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ON REPRESENTATION OF TERNARY STRUCTURES

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ABSTRACT. A construction is presented which gives a possibility of describing ternary relations. Graphical representation of ternary relations is noted.

0. Introduction

Let G be a nonempty set. If ϱ is a binary relation on G, then the pair $\mathbb{G} = (G, \varrho)$ is called a *binary structure*; if t is a ternary relation on G, then $\mathbb{G} = (G, t)$ is called a *ternary structure*.

A ternary relation t on G (and the structure $\mathbb{G} = (G, t)$) is called
symmetric if $(x, y, z) \in t \implies (z, y, x) \in t$,
asymmetric if $(x, y, z) \in t \implies (z, y, x) \notin t$,
cyclic if $(x, y, z) \in t \implies (y, z, x) \in t$,
transitive if $(x, y, z) \in t,\ (z, y, u) \in t \implies (x, y, u) \in t$.
If the last condition holds only for $y = z$, i.e., if

$$(x, y, y) \in t,\ (y, y, z) \in t \implies (x, y, z) \in t,$$

then the relation t and the structure $\mathbb{G}$ are called *weakly transitive*.

A ternary structure (G, t) is a called a *cyclically ordered set* ([3], [1], [2], [5]) if it is asymmetric, cyclic and transitive. In the process of constructing examples or counterexamples of ternary relations on finite sets, we meet the problem of graphical representation of such relations. If (G, t) is cyclic and $(x, y, z) \in t$,

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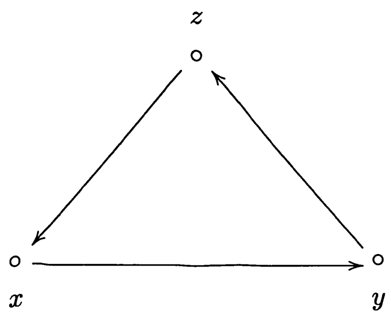
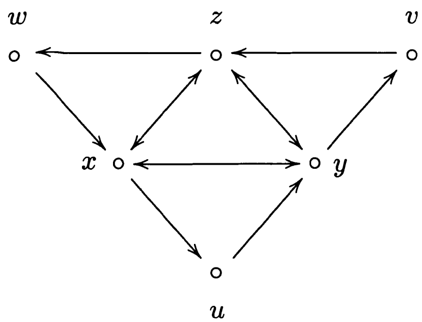
then the triplets (x,y,z), (y,z,x), (z,x,y) can be represented by an oriented triangle



The following example ([4]) shows that we can get into troubles even in that case: if $G=\{x,y,z,u,v,w\}$, $t=\{(x,y,z),\ (x,u,y),\ (y,v,z),\ (z,w,x)\}$ and t^c is a cyclic hull of t, then the graph of (G,t^c) is as follows:



Thus, in this graph, we have obtained an oriented triangle corresponding to triplets (x,z,y), (z,y,x), (y,x,z) which are not in t^c. In [4], we have represented ternary structures by double binary structures. A *double binary structure* is a triplet $\mathbb{G}=(G,\varrho,r)$, where G is a set, ϱ is a binary relation on G, and r is a binary relation on ϱ with the following property:

$$(x,y)\in\varrho,\ (u,v)\in\varrho,\ \big((x,y),(u,v)\big)\in r \implies y=u\,.$$

If (G,ϱ,r) is a double binary structure, then we can define a ternary relation t on G as follows:

$$(x,y,z)\in t \iff (x,y)\in\varrho,\ (y,z)\in\varrho,\ \big((x,y),(y,z)\big)\in r\,;$$

if (G,t) is a ternary structure, then it is possible to define a double binary structure (G,ϱ,r) by:

$$(x,y)\in\varrho \iff \text{there is } z\in G \text{ such that } (x,y,z)\in t \text{ or } (z,x,y)\in t \text{ and}$$
$$\text{for } (x,y)\in\varrho,\ (u,v)\in\varrho \text{ it is}$$
$$\big((x,y),\ (u,v)\big)\in r \iff y=u \text{ and } (x,y,v)\in t\,.$$

Special properties of ternary structures are transformed in corresponding properties of double binary structures ([4]). Further, double binary structures on finite sets can be graphically represented. In this paper, we describe an abstract construction which contains double binary structures as special cases.

1. E-systems

1.1. DEFINITION. Let E be a set, $G \neq \emptyset$ a set, and $p_1\colon E \to G$, $p_2\colon E \to G$ mappings. Let the pair of mappings $\{p_1, p_2\}$ distinguish elements of E, i.e., let $e_1, e_2 \in E$, $p_1(e_1) = p_1(e_2)$, $p_2(e_1) = p_2(e_2) \implies e_1 = e_2$ hold. Then the quadruple $\mathbb{G} = (E, G, p_1, p_2)$ will be called an *E-system.*

Let (E, G, p_1, p_2) be an E-system, and $e \in E$. If there exists an element $e' \in E$ such that $p_1(e') = p_2(e)$, $p_2(e') = p_1(e)$, then we denote it $e' = e^{-1}$.

