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# THE ENTROPY ON $F$-QUANTUM SPACES

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## Introduction

A usual mathematical model for the quantum statistical mechanics is the quantum logic theory, i.e. the theory of orthomodular lattices [1]. A state $m$ on an orthomodular $\sigma$-complete lattice $L(\vee, \wedge, \perp, 0, 1)$ is a maping $m: L \to \langle 0, 1\rangle$ satisfying the following two conditions:
1. $m(1) = 1$
2. If $a_i \leq a_j^{\perp}$ $(i \neq j)$, then $m\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} m(a_i)$.

Riečan and Dvurečenskij pointed out in [2] and [3] that the Piasecki $P$-measure has the same algebraic structure. The Piasecki $P$-measure $m: M \to \langle 0, 1\rangle$ (cf. [4]) is defined on an appropriate set of real functions $M \subset \langle 0, 1\rangle^X$ and satisfies the following conditions:
1. $m(f \vee f^{\perp}) = 1$ for every $f \in M$.
2. If $f_i \leq f_j^{\perp}$ $(i \neq j)$, then $m\left(\bigvee_{i=1}^{\infty} f_i\right) = \sum_{i=1}^{\infty} m(f)$.

Of course, here $f^{\perp} = 1 - f$ and $\bigvee_n f_n = \sup_n f_n$.

Riečan and Dvurečenskij introduced a new mathematical model of the statistical quantum theory based on the Piasecki measure, the so-called $F$-quantum space ([2], [3]). The aim of the present paper is to give a characterization of an informational ability of an $F$-state and of an $F$-dynamical system $(X, M, m, T)$. The main properties of such a quantity are stated. The connection with the classical cases is also mentioned.

## 1. Some definitions and notations

**Definition 1.1.** *By an F-quantum space we mean a couple* $(X, M)$, *where* $X$ *is a non-empty set and* $M$ *is a subset of* $\langle 0, 1\rangle^X$ *satisfying the following conditions*:

$$\text{If } 1(x) = 1 \text{ for any } x \in X, \text{ then } 1 \in M. \tag{1.1}$$

$$\textit{If } f \in M, \textit{ then } f' = 1 - f \in M. \tag{1.2}$$

$$\textit{If } f_n \in M \ (n = 1, 2, \ldots), \textit{ then } \bigvee_{n=1}^{\infty} f_n \in M. \tag{1.3}$$

$$\textit{If } 1/2(x) = 1/2 \textit{ for any } x \in X, \textit{ then } 1/2 \notin M. \tag{1.4}$$

If we define $\bigwedge_n f_n := \inf f_n$, then the meet $\wedge$ and the join $\vee$ are related to each other by simple relations:

$$1 - \bigwedge_n f_n = \bigvee_n (1 - f_n), \ \{f_n\} \subset M$$

$$1 - \bigvee_n f_n = \bigwedge_n (1 - f_n), \ \{f_n\} \subset M$$

$$f \wedge (g \vee h) = (f \wedge g) \vee (f \wedge h), f, g, h \in M.$$

We say that $f, g \in M$ are orthogonal (we write $f \perp g$) if $f \leq g'$.

**Definition 1.2.** *By an F-state on an F-quantum space* $(X, M)$ *we mean a mapping* $m: M \to \langle 0, 1 \rangle$ *satisfying the following conditions*:

$$m(f \vee (1 - f)) = 1 \textit{ for every } f \in M. \tag{1.5}$$

$$\textit{If } f_n \in M \ (n = 1, 2, \ldots), \ f_i \perp f_j \ (i \neq j), \textit{ then } m\left(\bigvee_{i=1}^{\infty} f_i\right) = \sum_{i=1}^{\infty} m(f_i). \tag{1.6}$$

**Lemma 1.1.** *An F-state m on an F-quantum space* $(X, M)$ *has the following properties*:

$$m(f) + m(f') = 1 \textit{ for every } f \in M. \tag{1.7}$$

$$\textit{If } f, g \in M, f \leq g, \textit{ then } m(g) = m(f) + m(g \wedge f'). \tag{1.8}$$

$$\textit{If } f, g \in M, f \leq g, \textit{ then } m(f) \leq m(g). \tag{1.9}$$

Proof. Since $f \perp f'$ for every $f \in M$ by (1.6) we obtain $1 = m(f) + m(f')$. Let $f, g \in M$, $f \leq g$. Then $f \perp g'$ and $m(f \vee g') = m(f) + m(g')$ by (1.6). Therefore $m(f' \wedge g) = m((f \vee g')') = 1 - m(f) - m(g') = m(g) - m(f)$. The property (1.8) implies the property (1.9).

Example 1.1. Let $(X, \mathcal{S}, P)$ be a probability space. Put $M = \{\chi_A, A \in \mathcal{S}\}$, where $\chi_A$ is the characteristic function of the set $A \in \mathcal{S}$ and $m(\chi_A) = P(A)$. Then $(X, M)$ is an $F$-quantum space and $m$ is an $F$-state on $(X, M)$.

Example 1. 2. Let $M$ be the set of all functions $f: X \to \langle 0, 1\rangle$ and $m$ be the Piasecki $P$-measure. Then $(X, M)$ is an $F$-quantum space and $m$ is an $F$-state.

