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EQUIVALENT ALGORITHMS FOR ESTIMATION IN LINEAR MODEL WITH CONDITION

LUBOMÍR KUBÁČEK

ABSTRACT. In the mixed linear model there exist different expressions for an estimator of a given linear function of parameters of the model. It is a welcome possibility how to check the numerical stability of calculation mainly in such cases where the size of the design matrix is large.

It is proved that analogous possibilities exist in the mixed linear model with linear condition on the first order parameters. Explicit formulae are given for the locally and uniformly best linear unbiased estimators of the first order parameters and for minimum norm quadratic estimators of the second order parameters.

Introduction

Let $Y$ be an $n$-dimensional random vector and $\mathcal{P} = \{P_{\beta, \vartheta} : \beta \in \mathcal{V}, \vartheta \in \mathcal{G}\}$ a class of probability measures with the properties: the mean value $E(Y | \beta, \vartheta) = X\beta$, $\beta \in \mathcal{V}$, and the covariance matrix $\text{Var}(Y | \beta, \vartheta)$ is $\Sigma(\vartheta) = \sum_{i=1}^{p} \vartheta_i V_i$. The $n \times k$ matrix $X$ and $n \times n$ symmetric matrices $V_1, \ldots, V_p$ are known. The notation $(Y, X\beta, \beta \in \mathcal{V}, \Sigma(\vartheta), \vartheta \in \mathcal{G})$ is used for this situation.

The set $\mathcal{V}$ is usually supposed to be equal to $\mathbb{R}^k$ (k-dimensional Euclidean space); $\mathcal{G}$ is an open set in $\mathbb{R}^p$ and fulfills the condition: $\vartheta = (\vartheta_1, \ldots, \vartheta_p)' \in \mathcal{G} \implies \Sigma(\vartheta)$ is positively semidefinite (p.s.d.); here ‘ denotes a transposition. In the following $\beta$ is called the parameter of the first order and $\vartheta$ the parameter of the second order.

In many situations $\mathcal{V} = \{u : u \in \mathbb{R}^k, b + Bu = O\} \subseteq \mathbb{R}^k$, where $B$ is a $q \times k$ matrix and $b \in \mathcal{M}(B)$ (column space of the matrix $B$, $\mathcal{M}(B) = \{Bv : v \in \mathbb{R}^k\}$). A model of measurement of angles in a plane triangle, i.e. $E[(Y_1, Y_2, Y_3)'|B] = \beta$, can serve as an example. Here obviously $\beta = (\beta_1, \beta_2, \beta_3)'$ fulfills the condition $\beta_1 + \beta_2 + \beta_3 - \pi = 0$, thus $B = (1, 1, 1)$ and $b = -\pi$.

If the parameter $\beta$ is expressed as $\beta = \beta_0 + K_B \gamma$, where $\beta_0$ is any solution of the equation $b + B\beta_0 = O$ and $K_B$ is the matrix of the type $k \times [k - R(B)]$ possessing the property $\mathcal{M}(K_B) = \text{Ker} B = \{u : Bu = O\}$, we obtain the model

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In this model, the standard formulae can be used for estimators of \( \gamma \) and \( \vartheta \) and thus no problems occur in determining the \( \vartheta_0 \)-LBLUE (locally best linear unbiased estimator) of an unbiasedly estimable linear function of \( \beta \) and \( \vartheta_0 \)-MINQUE (minim m norm quadratic unbiased estimator) of an unbiasedly and invariantly (with respect to the first order parameter) estimable linear function of \( \vartheta \). (See Chpt. 5 in [1].)

Nevertheless several interesting facts occur when problems connected with an estimation of the mentioned functions are studied in the model without reparametrization.

The aim of the present paper is to point out these facts mainly from the point of view of equivalence of different estimating procedures.