1.2. LEMMA. *Let $\mathbb{G} = (E, G, p_1, p_2)$ be an E-system. Put for any $x \in G$ $L(x) = \{e \in E;\ p_1(e) = x\}$, $R(x) = \{e \in E;\ p_2(e) = x\}$. Then the set systems $\{L(x);\ x \in G\}$, $\{R(x);\ x \in G\}$ have properties:*

$$\bigcup_{x\in G} L(x) = E, \qquad \bigcup_{x\in G} R(x) = E,$$
$$x, y \in G,\ x \neq y \implies L(x) \cap L(y) = \emptyset,\ R(x) \cap R(y) = \emptyset, \qquad (*)$$
$$x, y \in G \implies \operatorname{card}\{L(x) \cap R(y)\} \leq 1.$$

P r o o f . Let $e \in E$ be any element and $p_1(e) = x$. Then $e \in L(x)$, and thus $\bigcup_{x\in G} L(x) = E$; analogously, $\bigcup_{x\in G} R(x) = E$. Let $x, y \in G$, $x \neq y$, and suppose the existence of an $e \in L(x) \cap L(y)$. Then $p_1(e) = x$, $p_1(e) = y$, which is a contradiction. Hence $L(x) \cap L(y) = \emptyset$ and, similarly, $R(x) \cap R(y) = \emptyset$. Let $x, y \in G$ and e_1, $e_2 \in E$, $e_1 \in L(x) \cap R(y)$, $e_2 \in L(x) \cap R(y)$. Then $p_1(e_1) = x = p_1(e_2)$, $p_2(e_1) = y = p_2(e_2)$, and thus $e_1 = e_2$. Hence $\operatorname{card}\{L(x) \cap R(y)\} \leq 1$. □

1.3. LEMMA. *Let E and $G \neq \emptyset$ be sets. Let $\{L(x);\ x \in G\}$, $\{R(x);\ x \in G\}$ be systems of subsets of the set E which satisfy the condition $(*)$. Put for any $e \in E$ $p_1(e) = x$, where $e \in L(x)$, $p_2(e) = y$, where $e \in R(y)$. Then (E, G, p_1, p_2) is an E-system.*

P r o o f . If $e \in E$, then $(*)$ implies the existence of a unique $x \in G$ with $e \in L(x)$. Thus, $p_1\colon E \to G$ is a mapping, and $p_2\colon E \to G$ is a mapping. Let $e_1, e_2 \in E$, $p_1(e_1) = p_1(e_2) = x$, $p_2(e_1) = p_2(e_2) = y$. Then $e_1 \in L(x) \cap R(y)$, $e_2 \in L(x) \cap R(y)$ and, from this, $e_1 = e_2$. Thus, the pair of mappings $\{p_1, p_2\}$ distinguishes elements of E and (E, G, p_1, p_2) is an E-system. □

1.4. Example. Let (G,ϱ) be a binary structure. Put, for any $e=(x,y)\in\varrho$, $p_1(e)=x$, $p_2(e)=y$. Then (ϱ,G,p_1,p_2) is an E-system. Clearly, if $e=(x,y)\in\varrho$ and if there exists $e^{-1}\in\varrho$, then $e^{-1}=(y,x)$. Further, for any $x\in G$ we have $L(x)=[x]\varrho=\big(\{x\}\times G\big)\cap\varrho$, $R(x)=\varrho[x]=\big(G\times\{x\}\big)\cap\varrho$.

1.5. DEFINITION. Let $\mathbb{G}=(E,G,p_1,p_2)$, $\mathbb{H}=(F,H,q_1,q_2)$ be E-systems. Let $\varphi\colon E\to F$, $\psi\colon G\to H$ be mappings such that $\psi\circ p_1=q_1\circ\varphi$, $\psi\circ p_2=q_2\circ\varphi$, i.e., the diagrams

$$\begin{array}{ccc} E & \xrightarrow{\ \varphi\ } & F \\ {\scriptstyle p_i}\big\downarrow & & \big\downarrow{\scriptstyle q_i} \\ G & \xrightarrow[\ \psi\ ]{} & H \end{array}$$

are commutative for $i=1,2$. Then the couple (φ,ψ) will be called a *homomorphism* of $\mathbb{G}$ into $\mathbb{H}$.

In particular, if both mappings $\varphi\colon E\to F$, $\psi\colon G\to H$ are bijections, and, if (φ^{-1},ψ^{-1}) is a homomorphism of $\mathbb{H}$ into $\mathbb{G}$, we call (φ,ψ) an *isomorphism* of $\mathbb{G}$ onto $\mathbb{H}$. E-systems $\mathbb{G}$, $\mathbb{H}$ are isomorphic if there exists an isomorphism of $\mathbb{G}$ onto $\mathbb{H}$.

1.6. Remark. If in 1.5, $G=H$ and $\psi=\mathrm{id}_G$, then we denote the homomorphism (φ,id_G) simply by φ. Thus, if $\mathbb{G}=(E,G,p_1,p_2)$, $\mathbb{H}=(F,G,q_1,q_2)$ are E-systems and $\varphi\colon E\to F$, then φ is a homomorphism of $\mathbb{G}$ into $\mathbb{H}$ if $p_1(e)=q_1\big(\varphi(e)\big)$, $p_2(e)=q_2\big(\varphi(e)\big)$ for any $e\in E$. An isomorphism φ of $\mathbb{G}$ onto $\mathbb{H}$ is a bijective homomorphism of $\mathbb{G}$ onto $\mathbb{H}$.

1.7. Let $\mathbb{G}=(E,G,p_1,p_2)$ be an E-system. We define a binary relation ϱ on the set G so: for $x,y\in G$ there is $(x,y)\in\varrho \iff$ there exists an $e\in E$ such that $p_1(e)=x$, $p_2(e)=y$. The binary structure (G,ϱ) will be denoted $B(\mathbb{G})$. Thus, if $\mathcal{E}$ is the class of all E-systems, and $\mathcal{B}$ is the class of all binary structures, we have a mapping $B\colon\mathcal{E}\to\mathcal{B}$.