## 2. Definition of the entropy of an $F$-state

Let $(X, M)$ be an $F$-quantum space and $m$ an $F$-state on $(X, M)$. A finite set $\mathscr{A} = \{f_1, \ldots, f_n\}, f_i \in M$, is called an orthogonal resolution of the unit if for each $f_i, f_j \in \mathscr{A}$, $i \neq j$, there holds $f_i \perp f_j$ and $\bigvee_{i=1}^{n} f_i = 1$. Let us consider the set of all orthogonal resolutions of the unit and denote it by $\Phi$. Each $\mathscr{A} \in \Phi$ in the sense of the classical probability theory represents the random experiment with a finite number of outcomes with the probability distribution $p_i = m(f_i), f_i \in \mathscr{A}$, $p_i \geq 0$, $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} m(f_i) = m\left(\bigvee_{i=1}^{n} f_i\right) = m(1) = 1$.

**Definition 2.1.** *Let $\mathscr{A}$ be an orthogonal resolution of the unit, $\mathscr{A} = \{f_1, \ldots, f_n\}$. We define the entropy $H_m(\mathscr{A})$ of a resolution $\mathscr{A}$ in the F-state m by the Shannon formula*:

$$H_m(\mathscr{A}) = -\sum_{i=1}^{n} F(m(f_i)), \text{ where } F: \langle 0, \infty) \to R,\ F(x) = \begin{cases} x \log x & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases} \tag{2.1}$$

We define the entropy of an $F$-state $m$ as the maximal information which one can gain performing all experiments from the set $\Phi$.

**Definition 2.2.** *We define the entropy of an F-state m on an F-quantum space $(X, M)$ by*

$$h(m) = \sup\{H_m(\mathscr{A}):, \mathscr{A} \in \Phi\}. \tag{2.2}$$

In the following example there is mentioned the connection with the Shannon entropy of a probability distribution.

Example 2.1. Let $(X, \mathscr{S}, P)$ be a finite probability space, i.e. $X = \{x_1, \ldots, x_n\}$, $\mathscr{S} = 2^X, \bar{p} = \{p_1, \ldots, p_n\}$ is a probability distribution on $X$. If $A \in \mathscr{S}$, then $P(A) = \sum_{i: x_i \in A}' p_i$.

We define the $F$-quantum space $(X, M)$ and the $F$-state $m$ as in Example 1.1. Then the set $\Phi$ contains all resolutions of the type $\{\chi_{A_1}, \ldots, \chi_{A_k}\}$, *where* $A_i \subset X$ $(i = 1, \ldots, k)$, $A_i \cap A_j = \emptyset$ $(i \neq j)$ *and* $\bigcup_{i=1}^{k} A_i = X$. The entropy of a resolution

$\mathscr{A} = \{\chi_{A_1}, \ldots, \chi_{A_k}\}$ in the $F$-state $m$ is the number $H_m(\mathscr{A}) = -\sum_{i=1}^{k} F(P(A_i))$ and the entropy of an $F$-state $m$ is $h(m) = \sup\{H_m(\mathscr{A}); \mathscr{A} \in \Phi\} = -\sum_{i=1}^{n} F(p_i)$, which is in fact the Shannon entropy of the probability distribution $\bar{p} = \{p_1, \ldots, p_n\}$.

We shall now consider a $\sigma$-homomorphism $U: M \to M$, i.e. a mapping preserving the lattice operations as well as the mapping $f \to f'$, i.e.

$$U\left(\bigvee_{n=1}^{\infty} f_n\right) = \bigvee_{n=1}^{\infty} U(f_n) \quad \text{for every} \quad f_n \in M \ (n = 1, 2, \ldots) \tag{2.3}$$

$$U(1 - f) = 1 - U(f) \quad \text{for every} \quad f \in M \tag{2.4}$$

and furthermore

$$U(1) = 1. \tag{2.5}$$

We define $U^2 = U \circ U$ and by the mathematical induction $U^n = U \circ U^{n-1}$, $n = 1, 2, \ldots$, where $U^0$ is the identical mapping on $M$. It is easy to see that $U$ has the following properties: $U^n(1) = 1$, $U^n(0) = 0$, $U^n(1 - f) = 1 - U^n(f)$, $U^n\left(\bigvee_{i=1}^{\infty} f_i\right) = \bigvee_{i=1}^{\infty} U^n(f_i)$, $f \leq g$ implies $U^n(f) \leq U^n(g)$ for every $f, g \in M$ and for each sequence $\{f_i\} \subset M (n = 0, 1, 2, \ldots)$.

**Lemma 2.1.** *Let $\mathscr{A}$ be an orthogonal resolution of the unit and $U: M \to M$ be a $\sigma$-homomorphism. Then $U^n\mathscr{A} := \{U^n(f); f \in \mathscr{A}\}$ is also an orthogonal resolution of the unit ($n = 0, 1, 2, \ldots$).*

Proof. Let $\mathscr{A} = \{f_1, \ldots, f_k\}$, $\mathscr{A} \in \Phi$. Then $U^n\mathscr{A} = \{U^n(f_1), \ldots, U^n(f_k)\}$ and $\bigvee_{i=1}^{k} U^n(f_i) = U^n\left(\bigvee_{i=1}^{k} f_i\right) = U^n(1) = 1$ ($n = 0, 1, 2, \ldots$). Since for $i \neq j$ we have $f_i \leq 1 - f_j$, for $i \neq j$ we obtain $U^n(f_i) \leq U^n(1 - f_j) = 1 - U^n(f_j)$ ($n = 0, 1, 2, \ldots$).
So, $U^n\mathscr{A}$ is an orthogonal resolution of the unit ($n = 0, 1, 2, \ldots$).

