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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF THE $n$TH ORDER DIFFERENTIAL EQUATION WITH DELAYED ARGUMENT

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In the paper an investigation of the $n$th order nonlinear differential equation with delayed argument of a form

$$L_n y(t) + H(t, y(g(t))) = b(t)$$

is made, where $L_n y$ is a differential operator of a form

$$L_n y(t) = a_n(t)(a_{n-1}(t) \ldots (a_1(t)a_0(t)y(t)')' \ldots ')''),$$

the functions $a_0(t), a_1(t), \ldots, a_n(t), b(t), g(t)$ are continuous on $[t_0, \infty)$ and $H(t, y)$ is continuous on $[t_0, \infty) \times \mathbb{R}$. Further assume that $g(t) \leq t, g(t) \to \infty$ for $t \to \infty$ and that $a_i(t) > 0$ on $[t_0, \infty)$ for $i = 0, 1, \ldots, n$.

We shall use the following notation:

$$L_0 y(t) = a_0(t)y(t), \quad L_i y(t) = a_i(t)(L_{i-1} y(t))',$$

$$I_0(t, t_0) = 1, \quad I_i(t, t_0, a_1, \ldots, a_i) =$$

$$= \int_{t_0}^{t} \frac{1}{a_i(s)} I_{i-1}(s, t_0, a_2, \ldots, a_i) ds,$$

$$J_i(t, t_0) = \frac{1}{a_0(t)} I_i(t, t_0, a_1, \ldots, a_i),$$

$$K_i(t, t_0) = \frac{1}{a_n(t)} I_i(t, t_0, a_{n-1}, \ldots, a_{n-i}),$$

for $i = 1, 2, \ldots, n$.

In paper [1] some asymptotic properties of solutions of the equation (1) were studied whereby the function $H(t, y)$ satisfied the assumption:

$$|H(t, y)| \leq f(t, |y|),$$

for $f \geq 0$ on $[t_0, \infty)$.
where \( f(t, r) \) is a continuous function on \([t_0, \infty) \times R\), nondecreasing in \( r \) and such that \( \frac{f(t, r)}{r} \) is nonincreasing in \( r, r > 0 \).

We shall consider the solutions of the equation (1) that exist on \([t_0, \infty)\) and satisfy condition \( \sup\{|y(s)|, s \geq t\} > 0 \) for every \( t \geq t_0 \). Let further

\[ M = \{y(t) ; y(t) \text{ is an oscillatory solution of (1) such that } \lim_{t \to \infty} y(t) = 0\} . \]

**Theorem 1.** Let (6) be valid and furthermore let

\[ \int_{t_0}^{\infty} \frac{\left| b(t) \right|}{a_n(t)} \, dt < \infty \]

and

\[ \int_{t_0}^{\infty} \frac{f(t, J_{n-1}(g(t), t_0))}{a_n(t)} \, dt < \infty . \]

Then every solution of (1) has a property

\[ y(t) = O(J_{n-1}(t, t_0)) \quad \text{for } t \to \infty . \]

**Proof.** See the proof of theorem 1.1 in paper [1].

**Theorem 2.** Let the conditions of theorem 1 be satisfied and let there exist

\[ \lim_{t \to \infty} \frac{\int_{t}^{\infty} b(s) \, ds}{a_n(s)} = \infty . \]

The every solution of (1) is nonoscillatory.

**Proof.** Let \( y(t) \) be an oscillatory solution of (1). Then the functions \( L_n y(t) \) are also oscillatory for \( i = 0, 1, \ldots, n \) and then there exists a sequence \( \{t_n\}^\infty_{n=1}, t_n \to \infty \) for \( n \to \infty \) of zero points of the function \( L_n y(t) \). From (1) we have

\[ L_n y(t) = \int_{t_n}^{t} b(s) \, ds - \int_{t_n}^{t} \frac{H(s, y(g(s)))}{a_n(s)} \, ds \]

for every \( t \geq t_n \). Since the conditions of theorem 1 are satisfied there exists \( c > 1 \) such that

\[ |y(g(t))| \leq cJ_{n-1}(g(t), t_0) . \]

From the properties of (6) we have

\[ f(t, |y(g(t))|) \leq f(t, cJ_{n-1}(g(t), t_0)) \leq cf(t, J_{n-1}(g(t), t_0)) \]
From the relation (10) by means of (6) and (11) we obtain

$$L_{n-1}y(t) = \int_{t_0}^{t} \frac{b(s)}{a_n(s)} \, ds + c \int_{t_0}^{t} \frac{f(s, J_{n-1}(g(s), t_0))}{a_n(s)} \, ds,$$

wherefrom with regard to (7), (8) and the fact that $t_n \to \infty$ it follows that there exists $\lim_{t \to \infty} L_{n-1}y(t) = 0$. From the relation (10) we have that

$$\int_{t_0}^{x} \frac{b(s)}{a_n(s)} \, ds = \int_{t_0}^{x} \frac{H(s, y(g(s)))}{a_n(s)} \, ds,$$

wherefrom

$$\left| \int_{t_0}^{x} \frac{b(s)}{a_n(s)} \, ds \right| \leq c \int_{t_0}^{x} \frac{f(s, J_{n-1}(g(s), t_0))}{a_n(s)} \, ds,$$

where $c$ is a constant $1 < c < \infty$.