1.8. Let $\mathbb{G}=(G,\varrho)$ be a binary structure. Let (ϱ,G,p_1,p_2) be the E-system described in 1.4; we denote $E(\mathbb{G})$ this E-system. Thus, E is a mapping of $\mathcal{B}$ into $\mathcal{E}$, i.e., $E\colon\mathcal{B}\to\mathcal{E}$.

1.9. THEOREM. *Let $\mathbb{G}$ be a binary structure. Then $(B\circ E)(\mathbb{G})=\mathbb{G}$, i.e., $B\circ E=\mathrm{id}_{\mathcal{B}}$.*

Proof. Let $\mathbb{G}=(G,\varrho)$; then $E(\mathbb{G})=(\varrho,G,p_1,p_2)$, where $p_1(e)=x$, $p_2(e)=y$ for $e=(x,y)\in\varrho$, and $(B\circ E)(\mathbb{G})=(G,\varrho')$, where $(x,y)\in\varrho'\iff$ there exists $e\in\varrho$ with $p_1(e)=x$, $p_2(e)=y\iff e=(x,y)\in\varrho$. Thus $\varrho=\varrho'$ and $(B\circ E)(\mathbb{G})=\mathbb{G}$. □

1.10. THEOREM. *Let $\mathbb{G}$ be an E-system. Then $\mathbb{G}$ is isomorphic with $(E \circ B)(\mathbb{G})$.*

Proof. Let $\mathbb{G} = (E, G, p_1, p_2)$. Then $B(\mathbb{G}) = (G, \varrho)$, where $(x, y) \in \varrho \iff$ there exists $e \in E$ with $p_1(e) = x$, $p_2(e) = y$, and $(E \circ B)(\mathbb{G}) = (\varrho, G, q_1, q_2)$, where $q_1(f) = x$, $q_2(f) = y$ for $f = (x, y) \in \varrho$. Let define a mapping $\varphi \colon E \to \varrho$ by $\varphi(e) = \big(p_1(e), p_2(e)\big)$. φ is in fact a mapping of E into ϱ, and we show that it is an isomorphism of $\mathbb{G}$ onto $(E \circ B)(\mathbb{G})$. By 1.6, it suffices to show that φ is a bijection, and that $p_1(e) = q_1\big(\varphi(e)\big)$, $p_2(e) = q_2\big(\varphi(e)\big)$ for any $e \in E$. If $(x, y) \in \varrho$, then there exists $e \in E$ such that $p_1(e) = x$, $p_2(e) = y$, and then $\varphi(e) = (x, y)$. Thus, φ is surjective. If e_1, $e_2 \in E$, $\varphi(e_1) = \varphi(e_2)$, then $p_1(e_1) = p_1(e_2)$, $p_2(e_1) = p_2(e_2)$, and, by definition, we have $e_1 = e_2$. Thus φ is injective and hence, bijective. If $e \in E$, then $\varphi(e) = \big(p_1(e), p_2(e)\big)$, and hence $q_1\big(\varphi(e)\big) = p_1(e)$, $q_2\big(\varphi(e)\big) = p_2(e)$. By 1.6, φ is an isomorphism. □

2. E-systems with relation

2.1. DEFINITION. Let (E, G, p_1, p_2) be an E-system. Let r be a binary relation on the set E such that it holds

$$(e_1, e_2) \in r \implies p_2(e_1) = p_1(e_2).$$

Then the structure $\mathbb{G} = (E, G, p_1, p_2, r)$ will be called an *E-system with relation.*

2.2. Example. Let (G, ϱ, r) be a double binary structure and $E(G, \varrho) = (\varrho, G, p_1, p_2)$. Then $(\varrho, G, p_1, p_2, r)$ is an E-system with relation.

2.3. DEFINITION. Let $\mathbb{G} = (E, G, p_1, p_2, r)$, $\mathbb{H} = (F, H, q_1, q_2, s)$ be E-systems with relation, and (φ, ψ) be a homomorphism of the E-system (E, G, p_1, p_2) into the E-system (F, H, q_1, q_2). We call (φ, ψ) a *homomorphism* of $\mathbb{G}$ into $\mathbb{H}$ if $(e_1, e_2) \in r \implies \big(\varphi(e_1), \varphi(e_2)\big) \in s$. If (φ, ψ) is an isomorphism of (E, G, p_1, p_2) onto (F, H, q_1, q_2), and, if $(e_1, e_2) \in r \iff \big(\varphi(e_1), \varphi(e_2)\big) \in s$, then (φ, ψ) is an *isomorphism* of $\mathbb{G}$ onto $\mathbb{H}$. E-systems with relation $\mathbb{G}$, $\mathbb{H}$ are isomorphic if there exists an isomorphism of $\mathbb{G}$ onto $\mathbb{H}$.

2.4. Remark. Analogously as in 1.6, in case $G = H$ and $\psi = \mathrm{id}_G$, we write briefly φ in place of (φ, id_G). Thus, $\varphi \colon E \to F$ is a homomorphism of $\mathbb{G} = (E, G, p_1, p_2, r)$ into $\mathbb{H} = (F, G, q_1, q_2, s)$ if $p_1(e) = q_1\big(\varphi(e)\big)$, $p_2(e) = q_2\big(\varphi(e)\big)$ for any $e \in E$, and $(e_1, e_2) \in r \implies \big(\varphi(e_1), \varphi(e_2)\big) \in s$. φ is an isomorphism of $\mathbb{G}$ onto $\mathbb{H}$ if it is a bijective homomorphism, and $(e_1, e_2) \in r \iff \big(\varphi(e_1), \varphi(e_2)\big) \in s$.