**Lemma 2.2.** *Let $m$ be an $F$-state on an $F$-quantum space $(X, M)$ and $U: M \to M$ be a $\sigma$-homomorphism. Then the mapping $m \circ U^n: M \to \langle 0, 1 \rangle$, defined by $(m \circ U^n)(f) = m(U^n(f))$, $f \in M$,. ($n = 0, 1, 2, \ldots$) is an $F$-state on $(X, M)$.*

Proof. For every $f \in M$ we get

$$(m \circ U^n)(f \vee f') = m(U^n(f \vee f')) = m(U^n(f) \vee U^n(f')) =$$
$$= m(U^n(f) \vee (U^n(f))') = 1.$$

Let $f_i \in M$, $f_i \leq 1 - f_j (i \neq j)$. Then $U^n(f_i) \leq 1 - U^n(f_j)$ for $i \neq j$ and

$$(m \circ U^n)\left(\bigvee_{i=1}^{\infty} f_i\right) = m\left(U^n\left(\bigvee_{i=1}^{\infty} f_i\right)\right) = m\left(\bigvee_{i=1}^{\infty} U^n(f_i)\right) =$$

$$= \sum_{i=1}^{\infty} m(U^n(f_i)) = \sum_{i=1}^{\infty} (m \circ U^n)(f_i).$$

The basic properties of the entropy $H_m$ are stated in the next theorem.

**Theorem 2.1.** *The entropy $H_m\colon \Phi \to R$ has the following properties:*

$$H_m(\mathscr{A}) \geq 0 \textit{ for every } \mathscr{A} \in \Phi. \tag{2.6}$$

$$H_{m \circ U^n}(\mathscr{A}) = H_m(U^n \mathscr{A}) \textit{ for every } \mathscr{A} \in \Phi,\ n = 0, 1, 2, \ldots. \tag{2.7}$$

Proof. The property (2.6) is evident. Let $\mathscr{A} \in \Phi$, $\mathscr{A} = \{f_1, \ldots, f_k\}$. Then $H_{m.U^n}(\mathscr{A}) = -\sum_{i=1}^{k} F((m \circ U^n)(f_i)) = -\sum_{i=1}^{k} F(m(U^n f_i)) = H_m(U^n \mathscr{A})$.

**Corollary 2.1.** $h(m \circ U^n) = \sup\{H_m(U^n \mathscr{A});\ \mathscr{A} \in \Phi\}$.

In the set $\Phi$ of all the orthogonal resolutions of the unit one can define the operation $\vee$ in the following way: if $\mathscr{A}, \mathscr{B} \in \Phi$, $\mathscr{A} = \{f_1, \ldots, f_r\}$, $\mathscr{B} = \{g_1, \ldots, g_s\}$, then we put $\mathscr{A} \vee \mathscr{B} = \{f_i \wedge g_j;\ i = 1, \ldots, r, j = 1, \ldots, s\}$.
We shall read the symbol $\mathscr{A} \vee \mathscr{B}$ the common refinement of $\mathscr{A}$ and $\mathscr{B}$. If $\mathscr{A}_1$, $\mathscr{A}_2, \ldots \in \Phi$, then instead $\mathscr{A}_1 \vee \mathscr{A}_2$ we write $\bigvee_{i=1}^{2} \mathscr{A}_i$, and we define by the induction

$$\bigvee_{i=1}^{k+1} \mathscr{A}_i = \bigvee_{i=1}^{k} \mathscr{A}_i \vee \mathscr{A}_{k+1}, \quad \text{for} \quad k = 2, 3, 4, \ldots.$$

**Lemma 2.3.** *Let $\mathscr{A}$, $\mathscr{B}$ be the orthogonal resolutions of the unit. Then $\mathscr{A} \vee \mathscr{B}$ is an orthogonal resolution of the unit, too.*

Proof. Let $\mathscr{A}, \mathscr{B} \in \Phi$, $\mathscr{A} = \{f_1, \ldots, f_r\}$, $\mathscr{B} = \{g_1, \ldots, g_s\}$. Then $\mathscr{A} \vee \mathscr{B} = \{f_i \wedge g_j, i = 1, \ldots, r, j = 1, \ldots, s\}$ and $\bigvee_{j=1}^{s} \bigvee_{i=1}^{r} (f_i \wedge g_j) =$ $= \bigvee_{j=1}^{s}\left(\left(\bigvee_{i=1}^{r} f_i\right) \wedge g_j\right) = \bigvee_{j=1}^{s} g_j = 1$. Since for $i \neq j$ $f_i \leq f_j'$, we obtain $g_k \wedge f_i \leq$ $\leq f_i \leq f_j' \leq f_j' \vee g_l' = (f_j \wedge g_l)'$. Therefore $g_k \wedge f_i \perp f_j \wedge g_l$ for $i \neq j$ and $l, k = 1, 2, \ldots, s$.
Analogously we prove that $f_i \wedge g_k \perp f_j \wedge g_l$ for $l \neq k$ and $i, j = 1, 2, \ldots, r$.

The posibility of the definition of the entropy of the system $(X, M, m, T)$ is based on the following theorem.

**Theorem 2.2.** $H_m(\mathscr{A} \vee \mathscr{B}) \leq H_m(\mathscr{A}) + H_m(\mathscr{B})$ *for every* $\mathscr{A}, \mathscr{B} \in \Phi$.

