect to the operations $\vee = p$, $\wedge = q$ and, moreover, $Con(\mathfrak{A}) = Con(\mathfrak{A}, \{p, q\})$. A variety $\mathcal{V}$ with nullary operation 0 is *weakly regular* if each $\mathfrak{A} \in \mathcal{V}$ has this property.

Theorem 2. *Let $\mathcal{V}$ be a lattice variety with* 0. *The following conditions are equivalent:*

(1) *$\mathcal{V}$ is weakly regular;*

(2) *there exists a binary polynomial b such that $b(x, y) = 0$ if and only if $x = y$;*

(3) *there exist a binary polynomial p and 4-ary polynomials $r_1, \ldots, r_n$ such that*

$$p(x, x) = 0$$
$$r_1(0, p(x, y), x, y) = x, \quad r_n(p(x, y), 0, x, y) - y$$
$$r_{j-1}(p(x, y), 0, x, y) = r_j(0, p(x, y), x, y) \quad \textit{for} \quad j = 2, \ldots, n.$$

Proof. Since each $\mathfrak{A} \in \mathcal{V}$ has only lattice congruences, $\mathcal{V}$ is weakly regular if and only if each $\mathfrak{A} \in \mathcal{V}$ satisfies (2) of Theorem 1. By [2, Theorem 4] or [2, Theorem 6] it is equivalent to (3) or (2), respectively.

Examples. It is well known that *Boolean algebras* are regular and hence weakly regular. The polynomial $b(x, y)$ of Theorem 2 can be chosen in the form

$$b(x, y) = (x' \wedge y) \vee (x \wedge y').$$

In a similar way we can show that the variety of *Heyting algebras* is weakly regular.

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СЛАБО РЕГУЛЯРНЫЕ РЕШЕТКИ

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Резюме

Решетка L с 0 называется слабо регулярной, если любые две конгруэнции θ_1, $\theta_2 \in Con(L)$ совпадают тогда и только тогда, когда $[0]_{\theta_1} = [0]_{\theta_2}$. В этой статье дана характеристика таких решеток и многообразий таких решеток.