2.5. Let $\mathbb{G} = (E, G, p_1, p_2, r)$ be an E-system with relation. We define a ternary relation t on the set G as follows:

$$(x,y,z) \in t \iff \text{there exist } e_1, e_2 \in E \text{ such that } p_1(e_1) = x\,,\; p_2(e_1) = p_1(e_2) = y\,,\; p_2(e_2) = z \text{ and } (e_1, e_2) \in r\,.$$

The ternary structure (G,t) will be denoted $T(\mathbb{G})$. Thus, T is a mapping $T\colon \mathcal{R} \to \mathcal{T}$, where $\mathcal{R}$ is the class of E-systems with relation, $\mathcal{T}$ the class of ternary structures.

2.6. Let $\mathbb{G} = (G,t)$ be a ternary structure. We let define a binary relation ϱ on the set G so:

$$(x,y) \in \varrho \iff \text{there exists } z \in G \text{ such that } (x,y,z) \in t \text{ or } (z,x,y) \in t\,.$$

Let $E(\mathbb{G}) = (\varrho, G, p_1, p_2)$ be the E-system from 1.8. We define a binary relation r on the set ϱ in the following way:

$$\big((x,y),(u,v)\big) \in r \iff y = u \text{ and } (x,y,v) \in t\,.$$

The E-system with relation $(\varrho, G, p_1, p_2, r)$ will be denoted $R(\mathbb{G})$.

2.7. THEOREM. *Let $\mathbb{G}$ be a ternary structure. Then $(T \circ R)(\mathbb{G}) = \mathbb{G}$, i.e., $T \circ R = \mathrm{id}_{\mathcal{T}}$.*

P r o o f. Let $\mathbb{G} = (G,t)$. By definition, we have $R(\mathbb{G}) = (\varrho, G, p_1, p_2, r)$, where ϱ, r are defined in 2.6, and p_1, p_2 are defined in 1.4. Further, $(T \circ R)(\mathbb{G}) = (G, t')$, where t' is defined in 2.5. We show $t = t'$. Let $(x,y,z) \in t$. Then $(x,y) \in \varrho$, $(y,z) \in \varrho$ and $\big((x,y),(y,z)\big) \in r$. By 1.8 and 1.4, we have $p_1(x,y) = x$, $p_2(x,y) = y$, $p_1(y,z) = y$, $p_2(y,z) = z$, so that $(x,y,z) \in t'$. We have shown $t \subseteq t'$. If $(x,y,z) \in t'$, then there exist $e_1, e_2 \in \varrho$ such that $p_1(e_1) = x$, $p_2(e_1) = p_1(e_2) = y$, $p_2(e_2) = z$ and $(e_1,e_2) \in r$. By 1.4, it is $e_1 = (x,y)$, $e_2 = (y,z)$, and, by 2.6, $(x,y,z) \in t$. Thus $t' \subseteq t$, and hence $t = t'$. □

2.8. THEOREM. *Let $\mathbb{G}$ be an E-system with relation. Then the structures $\mathbb{G}$ and $(R \circ T)(\mathbb{G})$ are isomorphic.*

P r o o f. Let $\mathbb{G} = (E, G, p_1, p_2, r)$; then $T(\mathbb{G}) = (G,t)$, where t is defined in 2.5, and $(R \circ T)(\mathbb{G}) = (\varrho, G, q_1, q_2, s)$, where q_1, q_2, s are defined in 2.6. Let us define a mapping $\varphi\colon E \to \varrho$ in the same way as in the proof of 1.10, i.e., put $\varphi(e) = \big(p_1(e), p_2(e)\big)$ for $e \in E$. It was proved in 1.10 that φ is an isomorphism of the E-system (E, G, p_1, p_2) onto the E-system (ϱ, G, q_1, q_2). Let $e_1, e_2 \in E$, $(e_1, e_2) \in r$. Then $p_2(e_1) = p_1(e_2)$; denote $p_1(e_1) = x$, $p_2(e_1) = p_1(e_2) = y$, $p_2(e_2) = z$, so that $\varphi(e_1) = (x,y)$, $\varphi(e_2) = (y,z)$. By 2.5, it is $(x,y,z) \in t$, and, by 2.6, we have $\big((x,y),(y,z)\big) \in s$, i.e., $\big(\varphi(e_1), \varphi(e_2)\big) \in s$. On the other hand, if $e_1, e_2 \in E$, $\big(\varphi(e_1), \varphi(e_2)\big) \in s$, then, by 2.6, there exist $x, y, z \in G$ such that $\varphi(e_1) = (x, y) \in \varrho$, $\varphi(e_2) = (y, z) \in \varrho$ and $(x, y, z) \in t$. Then $x = p_1(e_1)$, $y = p_2(e_1) = p_1(e_2)$, $z = p_2(e_2)$ by definition of φ, and $(e_1, e_2) \in r$ by 2.5. Thus, φ is an isomorphism of (E, G, p_1, p_2, r) onto $(\varrho, G, q_1, q_2, s)$. □

3. Special relations

3.1. Definition. Let $\mathbb{G} = (E, G, p_1, p_2, r)$ be an E-system with relation. The relation r (and the E-system $\mathbb{G}$) is called

inversely symmetric if $(e_1, e_2) \in r \implies (e_2^{-1}, e_1^{-1}) \in r$,
inversely asymmetric if $(e_1, e_2) \in r \implies (e_2^{-1}, e_1^{-1}) \notin r$,
reversely transitive if $(e_1, e_2) \in r$, $(e_2^{-1}, e_3) \in r \implies (e_1, e_3) \in r$,
transferable if $(e_1, e_2) \in r \implies$ there exists $e_3 \in E$ with $(e_2, e_3) \in r$ and $(e_3, e_1) \in r$.