Proof. The function $F: \langle 0, \infty) \to R$,

$$F(x) = \begin{cases} x \log x & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \end{cases}$$

is convex and therefore for any convex combination $\sum_{i=1}^{k} \alpha_i x_i$ (i.e. such that $\alpha_1, \ldots, \alpha_k \geq 0$, $\sum_{i=1}^{k} \alpha_i = 1$) of the elements $x_1, \ldots, x_k \in \langle 0, 1 \rangle$ there holds

$$F\left(\sum_{i=1}^{k} \alpha_i x_i\right) \leq \sum_{i=1}^{k} \alpha_i F(x_i). \tag{2.8}$$

Let $\mathscr{A} = \{f_1, \ldots, f_n\}$, $\mathscr{B} = \{g_1, \ldots, g_k\}$. Put $\alpha_i = m(g_i)$ $(i = 1, \ldots, k)$, $x_i = m(f_j/g_i)$ $(i = 1, \ldots, k, j$ fixed), where we define

$$m(f_j/g_i) := \begin{cases} \dfrac{m(f_j \wedge g_i)}{m(g_i)} & \text{if} \quad m(g_i) > 0, \\ 0 & \text{if} \quad m(g_i) = 0. \end{cases}$$

Then

$$\sum_{i=1}^{k} \alpha_i x_i = \sum_{i=1}^{k} m(g_i) \,.\, m(f_j/g_i) = \sum_{i: m(g_i) > 0} m(g_i) \cdot \frac{m(f_j \wedge g_i)}{m(g_i)} = \sum_{i=1}^{k} m(f_j \wedge g_i) =$$

$$= m\left(f_j \wedge \left(\bigvee_{i=1}^{k} g_i\right)\right) = m(f_j).$$

By (2.8) we obtain $F(m(f_j)) \leq \sum_{i=1}^{k} m(g_i) \,.\, F(m(f_j/g_i))$, for $j = 1, \ldots, n$. If $m(g_i) = 0$, then also $m(g_i) \,.\, F(m(f_j/g_i)) = 0$. If $m(f_j \wedge g_i) > 0$, then $m(g_i) \,.$

$$.\, F(m(f_j/g_i)) = m(g_i) \cdot \frac{m(f_j \wedge g_i)}{m(g_i)} \cdot \log \frac{m(f_j \wedge g_i)}{m(g_i)} = m(f_j \wedge g_i) \,.$$

$$.\, \log m(f_j \wedge g_i) - m(f_j \wedge g_i) \,.\, \log m(g_i).$$

Denote by

$$\alpha = \{(i, j);\ 1 \leq j \leq n,\ 1 \leq i \leq k,\ m(f_j \wedge g_i) > 0\},$$

$$\beta = \{i;\ 1 \leq i \leq k,\ m(g_i) > 0\}.$$

Then

$$H_m(\mathscr{A}) = -\sum_{j=1}^{n} F(m(f_j)) \geq -\sum_{j=1}^{n} \sum_{i=1}^{k} m(g_i) \,.\, F(m(f_j/g_i)) =$$

$$= - \sum_{(i,j)\in\alpha} m(f_j \wedge g_i) \log m(f_j \wedge g_i) + \sum_{(i,j)\in\alpha} m(f_j \wedge g_i)) \log m(g_i) =$$

$$= - \sum_{j=1}^{n} \sum_{i=1}^{k} F(m(f_j \wedge g_i)) + \sum_{i\in\beta} \log m(g_i) \sum_{j=1}^{n} m(f_j \wedge g_i) = H_m(\mathscr{A} \vee \mathscr{B}) +$$

$$+ \sum_{i\in\beta} m(g_i) \log m(g_i) = H_m(\mathscr{A} \vee \mathscr{B}) - \left( - \sum_{i=1}^{k} F(m(g_i)) \right) = H_m(\mathscr{A} \vee \mathscr{B}) - H_m(\mathscr{B}).$$

## 3. The entropy of the $F$-dynamical system

By an $F$-dynamical system we mean the quadruple $(X, M, m, T)$, where $(X, M)$ is an $F$-quantum space, $m$ is an $F$-state on $(X, M)$ and $T$ is an $F$-state $m$ preserving the transformation, i.e. $T: X \to X$ satisfies the following condition:

$$f \in M \quad \text{implies} \quad f \circ T \in M \quad \text{and} \quad m(f \circ T) = m(f). \tag{3.1}$$

Example 3.1. Let $(X, \mathscr{S}, P, T)$ be a dynamical system in the sense of the classical probability theory, i.e. $(X, \mathscr{S}, P)$ is a probability space and $T$ is a measure preserving transformation (i.e. $E \in \mathscr{S}$ implies $T^{-1}(E) \in \mathscr{S}$ and $P(T^{-1}(E)) = P(E)$). Then the quadruple $(X, M, m, T)$, where $(X, M)$ and $m$ are defined as in the Example 1.1, is an $F$-dynamical system. It is easy to see that *satisfies also the condition* (3.1). *Namely, if* $f \in M$, *then* $f = \chi_E$, *where* $E \in \mathscr{S}$ $m(f \circ T) = m(\chi_E \circ T) = m(\chi_{T^{-1}(E)}) = P(T^{-1}(E)) = P(E) = m(\chi_E) = m(f)$.