Hence

$$\frac{1}{\int_{t_0}^{x} \frac{f(s, J_{n-1}(g(s), t_0))}{a_n(s)} \, ds} \leq c,$$

which contradicts assumption (9). This completes the proof of the theorem.

Example 1. We shall consider

$$t^3y''(t) + \frac{60}{(t^2 + 1)^{1/3}}t^{1/3} = -6t^{-2}, \quad t > 0.$$  (13)

The conditions of theorem 2 are satisfied and thus every solution of (13) is nonoscillatory. The solution of the equation is, e.g. $y(t) = t^{-2} + t^{-4}$.

**Theorem 3.** Let (7) be satisfied and let $H(t, y) = a(t)h(y)$, whereby

$$\lim_{y \to 0} h(y) = 0.$$  (14)

and

$$\int_{t_0}^{\infty} \frac{|a(t)|}{a_n(t)} \, dt < \infty.$$  (15)

If

$$\lim_{t \to \infty} \inf \frac{1}{a_n(s)} \int_{t}^{\infty} \frac{b(s)}{a_n(s)} \, ds = \beta > 0,$$

then the set $M$ is empty.
Proof. Let there exist the oscillatory solution $y(t)$ of (1) such that $\lim_{t \to \infty} y(t) = 0$. From the condition (14) it follows that to any arbitrary positive number $\gamma$ there exists $T$ such that $|h(y(g(t)))| < \gamma$ for every $t > T$. Let $\gamma < \beta$. From the relation (10) we have that

$$|L_{n-1}y(t)| = \left| \int_{t_n}^{t} \frac{b(s)}{a_n(s)} \, ds \right| + \gamma \int_{t_n}^{t} \frac{|a(s)|}{a_n(s)} \, ds,$$

from what with respect to (7), (15) and the fact that $t_n \to \infty$ for $n \to \infty$ it follows that $\lim_{t \to \infty} L_{n-1}y(t) = 0$. Then from the relation (12) we have that

$$\int_{t_n}^{t} \frac{b(s)}{a_n(s)} \, ds \leq \gamma \int_{t_n}^{t} \frac{|a(s)|}{a_n(s)} \, ds \quad \text{for } t_n > T.$$

But from the last relation it results that

$$\liminf_{n \to \infty} \frac{\int_{t_n}^{t} \frac{b(s)}{a_n(s)} \, ds}{\int_{t_n}^{t} \frac{|a(s)|}{a_n(s)} \, ds} \leq \gamma < \beta,$$

which contradicts assumption (16), hence the set $M$ is empty. This completes the proof of the theorem.

**Theorem 4.** Let the conditions of theorem 1 be satisfied and if furthermore

$$\lim_{t \to \infty} \left| \int_{t_0}^{t} \frac{J_{n-1}(g(t),s) b(s)}{a_n(s)} \, ds \right| < \infty \quad \text{(17)}$$

and

$$\lim_{t \to \infty} \int_{t_0}^{t} \frac{J_{n-1}(g(t),s) f(s,J_{n-1}(g(s),t_0)))}{a_n(s)} \, ds < \infty \quad \text{(18)}$$

then every oscillatory solution is bounded.

Proof. Let $y(t)$ be an oscillatory solution of (1). Then there exist $T_i$, $i = 1, \ldots, n$ such that $L_{n-1}y(T_i) = 0$, $T_i \leq T_i \leq \ldots \leq T_n$. We integrate (1) $n$-times successively from $T_i$ to $t > T_n$, multiplying the result of the $i$th integration by the function $\frac{1}{a_{n-i+1}(t)}$. We obtain

$$a_0(t)y(t) = \int_{T_n}^{t} \frac{1}{a_1(s_1)} \int_{T_{n-1}}^{s_1} \frac{1}{a_2(s_2)} \cdots \int_{T_1}^{s_{n-1}} \frac{b(s) - H(s,y(g(s)))}{a_n(s)} \, ds \, ds_{n-1} \ldots \, ds_1,$$
wherefrom

\[ |y(t)| \leq \frac{1}{a_0(t)} \int_{T_i}^{t} \frac{1}{a_1(s_1)} \int_{T_i}^{s_1} \frac{1}{a_2(s_2)} \cdots \int_{T_i}^{s_n-1} \frac{|b(s)| + |H(s, y(g(s))|}{a_n(s)} \, ds \, ds_{n-1} \cdots ds_1. \]  

(19)