3.2. Theorem. *Let $\mathbb{G}$ be an E-system with relation. Then $\mathbb{G}$ is inversely symmetric if and only if $T(\mathbb{G})$ is a symmetric ternary structure.*

Proof. Let $\mathbb{G} = (E, G, p_1, p_2, r)$, $T(\mathbb{G}) = (G, t)$. Let $\mathbb{G}$ be inversely symmetric and let $x, y, z \in G$, $(x, y, z) \in t$. Then there exist $e_1, e_2 \in E$ with $p_1(e_1) = x$, $p_2(e_1) = p_1(e_2) = y$, $p_2(e_2) = z$ and $(e_1, e_2) \in r$. By assumption, then $(e_2^{-1}, e_1^{-1}) \in r$, and we have $p_1(e_2^{-1}) = z$, $p_2(e_2^{-1}) = p_1(e_1^{-1}) = y$, $p_2(e_1^{-1}) = x$. This implies $(z, y, x) \in t$ and t is symmetric. Let t be symmetric and let $e_1, e_2 \in E$, $(e_1, e_2) \in r$. By 2.1, it is $p_2(e_1) = p_1(e_2)$, and, if we denote $p_1(e_1) = x$, $p_2(e_1) = p_1(e_2) = y$, $p_2(e_2) = z$, then $(x, y, z) \in t$ by 2.5. Then $(z, y, x) \in t$, which means that there exist $e_3, e_4 \in E$ with $p_1(e_3) = z$, $p_2(e_3) = p_1(e_4) = y$, $p_2(e_4) = x$ and $(e_3, e_4) \in r$. From this, it follows $e_3 = e_2^{-1}$, $e_4 = e_1^{-1}$, thus $(e_2^{-1}, e_1^{-1}) \in r$, and r is inversely symmetric. □

3.3. Theorem. *Let $\mathbb{G}$ be a ternary structure. Then $\mathbb{G}$ is symmetric if and only if $R(\mathbb{G})$ is inversely symmetric.*

Proof. By 2.7, it is $(T \circ R)(\mathbb{G}) = \mathbb{G}$. Thus, if $R(\mathbb{G})$ is inversely symmetric, then, by 3.2, $\mathbb{G} = (T \circ R)(\mathbb{G})$ is symmetric. Conversely, if $\mathbb{G} = (T \circ R)(\mathbb{G})$ is symmetric, then, by 3.2, $R(\mathbb{G})$ is inversely symmetric. □

3.4. Theorem. *Let $\mathbb{G}$ be an E-system with relation. Then $\mathbb{G}$ is inversely asymmetric if and only if the ternary structure $T(\mathbb{G})$ is asymmetric.*

Proof. Let $\mathbb{G} = (E, G, p_1, p_2, r)$ and $T(\mathbb{G}) = (G, t)$. Let $\mathbb{G}$ be inversely asymmetric, and let $x, y, z \in G$, $(x, y, z) \in t$, $(z, y, x) \in t$. Then there exist $e_1, e_2 \in E$ such that $p_1(e_1) = x$, $p_2(e_1) = p_1(e_2) = y$, $p_2(e_2) = z$, $(e_1, e_2) \in r$, and there exist $e_3, e_4 \in E$ such that $p_1(e_3) = z$, $p_2(e_3) = p_1(e_4) = y$, $p_2(e_4) = x$, $(e_3, e_4) \in r$. From this, $e_3 = e_2^{-1}$, $e_4 = e_1^{-1}$ so that $(e_2^{-1}, e_1^{-1}) \in r$, which contradicts the inverse asymmetry of r. Thus t

is asymmetric. Conversely, let t be asymmetric and suppose the existence of $e_1, e_2 \in E$ with $(e_1, e_2) \in r$, $(e_2^{-1}, e_1^{-1}) \in r$. Denote $p_1(e_1) = x$, $p_2(e_1) = p_1(e_2) = y$, $p_2(e_2) = z$. Then $(x, y, z) \in t$, and, further, $p_1(e_2^{-1}) = z$, $p_2(e_2^{-1}) = p_1(e_1^{-1}) = y$, $p_2(e_1^{-1}) = x$, which implies $(z, y, x) \in t$, a contradiction. Thus, r is inversely asymmetric. □

3.5. THEOREM. *Let $\mathbb{G}$ be a ternary structure. Then $\mathbb{G}$ is asymmetric if and only if $R(\mathbb{G})$ is inversely asymmetric.*

Proof follows from 3.4 and from $(T \circ R)(\mathbb{G}) = \mathbb{G}$ similarly as the proof of 3.3. □

3.6. THEOREM. *Let $\mathbb{G}$ be an E-system with relation. Then $\mathbb{G}$ is transferable if and only if the ternary structure $T(\mathbb{G})$ is cyclic.*