**Lemma 3.1.** *Let $(X, M, m, T)$ be an F-dynamical system. Then the maping $U: M \to M$, $U(f) = f \circ T$, $f \in M$, is a σ-homomorphism of M.*

Proof. Since for every $x \in X$

$$\left[\left(\bigvee_{n=1}^{\infty} f_n\right) \circ T\right](x) = \left(\bigvee_{n=1}^{\infty} f_n\right)(T(x)) = \bigvee_{n=1}^{\infty} (f_n(T(x))) = \bigvee_{n=1}^{\infty} (f_n \circ T)(x),$$

we obtain

$$U\left(\bigvee_{n=1}^{\infty} f_n\right) = \left(\bigvee_{n=1}^{\infty} f_n\right) \circ T = \bigvee_{n=1}^{\infty} (f_n \circ T) = \bigvee_{n=1}^{\infty} U(f_n).$$

Moreover, for every $x \in X$

$$[(1 - f) \circ T](x) = (1 - f)(T(x)) = 1 - f(T(x)) = 1 - (f \circ T)(x)$$

and therefore $U(1 - f) = (1 - f) \circ T = 1 - f \circ T = 1 - U(f)$. It is easy to see that $U$ fulfils also the condition (2.5).

**Lemma 3.2.** *Let $\mathscr{A} = \{f_1, \ldots, f_k\}$ be an orthogonal resolution of the unit. Then $T^n\mathscr{A} := \{f_1 \circ T^n, \ldots, f_k \circ T^n\}$ $(n = 0, 1, 2, \ldots)$ is an orthogonal resolution of the unit, too.*

Proof.

$$\bigvee_{i=1}^{k} (f_i \circ T^n) = \left(\bigvee_{i=1}^{k} f_i\right) \circ T^n = 1 \circ T^n = 1.$$

Since $(f_i \circ T^n) \wedge (1 - f_j \circ T^n) = (f_i \wedge (1 - f_j)) \circ T^n = f_i \circ T^n$ $(i \neq j)$ there holds for $i \neq j$ $f_i \circ T^n \leq 1 - f_j \circ T^n$. So that $T^n\mathscr{A}$ is an orthogonal resolution of the unit.

**Lemma 3.3.** *$H_m(T^n\mathscr{A}) = H_m(\mathscr{A})$, where $T^n\mathscr{A} = \{f_1 \circ T^n, \ldots, f_k \circ T^n\}$ $(n = 0, 1, 2, \ldots)$ for every $\mathscr{A} \in \Phi$, $\mathscr{A} = \{f_1, \ldots, f_k\}$.*

Proof. Since $m(f \circ T^n) = m(f)$ for $n = 0, 1, 2, \ldots$ and every $f \in M$, we obtain $H_m(T^n\mathscr{A}) = -\sum_{i=1}^{k} F(m(f_i \circ T^n)) = -\sum_{i=1}^{k} F(m(f_i)) = H_m(\mathscr{A})$.

**Lemma 3.4.** ([5]) *Let $(a_n)_{n=1}^{\infty}$ be a sequence of nonnegative numbers such that $a_{r+s} \leq a_r + a_s$ for each $r, s = 1, 2, \ldots$. Then there exists $\lim_{n\to\infty} \frac{1}{n} a_n$.*

**Lemma 3.5.** *For every $\mathscr{A} \in \Phi$ there exists $\lim_{n\to\infty} \frac{1}{n} H_m\left(\bigvee_{j=0}^{n-1} T^j\mathscr{A}\right)$.*

Proof. Put $a_n = H_m\left(\bigvee_{j=0}^{n-1} T^j\mathscr{A}\right)$. According to Theorem 2.2 and Lemma 3.3 we obtain

$$a_{r+s} = H_m\left(\bigvee_{j=0}^{r+s-1} T^j\mathscr{A}\right) = H_m\left(\bigvee_{j=0}^{s-1} T^j\mathscr{A} \vee \bigvee_{j=s}^{r+s-1} T^j\mathscr{A}\right) \leq H_m\left(\bigvee_{j=0}^{s-1} T^j\mathscr{A}\right) +$$

$$+ H_m\left(\bigvee_{j=s}^{r+s-1} T^j\mathscr{A}\right) = a_s + H_m\left(T^s\left(\bigvee_{i=0}^{r-1} T^i\mathscr{A}\right)\right) = a_s + H_m\left(\bigvee_{i=0}^{r-1} T^i\mathscr{A}\right) =$$

$$= a_s + a_r.$$

By the preceding lemma there exists $\lim_{n\to\infty} \frac{1}{n} a_n$.

**Definition 3.1.** *Let $(X, M, m, T)$ be an F-dynamical system. Then for every $\mathscr{A} \in \Phi$ we define $h_m(T, \mathscr{A}) = \lim_{n\to\infty} \frac{1}{n} H_m\left(\bigvee_{j=0}^{n-1} T^j\mathscr{A}\right)$. The entropy of the F-dynamical system $(X, M, m, T)$ is defined by $h_m(T) = \sup\{h_m(T, \mathscr{A}); \mathscr{A} \in \Phi\}$.*

In the following we shall see that the Definition 3.1 is a generalization of the classical Kolmogorov-Sinaj entropy of a dynamical system $(X, \mathcal{S}, P, T)$. A starting point in its definition is the notion of the entropy of a measurable partition. If $A = \{A_1, \ldots, A_n\}$ is a measurable partition of the space $(X, \mathcal{S}, P)$, then the entropy of the partition $A$ is defined by $H(A) = -\sum_{i=1}^{n} F(P(A_i))$. If we consider the F-quantum space $(X, M)$ and the $F$-state $m$ from Example 1.1, then for every measurable partition $A = \{A_1, \ldots, A_n\}$ of the space $(X, \mathcal{S}, P)$ there exists the partition $\mathcal{A} \in \Phi$, $\mathcal{A} = \{\chi_{A_1}, \ldots, \chi_{A_n}\}$ and there holds further $H_m(\mathcal{A}) = -\sum_{i=1}^{n} F(m(\chi_{A_i})) = -\sum_{i=1}^{n} F(P(A_i)) = H(A)$. If $A = \{A_1, \ldots, A_n\}$, $\boldsymbol{B} = \{B_1, \ldots, B_k\}$ are two measurable partitions of the space $(X, \mathcal{S}, P)$, then the common refinement of $A$ and $\boldsymbol{B}$ is defined as the set $A \vee \boldsymbol{B} = \{A_i \cap B_j;\ i = 1, \ldots, n, j = 1, \ldots, k\}$. If we put $\mathcal{A} = \{\chi_{A_1}, \ldots, \chi_{A_n}\}$, $\mathcal{B} = \{\chi_{B_1}, \ldots, \chi_{B_k}\}$, then the following equality holds:

$$H_m(\mathcal{A} \vee \mathcal{B}) = -\sum_{i=1}^{n} \sum_{j=1}^{k} F(m(\chi_{A_i} \wedge \chi_{B_j})) = -\sum_{i=1}^{n} \sum_{j=1}^{k} F(P(A_i \cap B_j)) = \\ = H(A \vee \boldsymbol{B}).$$

The Kolmogorov-Sinaj entropy of the dynamical system $(X, \mathcal{S}, P, T)$ is defined by $h(T) = \sup\{h(T, A);\ A$ is a finite measurable partition of $X\}$, where $h(T, A) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}A\right)$ and finally $T^{-i}A = \{T^{-i}(A_1), \ldots, T^{-i}(A_k)\}$ for every measurable partition $A = \{A_1, \ldots, A_k\}$. Since $H_m(\mathcal{A} \vee T\mathcal{A}) = -\sum_{i,j=1}^{k} F(m(\chi_{A_i} \wedge \chi_{T^{-1}(A_j)})) = -\sum_{i,j=1}^{k} F(P(A_i \cap T^{-1}(A_j)) = H(A \vee T^{-1}A)$, by induction we obtain $H_m\left(\bigvee_{i=0}^{n-1} T^i\mathcal{A}\right) = H\left(\bigvee_{i=0}^{n-1} T^{-i}A\right)$, hence

$$h_m(T, \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} H_m\left(\bigvee_{i=0}^{n-1} T^i\mathcal{A}\right) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}A\right) = h(T, A)$$

and finally
$h_m(T) = \sup\{h_m(T, \mathcal{A});\ \mathcal{A} \in \Phi\} = \sup\{h(T, A);\ A$ is a finite measurable partition$\} = h(T)$.

**Lemma 3.6.** *Let $(X, M, m, T)$ be an F-dynamical system. Then the function $T \circ m: M \to \langle 0, 1\rangle$ defined by*

$$(T \circ m)(f) = m(f \circ T)$$

*is an F-state on $(X, M)$.*

Proof. For every $f \in M$ there holds

$$(T \circ m)(f \vee f') = m((f \vee f') \circ T) = m(f \circ T \vee f' \circ T) = m(f \circ T \vee (f \circ T)') = 1.$$

Let $f_i \in M$, $f_i \perp f_j$ $(i \neq j)$. Then for every $x \in X$ and $i \neq j$ $f_i(x) \leq 1 - f_j(x)$ and therefore we obtain

$$(f_i \circ T)(x) = f_i(T(x)) \leq 1 - f_j(T(x)) = 1 - (f_j \circ T)(x).$$

$$(T \circ m)\left(\bigvee_{i=1}^{\infty} f_i\right) = m\left(\left(\bigvee_{i=1}^{\infty} f_i\right) \circ T\right) = m\left(\bigvee_{i=1}^{\infty} (f_i \circ T)\right) = \sum_{i=1}^{\infty} m(f_i \circ T) =$$
$$= \sum_{i=1}^{\infty} (T \circ m)(f_i)$$

**Lemma 3.7.** *For every $\mathscr{A} \in \Phi$ there holds $H_{T \circ m}(\mathscr{A}) = h_m(T\mathscr{A}) = H_m(\mathscr{A})$.*

**Theorem 3.1.** $h_{T \circ m}(T) = h_m(T)$.

Proof. For every $\mathscr{A} \in \Phi$ we have by the preceding lemma

$$h_{T \circ m}(T, \mathscr{A}) = \lim_{n \to \infty} \frac{1}{n} H_{T \circ m}\left(\bigvee_{j=0}^{n-1} T^j \mathscr{A}\right) = \lim_{n \to \infty} \frac{1}{n} H_m\left(\bigvee_{j=0}^{n-1} T^j \mathscr{A}\right) = h_m(T, \mathscr{A}).$$

$$h_{T \circ m}(T) = \sup\{h_{T \circ m}(T, \mathscr{A});\ \mathscr{A} \in \Phi\} = \sup\{h_m(T, \mathscr{A});\ \mathscr{A} \in \Phi\} = h_m(T).$$

## 4. The connection with the general scheme

Riečan in [6] notices some common properties of the topological and the Kolmogorov-Sinaj entropy and introduces a general scheme which includes the mentioned entropy. A similar character have also the papers [7], [8] and [9]. Grošek in [7] pays first of all attention to algebraic aspects of the entropy. In this section we give the definition of the so-called generalized base of the $l$-entropy (see [7]). At the same time we show that the entropy of the system $(X, M, m, T)$ is a special case of the $l$-entropy. First we give the definitions of some algebraic notions which we shall use in the following.