From the last relation we have by using (6) and notation (3), (4)

\[ |y(t)| \leq \int_{T_i}^{t} \frac{J_{n-1}(t, s)|b(s)|}{a_n(s)} \, ds + \int_{T_i}^{t} \frac{J_{n-1}(t, s)f(s, |y(g(s))|)}{a_n(s)} \, ds. \]

Since \( g(t) \leq t, g(t) \to \infty \) for \( t \to \infty \) from the last relation is

\[ |y(g(t))| \leq \int_{T_i}^{g(t)} \frac{J_{n-1}(g(t), s)|b(s)|}{a_n(s)} \, ds +
\]

\[ + c \int_{T_i}^{g(t)} \frac{J_{n-1}(g(t), s)f(s, J_{n-1}(g(s), t_0))}{a_n(s)} \, ds \quad (20) \]

for every \( t \geq T^* \) such that \( g(t) \geq T_i \). With regard to (17) and (18) we shall obtain the assertion of the theorem.

**Theorem 5.** Let the conditions of theorem 4 be fulfilled and if furthermore

\[ \lim_{t \to \infty} J_{n-1}(g(t), t_0) < \infty, \]  

(21)

then for every oscillatory solution \( y(t) \) of (1) \( \lim_{t \to \infty} y(t) = 0 \).

**Proof.** Since \( J_{n-1}t, s \) is a nonincreasing function in \( s \) from relation (20) we have

\[ |y(g(t))| \leq J_{n-1}(g(t), T_i) \left( \int_{T_i}^{g(t)} \frac{|b(s)|}{a_n(s)} \, ds +
\]

\[ + c \int_{T_i}^{g(t)} \frac{f(s, J_{n-1}(g(s), t_0))}{a_n(s)} \, ds \right), \]

which, with respect to (21), (7), (8) and the fact that \( T_i \) may be arbitrary large, leads to the assertion of the theorem.

**Theorem 6.** Let the conditions of theorem 1, (14), (16) and (21) be fulfilled. Then every solution of equation (1) is nonoscillatory.

**Proof.** Let there exist an oscillatory solution \( y(t) \) of (1). From Theorem 5 it results that then there exists \( \lim_{t \to \infty} y(t) = 0 \) i.e. \( y(t) \in M \). Since the assumptions of theorem 3 be fulfilled we have a contradiction. This completes the proof of the theorem.
Theorem 7. Let the conditions of theorem 1 be satisfied and let furthermore
\[ \int_{t_0}^\infty K_{n-1}(t, t_0) |b(t)| \, dt < \infty, \quad (22) \]
\[ \int_{t_0}^\infty K_{n-1}(t, t_0) f(t, J_{n-1}(g(t), t_0)) \, dt < \infty, \quad (23) \]
\[ \liminf_{t \to \infty} a_0(t) > 0. \quad (24) \]

Then for every oscillatory solution \( y(t) \) of (1) \( \lim_{t \to \infty} y(t) = 0. \)

Proof. See the proof of theorem 1.2 in paper [1].

Theorem 8. Let the assumptions of theorem 7 and (14), (16) be satisfied. Then every solution of (1) is nonoscillatory.

Proof. It follows from theorems 7 and 3.

Note. Sufficient conditions for the nonoscillation of equation (1) presented in theorems 2, 6 and 8 are not equivalent, which results from the next examples.

Example 2. Consider an equation
\[ (t^3 y''(t))'' + \frac{1}{t^3} y^2(t) = t^{-\frac{5}{2}}, \quad t > 0. \quad (25) \]
The conditions of theorems 2 and 6 are not satisfied, but the conditions of theorem 8 are satisfied and thus every solution of (25) is nonoscillatory. The equation has a nonoscillatory solution, e.g. \( y(t) = t. \)

Example 3. Consider an equation
\[ (t^3 y'(t))'' + \frac{6}{t^2(t^{3/2} + 1)^{1/3}} y^\frac{1}{3}(t) = \frac{3}{8} t^{-\frac{3}{2}}, \quad t > 0. \quad (26) \]
In the equation the conditions of theorem 8 are not satisfied but the conditions of theorem 6, resp. 2 are fulfilled and thus every solution of (26) is nonoscillatory. The equation has nonoscillatory solutions, e.g.
\[ y(t) = t^{-3} + t^{-\frac{3}{2}}. \]

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АСИМПТОТИЧЕСКИЕ СВОЙСТВА РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ n-ОГО ПОРЯДКА С ОТКЛОНЯЮЩИМСЯ АРГУМЕНТОМ

Marián Rusnák, Vincent Šoltés

Резюме

В работе исследуются асимптотические свойства решений дифференциального уравнения n-ого порядка в форме

\[ L_n y(t) + H(t, y(g(t))) = b(t), \quad \text{для } n \geq 2, \]

где \( L_n y(t) = a_n(t)(a_{n-1}(t)(... (a_1(t)(a_0(t)y(t))'))')' \).

Для каждого уравнения приводятся достаточные условия, при которых каждое решение является неколеблющимся.