Proof. Let $\mathbb{G} = (E, G, p_1, p_2, r)$, $T(\mathbb{G}) = (G, t)$, and suppose that $\mathbb{G}$ is transferable. Let $x, y, z \in G$, $(x, y, z) \in t$. Then there exist $e_1, e_2 \in E$ with $p_1(e_1) = x$, $p_2(e_1) = p_1(e_2) = y$, $p_2(e_2) = z$ and $(e_1, e_2) \in r$. As r is transferable, there exists $e_3 \in E$ with $(e_2, e_3) \in r$ and $(e_3, e_1) \in r$. From this, $p_1(e_3) = p_2(e_2) = z$, $p_2(e_3) = p_1(e_1) = x$, and we have $p_1(e_2) = y$, $p_2(e_2) = p_1(e_3) = z$, $p_2(e_3) = x$, $(e_2, e_3) \in r$. This implies $(y, z, x) \in t$ and t is cyclic. Conversely, let t be cyclic and let $e_1, e_2 \in E$, $(e_1, e_2) \in r$. If we denote $p_1(e_1) = x$, $p_2(e_1) = p_1(e_2) = y$, $p_2(e_2) = z$, we have $(x, y, z) \in t$. Hence $(y, z, x) \in t$ so that there exist $e', e_3 \in E$ such that $p_1(e') = y$, $p_2(e') = p_1(e_3) = z$, $p_2(e_3) = x$ and $(e', e_3) \in r$. As $p_1(e') = y = p_1(e_2)$, $p_2(e') = z = p_2(e_2)$, it is $e' = e_2$; thus $(e_2, e_3) \in r$. Further, $(z, x, y) \in t$ so that there exist $e'', e''' \in E$ with $p_1(e'') = z$, $p_2(e'') = p_1(e''') = x$, $p_2(e''') = y$ and $(e'', e''') \in r$. As $p_1(e'') = p_1(e_3)$, $p_2(e'') = p_2(e_3)$, it is $e'' = e_3$, and, similarly, we have $e''' = e_1$. Thus $(e_3, e_1) \in r$ and r is transferable. □

3.7. THEOREM. *Let $\mathbb{G}$ be a ternary structure. Then $\mathbb{G}$ is cyclic if and only if $R(\mathbb{G})$ is transferable.*

Proof follows from 3.6 and from $\mathbb{G} = (T \circ R)(\mathbb{G})$. □

3.8. THEOREM. *Let $\mathbb{G}$ be an E-system with relation. Then $\mathbb{G}$ is reversely transitive if and only if the ternary structure $T(\mathbb{G})$ is transitive.*

Proof. Denote $\mathbb{G} = (E, G, p_1, p_2, r)$, $T(\mathbb{G}) = (G, t)$ and suppose that $\mathbb{G}$ is reversely transitive. Let $x, y, z, u \in G$, $(x, y, z) \in t$, $(z, y, u) \in t$. Then there exist $e_1, e_2 \in E$ with $p_1(e_1) = x$, $p_2(e_1) = p_1(e_2) = y$, $p_2(e_2) = z$, $(e_1, e_2) \in r$, and there exist $e', e_3 \in E$ with $p_1(e') = z$, $p_2(e') = p_1(e_3) = y$, $p_2(e_3) = u$, $(e', e_3) \in r$. As $p_1(e') = p_2(e_2)$, $p_2(e') = p_1(e_2)$, it is $e' = e_2^{-1}$. Thus $(e_1, e_2) \in r$, $(e_2^{-1}, e_3) \in r$, and reverse transitivity of r implies $(e_1, e_3) \in r$. As $p_1(e_1) = x$, $p_2(e_1) = p_1(e_3) = y$, $p_2(e_3) = u$, we have $(x, y, u) \in t$ and t is transitive. Let t be transitive and let $e_1, e_2, e_3 \in E$, $(e_1, e_2) \in r$, $(e_2^{-1}, e_3) \in r$. If we denote $p_1(e_1) = x$, $p_2(e_1) = p_1(e_2) = y$, $p_2(e_2) = z$, then $(x, y, z) \in t$. Further, $p_1(e_2^{-1}) = z$, $p_2(e_2^{-1}) = y$, and from $(e_2^{-1}, e_3) \in r$, follows $p_1(e_3) = y$. Denote $p_2(e_3) = u$; then $(z, y, u) \in t$ and transitivity of t implies $(x, y, u) \in t$. As $p_1(e_1) = x$, $p_2(e_1) = p_1(e_3) = y$, $p_2(e_3) = u$, we see $(e_1, e_3) \in r$ and r is reversely transitive. □

3.9. THEOREM. *Let $\mathbb{G}$ be a ternary structure. Then $\mathbb{G}$ is transitive if and only if $R(\mathbb{G})$ is reversely transitive.*

Proof follows from 3.8 and 2.7. □

As a consequence of Theorems 3.5, 3.7 and 3.9, we get

3.10. THEOREM. *Let $\mathbb{G}$ be a ternary structure. Then $\mathbb{G}$ is a cyclically ordered set if and only if the structure $R(\mathbb{G})$ is inversely asymmetric, transferable and reversely transitive.*

Similarly, 3.4, 3.6 and 3.8 imply

3.11. THEOREM. *Let $\mathbb{G}$ be an E-system with relation. Then $\mathbb{G}$ is inversely asymmetric, transferable and reversely transitive if and only if $T(\mathbb{G})$ is a cyclically ordered set.*

3.12. Let $\mathbb{G} = (E, G, p_1, p_2, r)$ be an E-system with relation. The relation r (and the structure $\mathbb{G}$) will be called *conditionally transitive* if

$$(e_1, e_2) \in r,\ (e_2, e_3) \in r,\ p_2(e_1) = p_1(e_3) \implies (e_1, e_3) \in r\,.$$

3.13. THEOREM. *Let $\mathbb{G}$ be an E-system with relation. Then $\mathbb{G}$ is conditionally transitive if and only if the ternary structure $T(\mathbb{G})$ is weakly transitive.*