1. Notations and auxiliary statements

In the following \( P^W_A \) denotes the projection matrix on \( M(A) \) with respect to the norm given by the relation \( \|x\|_W = \sqrt{x'Wx} \); thus \( W \) is positively definite (p.d.). It is easy to prove that \( P^W_A = A(A'WA)^{-}A'W \) ( denotes a generalized inverse [3]). This expression is well defined when either \( M(A) \subset M(W) \) or \( M(A') \subset M(A'WA) \) even if \( W \) is p.s.d. For \( W = I \) (unit matrix), the notation \( P_A \) is used. The notation LM (linear model) is used for \( (Y, X\beta, \beta \in \mathbb{R}^k, \Sigma(\vartheta), \vartheta \in \vartheta) \), i.e. for the case \( \mathcal{V} = \mathbb{R}^k \).

**Lemma 1.1.** Let \( A, B \) be \( n \times n \) symmetric and p.s.d. matrices. Then \( M(A + B) = M(A,B) \).

**Proof.** see in [3, p. 122].

**Lemma 1.2.** In LM the following rules can be used:

R1: A function \( h(\beta) = h'(\beta', \beta \in \mathbb{R}^k \), is linearly unbiasedly estimable iff \( h \in M(X') \).

**Proof** is obvious.

R2: If \( h_0(\beta) = 0 \), then the class of all its linear unbiased estimators is \( \mathcal{U}_0 = \{L_0'Y: L_0 \in M(M_X)\} \), where \( M_X = I - P_X \).

**Proof** is obvious.

R3: A statistic \( L'Y \) is \( \vartheta_0 \)-LBLUE of its mean if and only if the condition

\[ M_X \Sigma(\vartheta_0) L = 0 \]

is fulfilled.

**Proof.** The statement is a consequence of [2, (i) p. 257] where the condition for the considered case is \( \mathcal{U}_0 = \{L_0'M_XY: \lambda \in \mathbb{R}^n\} \), this can be rewritten as \( M_X \Sigma(\vartheta_0) L = 0 \).
R₄: A statistic L'ₓ is UBLUE (uniformly - with respect to Ṵ ∈ Ṵ - best linear unbiased estimator) of its mean value iff the condition

\[ L \in \text{Ker} \left( \sum_{i=1}^{p} V_i M_x V_i \right) \]

is fulfilled (Ker(·) means nullspace).

Proof cf. [1, p. 203].

R₅: A function \( f(β) = f' β, β ∈ \mathbb{R}^k \), has the UBLUE iff

\[ f \in \mathcal{M} \left[ \mathcal{X}' \text{Ker} \left( \sum_{i=1}^{p} V_i M_x V_i \right) \right] \]

(here \( \text{Ker} \left( \sum_{i=1}^{p} V_i M_x V_i \right) \) denotes a matrix whose columns generate the subspace \( \text{Ker} \left( \sum_{i=1}^{p} V_i M_x V_i \right) \)).

Proof cf. [1, p 204].

Lemma 1.3. Let \( A, B, C \) be known matrices and \( AXB = C \) an equation for an unknown \( X \).

(a) This equation has a solution iff \( AA^-CB^-B = C \).

(b) If the condition in (a) is fulfilled, then the class of all solutions is

\[ \mathcal{X} = \{ A_0^-CB_0^- + Z - A_0^-AZBB_0^- : Z \text{ is an arbitrary matrix} \} \]

where \( A_0^- \), \( B_0^- \) are arbitrary but fixed generalized inverses (g-inverses) of matrices \( A, B \).

Proof see in [3, theorem 2.3.2].

In what follows \( A^+ \) denotes the Moore-Penrose g-inverse of the matrix \( A \), i.e. it is a matrix with the following properties \( AA^+A = A, A^+AA^+ = A^+, AA^+ = (AA^+)' \) and \( A^+A = (A^+A)' \).