A triplet $(S, \vee, \leq)$ is called a quasi-ordered semigroup if the couple $(S, \vee)$ is a semigroup, the set $S$ is quasi-ordered by relation $\leq$ and for every $x, y, z \in S$ there holds

$$x \leq y \text{ implies } x \vee z \leq y \vee z \text{ and } z \vee x \leq z \vee y. \tag{4.1}$$

The set $S$ is called a strong quasi-ordered semigroup if $S$ is a quasi-ordered semigroup and the ordering $\leq$ on the set $S$ satisfies the condition

$$x \leq x \vee y \quad \text{for every} \quad x, y \in S. \tag{4.2}$$

**Lemma 4.1.** *If the quasi-ordered semigroup $S$ contains the unit-element such that it is at the same time also the minimum of the set $S$, then $S$ is a strong quasi-ordered semigroup.*

Proof. Let $x, y \in S$. Then $1 \leq y$ and by (4.1) $x \vee 1 \leq x \vee y$. Since $x \vee 1 = x$, we obtain $x \leq x \vee y$.

A mapping $T: S \to S$ is called an isotone endomorphiom if for every $x, y \in S$ the following conditions hold:

$$T(x \vee y) = T(x) \vee T(y) \tag{4.3}$$

$$x \leq y \quad \text{implies} \quad T(x) \leq T(y) \tag{4.4}$$

**Definition 4.1.** *Let $S$ be a strong quasi-ordered commutative semigroup, $T$ be an isotone endomorphism on $S$. By a generalized entropy with respect to the endomorphism $T$ we shall mean a function $H: S \to \langle 0, \infty)$ satisfying for every $x$, $y \in S$ the following conditions:*

$$x \leq y \quad \text{implies} \quad H(x) \leq H(y) \tag{4.5}$$

$$H(T(x)) \leq H(x) \tag{4.6}$$

$$H(x \vee T(x) \vee \ldots \vee T^n(x)) \leq H(x \vee T(x) \vee \ldots \vee T^j(x)) + H(T^{j+1}(x) \vee \ldots \vee T^n(x)) \tag{4.7}$$

*for every $j, n \in N$, $0 \leq j \leq n$.*

**Definition 4.2.** *By a generalized l-entropy of the element $x \in S$ with respect to the isotone endomorphism $T$ we mean a function $h_T: S \to \langle 0, \infty)$ defined by $h_T(x) = \lim_{n \to \infty} \frac{1}{n} H_n(x)$, where $H_n(x) = H(x \vee T(x) \vee \ldots \vee T^{n-1}(x))$, $x \in S$. By a generalized base of the l-entropy $h_T$ we mean an ordered triplet $(S, T, H)$, where $S$ is a strong quasi-ordered commutative semigroup, $T$ is an isotone endomorphism on $S$ and $H$ is a generalized entropy. We define the generalized entropy of the endomorphism $T$ at the base $(S, T, H)$ by*

$$h_T^* = \sup\{h_T(x);\ x \in S\}.$$

Let $(X, M, m, T)$ be an $F$-dynamical system. Let $\Phi$ be the set of all orthogonal resolutions of the unit. In the set $\Phi$ we define the relation $\leq$ in the following way: for every $\mathscr{A}, \mathscr{B} \in \Phi$, $\mathscr{A} \leq \mathscr{B}$ iff there exists $\mathscr{C} \in \Phi$ such that $\mathscr{B} = \mathscr{A} \vee \mathscr{C}$. We say then that $\mathscr{B}$ is the refinement of $\mathscr{A}$.

**Proposition 4.1.** *The set $\Phi$ of all orthogonal resolutions of the unit is a strong quasi-ordered commutative semigroup.*

Proof. Evidently, the operation $\vee$ is commutative and associative and according to Lemma 2.3 the set $\Phi$ with the operation $\vee$ is a commutative semigroup. We prove that the relation $\leq$ is a quasi-ordering on $\Phi$ as well as the condition (4.1) holds. For every $\mathscr{A} \in \Phi$ there exists $\mathscr{C} \in \Phi$ such that $\mathscr{A} = \mathscr{A} \vee \mathscr{C}$. Indeed, it suffices to put $\mathscr{C} = \mathscr{E} := \{1\}$. The relation $\leq$ is reflexive. We prove that it is transitive, too. If $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3 \in \Phi$ such that $\mathscr{A}_1 \leq \mathscr{A}_2$ and $\mathscr{A}_2 \leq \mathscr{A}_3$, then there are $\mathscr{B}, \mathscr{C} \in \Phi$ such that $\mathscr{A}_2 = \mathscr{A}_1 \vee \mathscr{B}$, $\mathscr{A}_3 = \mathscr{A}_2 \vee \mathscr{C}$. We have $\mathscr{A}_3 = (\mathscr{A}_1 \vee \mathscr{B}) \vee \mathscr{C} = \mathscr{A}_1 \vee (\mathscr{B} \vee \mathscr{C})$. Hence $\mathscr{A}_1 \leq \mathscr{A}_3$. We prove (4.1). If $\mathscr{A}$, $\mathscr{B}, \mathscr{C} \in \Phi$, where $\mathscr{A} \leq \mathscr{B}$, then there exists $\mathscr{D} \in \Phi$ such that $\mathscr{B} = \mathscr{A} \vee \mathscr{D}$. We obtain

$$\mathscr{B} \vee \mathscr{C} = (\mathscr{A} \vee \mathscr{D}) \vee \mathscr{C} = \mathscr{A} \vee (\mathscr{D} \vee \mathscr{C}) = \mathscr{A} \vee (\mathscr{C} \vee \mathscr{D}) = (\mathscr{A} \vee \mathscr{C}) \vee \mathscr{D}.$$

Hence $\mathscr{A} \vee \mathscr{C} \leq \mathscr{B} \vee \mathscr{C}$. The partition $\mathscr{E} = \{1\}$ is the unit-element and at the same time the minimum of the set $\Phi$. For every $\mathscr{A} \in \Phi$ there holds $\mathscr{E} \leq \mathscr{A}$ because $\mathscr{A} = \mathscr{A} \vee \mathscr{E}$. So, by Lemma 4.1 the set $\Phi$ is a strong quasi-ordered commutative semigroup.