Proof. Denote $\mathbb{G} = (E, G, p_1, p_2, r)$, $T(\mathbb{G}) = (G, t)$. Let $\mathbb{G}$ be conditionally transitive, and let $x, y, z \in G$, $(x, y, y) \in t$, $(y, y, z) \in t$. Then there are $e_1, e_2 \in E$ with $p_1(e_1) = x$, $p_2(e_1) = p_1(e_2) = y$, $p_2(e_2) = y$, $(e_1, e_2) \in r$, and $e', e_3 \in E$ with $p_1(e') = y$, $p_2(e') = p_1(e_3) = y$, $p_2(e_3) = z$, $(e', e_3) \in r$. As $p_1(e') = p_1(e_2)$, $p_2(e') = p_2(e_2)$, it is $e' = e_2$. Thus $(e_1, e_2) \in r$, $(e_2, e_3) \in r$ and $p_2(e_1) = y = p_1(e_3)$. By assumption, we have $(e_1, e_3) \in r$, and, as $p_1(e_1) = x$, $p_2(e_1) = p_1(e_3) = y$, $p_2(e_3) = z$, it is $(x, y, z) \in t$, and t is weakly transitive. Let t be weakly transitive and let $e_1, e_2, e_3 \in E$, $(e_1, e_2) \in r$, $(e_2, e_3) \in r$, $p_2(e_1) = p_1(e_3)$. Denote $p_1(e_1) = x$, $p_2(e_1) = y$. From $(e_1, e_2) \in r$, it follows $p_1(e_2) = y$, and $(e_2, e_3) \in r$ implies $p_2(e_2) = p_1(e_3) = p_2(e_1) = y$. Thus $p_1(e_1) = x$, $p_2(e_1) = p_1(e_2) = y$, $p_2(e_2) = y$, $(e_1, e_2) \in r$, which implies $(x, y, y) \in t$. Further, denote $p_2(e_3) = z$ so that $p_1(e_2) = y$, $p_2(e_2) = p_1(e_3) = y$, $p_2(e_3) = z$, $(e_2, e_3) \in r$, and thus $(y, y, z) \in t$. The weak transitivity of t implies $(x, y, z) \in t$. At the same time, $p_1(e_1) = x$, $p_2(e_1) = p_1(e_3) = y$, $p_2(e_3) = z$ so that $(e_1, e_3) \in r$, and r is conditionally transitive. □

3.14. THEOREM. *Let $\mathbb{G}$ be a ternary structure. Then $\mathbb{G}$ is weakly transitive if and only if the structure $R(\mathbb{G})$ is conditionally transitive.*

Proof follows from 3.13 and 2.7. □

3.15. Let $\mathbb{G} = (E, G, p_1, p_2, r)$ be an E-system with relation and let $e \in E$. We say that e is *right isolated* if $(e, e') \in r$ holds for no $e' \in E$. The relation r (and the structure $\mathbb{G}$) will be called *relatively complete* if the following holds:

$$e_1, e_2 \in E \text{ are not right isolated, } p_2(e_1) = p_1(e_2) \implies (e_1, e_2) \in r.$$

Let $\mathbb{G} = (G, t)$ be a ternary structure. J. Šlapal [6] calls the relation t (and the structure $\mathbb{G}$) *feebly regular* if it holds

$$x, y, z, u, v \in G,\ (x, y, u) \in t,\ (y, z, v) \in t \implies (x, y, z) \in t.$$

3.16. THEOREM. *Let $\mathbb{G}$ be an E-system with relation. Then $\mathbb{G}$ is relatively complete if and only if the ternary structure $T(\mathbb{G})$ is feebly regular.*

Proof. Put $\mathbb{G} = (E, G, p_1, p_2, r)$, $T(\mathbb{G}) = (G, t)$. Let $\mathbb{G}$ be relatively complete, and let $x, y, z, u, v \in G$, $(x, y, u) \in t$, $(y, z, v) \in t$. Then there are $e_1, e_2 \in E$ with $p_1(e_1) = x$, $p_2(e_1) = p_1(e_2) = y$, $p_2(e_2) = u$, $(e_1, e_2) \in r$, and there are $e_3, e_4 \in E$ with $p_1(e_3) = y$, $p_2(e_3) = p_1(e_4) = z$, $p_2(e_4) = v$, $(e_3, e_4) \in r$. Thus neither e_1 nor e_3 is right isolated, and $p_2(e_1) = p_1(e_3)$. By assumption, $(e_1, e_3) \in r$, and, as $p_1(e_1) = x$, $p_2(e_1) = p_1(e_3) = y$, $p_2(e_3) = z$, there is $(x, y, z) \in t$, and t is feebly regular. Let t be feebly regular, and let $e_1, e_2 \in E$ be not right isolated and $p_2(e_1) = p_1(e_2)$. Denote $p_1(e_1) = x$, $p_2(e_1) = p_1(e_2) = y$, $p_2(e_2) = z$. By assumption, there exist $e_3, e_4 \in E$ such that $(e_1, e_3) \in r$, $(e_2, e_4) \in r$. Then $p_1(e_3) = p_2(e_1) = y$; if $p_2(e_3) = u$, we have $(x, y, u) \in t$. Similarly, $p_1(e_4) = p_2(e_2) = z$, and if $p_2(e_4) = v$, then $(y, z, v) \in t$. As t is feebly regular, it is $(x, y, z) \in t$, from which $(e_1, e_2) \in r$. Thus r is relatively complete. □

3.17. THEOREM. *Let $\mathbb{G}$ be a ternary structure. Then $\mathbb{G}$ is feebly regular if and only if the structure $R(\mathbb{G})$ is relatively complete.*

Proof follows from 3.16. and 2.7. □

4. Graphical representation

4.1. Let (E, G, p_1, p_2, r) be an E-system with relation, and let E, G be finite sets. We can assume without loss of generality $G \subseteq \mathbb{R}$ (the set of reals). Elements of the set E will be represented by points in a plane; concretely, an element $e \in E$ will coincide with the point $\big(p_1(e), p_2(e)\big)$. The relation r will be represented, in an obvious way, by means of oriented segments.