Lemma 1.4. Let \( A \) be any \( n \times k \) matrix, \( W \) be any \( n \times n \) p.s.d. matrix and let \( \mathcal{M}(A) \subset \mathcal{M}(W) \). Then

\[ (M_AWM_A)^+ = \begin{cases} W^{-1} - W^{-1}A(A'W^{-1}A)^{-}A'W^{-1} & \text{for } W \text{ p.d.}, \\ W^+ - W^+A(A'W^+A)^{-}A'W^+ & \text{otherwise}. \end{cases} \]

If the condition \( \mathcal{M}(A) \subset \mathcal{M}(W) \) is not fulfilled, then

\[ (M_AWM_A)^+ = (W + AVA')^+ - (W + AVA')^+A[A'(W + AVA')^+A]^{-} \cdot A'(W + AVA')^+ \]

where \( V \) is any \( k \times k \) matrix with the property \( \mathcal{M}(AVA') = \mathcal{M}(A) \).

Proof. It is sufficient to verify the four above mentioned properties of the Moore-Penrose g-inverse.
Lemma 1.5. Let $N$ be an $n \times n$ p.s.d. matrix; i.e. $\exists \{J, n \times R(N) \text{ matrix}\}$ $N = JJ'$. Then $N^+ = KK'$, where $J'KK'J = I$.

Proof is obvious.

Definition 1.6. In LM a function $g(\vartheta) = g'\vartheta$, $\vartheta \in \vartheta$, is unbiasedly and invariantly estimable by a quadratic estimator if there exists a matrix $U$ possessing the properties

(a) unbiasedness $\forall \{\beta \in \mathbb{R}^k\} \forall \{\vartheta \in \vartheta\}$ $E(Y'UY | \beta, \vartheta) = g'\vartheta$,

(b) invariance $\forall \{\beta \in \mathbb{R}^k\} (Y + X\beta)'U(Y + X\beta) = Y'UY$.

Lemma 1.7. In LM a function $g(\vartheta) = g'\vartheta$, $\vartheta \in \vartheta$, is unbiasedly and invariantly estimable by an estimator $Y'UY$ iff $UX = 0$, $\text{Tr}(UV_i) = g_i$, $i = 1, \ldots, p$, i.e. iff $g \in \mathcal{M}(K^{(l)})$, where $\{K^{(l)}\}_{i,j} = \text{Tr}(M_XV_i M_XV_j)$, $i,j = 1, \ldots, p$.

Proof see in [5].

Lemma 1.8. The $\vartheta_0 - \text{MINQUE}$ of the function $g(\cdot)$ from Lemma 1.7 is

$$
\widehat{g'}\bar{\vartheta} = \sum_{i=1}^{p} \lambda_i Y' [M_X \Sigma(\vartheta_0) M_X]^+ V_i [M_X \Sigma(\vartheta_0) M_X]^+ Y,
$$

where $\lambda = (\lambda_1, \ldots, \lambda_p)'$ is a solution of the equation

$$
\lambda = g,
$$

$$
\left( [M_X \Sigma(\vartheta_0) M_X]^+ \right) = \text{Tr} \left\{ [M_X \Sigma(\vartheta_0) M_X]^+ V_i [M_X \Sigma(\vartheta_0) M_X]^+ V_j \right\},
$$

$i,j = 1, \ldots, p$.

In the case of normal distributions this estimator is $\vartheta_0 - \text{LMVQUIE}$, i.e. locally minimum variance quadratic unbiased invariant estimator.

Proof see in [5].

2. Estimators of the first order parameters

Let a matrix $K_B = \text{Ker}(B)$ be of the full rank in columns and $\mathcal{M}(K_B) = \text{Ker}(B)$, i.e. it is of the type $k \times [k - R(B)]$ ($R(B)$ is the rank of the matrix $B$).
Let $V = \{ u : u \in \mathbb{R}^k, \ b + Bu = 0 \}$ and $\beta_0$ be an arbitrary but fixed solution of the equation $b + B\beta_0 = 0$. Then it is obvious that models

$$(Y, X\beta, \beta \in V, \Sigma(\vartheta), \vartheta \in \vartheta),$$

$$(Y - X\beta_0, XK_B\gamma, \gamma \in \mathbb{R}^{k-R(B)}, \Sigma(\vartheta), \vartheta \in \vartheta)$$

and

$$\left( \begin{pmatrix} Y \\ -b \end{pmatrix}, \begin{pmatrix} X \\ B \end{pmatrix} \beta, \beta \in V, \begin{pmatrix} \Sigma(\vartheta), & 0 \\ 0, & 0 \end{pmatrix}, \vartheta \in \vartheta \right)$$

are equivalent. The symbol LMC (linear model with condition) means any of the models (2.1), (2.2), (2.3).