**Proposition 4.2.** *The mapping $T: \Phi \to \Phi$ defined by $T\mathscr{A} = \{f_1 \circ T, \ldots, f_n \circ T\}$, where $\mathscr{A} \in \Phi$, $\mathscr{A} = \{f_1, \ldots, f_n\}$, is an isotone endomorphism on the set $\Phi$.*

Proof. According to Lemma 3.2 if $\mathscr{A} \in \Phi$, then $T\mathscr{A} \in \Phi$, too. Let $\mathscr{A}$, $\mathscr{B} \in \Phi$, $\mathscr{A} = \{f_1, \ldots, f_n\}$, $\mathscr{B} = \{g_1, \ldots, g_k\}$. Then

$$\mathscr{A} \vee \mathscr{B} = \{f_i \wedge g_j, i = 1, \ldots, n, j = 1, \ldots, k\}.$$

$$T(\mathscr{A} \vee \mathscr{B}) = \{(f_i \wedge g_j) \circ T; i = 1, \ldots, n, j = 1, \ldots, k\} =$$

$$= \{(f_i \circ T) \wedge (g_j \circ T), i = 1, \ldots, n, j = 1, \ldots, k\} = T\mathscr{A} \vee T\mathscr{B}.$$

If $\mathscr{A}, \mathscr{B} \in \Phi$, $\mathscr{A} \leq \mathscr{B}$, then there exists $\mathscr{C} \in \Phi$ such that $\mathscr{B} = \mathscr{A} \vee \mathscr{C}$. $T\mathscr{B} = T(\mathscr{A} \vee \mathscr{C}) = T\mathscr{A} \vee T\mathscr{C}$. This implies $T\mathscr{A} \leq T\mathscr{B}$.

**Theorem 4.1.** *The function $H_m: \Phi \to \langle 0, \infty)$ defined by $H_m(\mathscr{A}) = -\sum_{i=1}^{n} F(m(f_i))$, $\mathscr{A} \in \Phi$, $\mathscr{A} = \{f_1, \ldots, f_n\}$, is a generalized entropy with respect to the endomorphism $T$ from the Proposition* 4.2.

Proof. We prove that (4.5) holds. Let $\mathscr{A}, \mathscr{B} \in \Phi$, $\mathscr{A} \leq \mathscr{B}$, i.e. $\mathscr{B} = \mathscr{A} \vee \mathscr{C} = \{f_i \wedge g_j, i = 1, \ldots, n, j = 1, \ldots, k\}$. Put $\alpha = \{(i, j); i = 1, \ldots, n, j = 1, \ldots, k, m(f_i \wedge g_j) > 0\}$. Then

$$H_m(\mathscr{B}) = -\sum_{i=1}^{n} \sum_{j=1}^{k} F(m(f_i \wedge g_j)) = -\sum_{(i,j) \in \alpha} m(f_i \wedge g_j) \log m(f_i \wedge g_j) =$$

$$= -\sum_{(i,j) \in \alpha} m(f_i \wedge g_j) \log m(g_j/f_i) - \sum_{(i,j) \in \alpha} m(f_i \wedge g_j) \log m(f_i) =$$

$$= - \sum_{(i,j) \in \alpha} m(f_i \wedge g_j) \log m(g_j/f_i) - \sum_{i:(i,j) \in \alpha} \log m(f_i) \sum_{j=1}^{k} m(f_i \wedge g_j) \geq$$

$$\geq - \sum_{i:(i,j) \in \alpha} m(f_i) \log m(f_i) = - \sum_{i=1}^{n} F(m(f_i)) = H_m(\mathscr{A}).$$

The condition (4.6) is proved in Lemma 3.3 and the condition (4.7) follows from Theorem 2.2.

At the same time we obtain that the function $h_m(T, \mathscr{A}) = \lim_{n \to \infty} \frac{1}{n} H_m\left(\bigvee_{j=0}^{n-1} T^j \mathscr{A}\right)$, $\mathscr{A} \in \Phi$, is a generalized $l$-entropy of the element $\mathscr{A} \in \Phi$ with respect to the endomorphism $T$. The triplet $(\Phi, T, H_m)$ is a generalized base of the $l$-entropy

$$h_T(\,.\,) = h_m(T, .): \Phi \to \langle 0, \infty).$$

The entropy $h_m(T)$ of the $F$-dynamical system $(X, M, m, T)$ is a generalized entropy of the endomorphism $T$ at the base $(\Phi, T, H_m)$:

$$h_m(T) = h_T^* = \sup \{h_T(\mathscr{A}); \mathscr{A} \in \Phi\}.$$

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# ЭНТРОПИЯ НА $F$-КВАНТОВЫХ ПРОСТРАНСТВАХ

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Резюме

В статье рассматриваются энтропия на $F$-квантовых пространствах, энтропия $F$-состояния и энтропия $F$-динамической системы. В работе показано, что приведенные определения являются обобщением энтропий Шаннона и Колмогоровова-Синия.