4.2. Example. Let $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, $G = \{1, 2, 3, 4, 5\}$,
$p_1(e_1) = 1$, $p_1(e_2) = 2$, $p_1(e_3) = 2$, $p_1(e_4) = 2$, $p_1(e_5) = 3$, $p_1(e_6) = 4$,
$p_2(e_1) = 2$, $p_2(e_2) = 3$, $p_2(e_3) = 4$, $p_2(e_4) = 5$, $p_2(e_5) = 4$, $p_2(e_6) = 5$,
$r = \{(e_1, e_2), (e_1, e_3), (e_1, e_4), (e_5, e_6)\}$

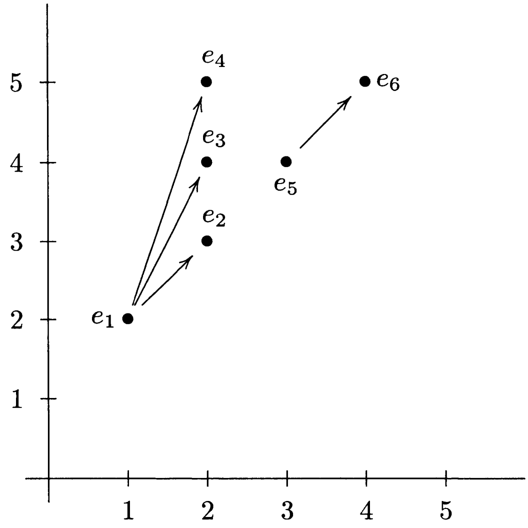


Figure 1.

4.3. Let $\mathbb{G} = (G, t)$ be a ternary structure, where G is a finite set. We can assume $G \subseteq \mathbb{R}$. Let $R(\mathbb{G}) = (\varrho, G, p_1, p_2, r)$ be the E-system with relation from 2.6. We construct the graphical representation of $R(\mathbb{G})$ as it is described in 4.1. From this representation, we can easily obtain the relation t, for by definition of the mapping R it holds

$$(e_1, e_2) \in r \iff \big(p_1(e_1), p_2(e_1), p_2(e_2)\big) \in t\,.$$

4.4. Example. Let $G = \{x, y, z, u, v\}$, $s = \{(x, y, z), (x, y, u), (x, y, v), (z, u, v)\}$, t be a cyclic hull of s and $\mathbb{G} = (G, t)$. Then

$$E = \{(x, y), (y, z), (z, x), (y, u), (u, x), (y, v), (v, x), (z, u), (u, v), (v, z)\}\,.$$

Denote $(x, y) = e_1$, $(y, z) = e_2$, $(z, x) = e_3$, $(y, u) = e_4$, $(u, x) = e_5$, $(y, v) = e_6$, $(v, x) = e_7$, $(z, u) = e_8$, $(u, v) = e_9$, $(v, z) = e_{10}$, and we have
$p_1(e_1) = x$, $p_1(e_2) = y$, $p_1(e_3) = z$, $p_1(e_4) = y$, $p_1(e_5) = u$,
$p_1(e_6) = y$, $p_1(e_7) = v$, $p_1(e_8) = z$, $p_1(e_9) = u$, $p_1(e_{10}) = v$,
$p_2(e_1) = y$, $p_2(e_2) = z$, $p_2(e_3) = x$, $p_2(e_4) = u$, $p_2(e_5) = x$,
$p_2(e_6) = v$, $p_2(e_7) = x$, $p_2(e_8) = u$, $p_2(e_9) = v$, $p_2(e_{10}) = z$,

$$r = \{(e_1,e_2),(e_2,e_3),(e_3,e_1),(e_1,e_4),(e_4,e_5),(e_5,e_1),(e_1,e_6),(e_6,e_7),(e_7,e_1),(e_8,e_9),(e_9,e_{10}),(e_{10},e_8)\}.$$

The graphical representation of $R(\mathbb{G})$ is the following:

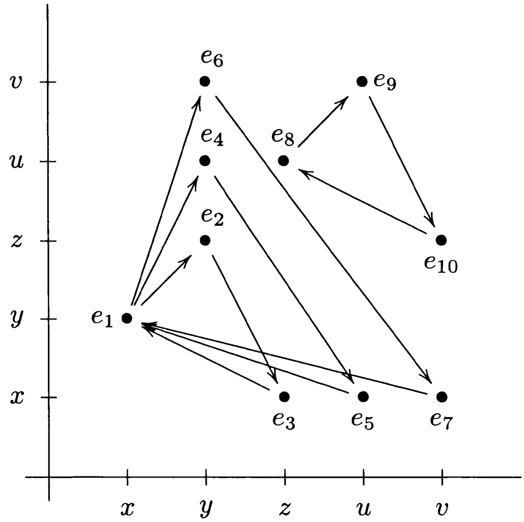


Figure 2.

Now, as, for example, $(e_4,e_5) \in r$, we have $\bigl(p_1(e_4),p_2(e_4),p_2(e_5)\bigr) = (y,u,x) \in t$.

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