**Lemma 2.1.** In LMC a function $f(\beta) = f'\beta$, $\beta \in V$, is unbiasedly estimable iff $K_B'f \in \mathcal{M}(K_B'X')$.

**Proof.** As $\beta = \beta_0 + K_B\gamma$, $\gamma \in \mathbb{R}^{k-R(B)}$, the function $f(\cdot)$ can be written as $f(\beta) = f'\beta_0 + f'K_B\gamma$. It is unbiasedly estimable iff there exists a vector $L \in \mathbb{R}^n$ and a real number $l \in \mathbb{R}$ with the property $\forall \{ \gamma \in \mathbb{R}^{k-R(B)} \} \{ E(L'Y + l|\beta_0, \gamma) = L'X(\beta_0 + K_B\gamma) + l = f'\beta_0 + f'K_B\gamma \iff K_B'f = K_B'X'L \& l = f'\beta_0 - L'X\beta_0 \}$. Obviously $\exists \{ L : L \in \mathbb{R}^n, K_B'f - K_B'X'L \} \iff K_B'f \in \mathcal{M}(K_B'X') \implies l = f'\beta_0 - L'X\beta_0$. □

The equivalence $\forall \{ \gamma \in \mathbb{R}^{k-R(B)} \} L'XK_B\gamma = f'K_B\gamma \iff K_B'f = K_B'X'L$ is a consequence of a possibility to change the vector $\gamma$ in the whole space $\mathbb{R}^{k-R(B)}$. This is impossible in LMC with respect to $\beta \in V \nsubseteq \mathbb{R}^k$. Nevertheless, the following theorem states that the rule $R_1$ is valid in LMC (2.3.).

**Theorem 2.2.** In LMC a function $f(\beta) = f'\beta$, $\beta \in V$, is unbiasedly estimable iff $f \in \mathcal{M}(X', B')$.

**Proof.** It is sufficient to prove $f \in \mathcal{M}(X', B') \iff K_B'f \in \mathcal{M}(K_B'X')$ with respect to Lemma 2.1. Let $f = X'\mu + B'v$; then $K_B'f = K_B'X'\mu$ and thus $K_B'f \in \mathcal{M}(K_B'X')$. Let $K_B'f = K_B'X'z$. Then $f \in \{ X'\mu + z - (K_B')^{-1}K_B'z : z \in \mathbb{R}^k \}$ (cf. Lemma 1.3), since $X'\mu$ is a particular solution to $f$. Further $\mathcal{M}[1 - (K_B')^{-1}K_B'] = \mathcal{M}(B')$. Thus $z - (K_B')^{-1}K_B'z \in \mathcal{M}(B')$ and $f = X'\mu + B'v$. □

**Theorem 2.3.** The class of all linear unbiased estimators of the function $f_0(\beta) = 0$, $\beta \in V$, in LMC is

$$U_0 = \left\{ L_{01}'Y + L_{02}'(-b) : (L_{01}', L_{02}')' \in \mathcal{M}(X_B') \right\}$$
Thus the rule $R_2$ is valid in LMC (2.3).

Proof. With respect to Lemma 1.2 ($R_2$), the class $\mathcal{U}_0$ in LMC is

$$\{L_0'(Y - X\beta_0) : L_0' \in \mathcal{M}(M_{X\delta B})\}.$$ Let $(L_0', L_{02}') \in \mathcal{M}(M_{X\delta B}) \iff X'L_0 + B'L_{02} = O \iff K_B X'L_0 = 0 \iff L_0' \in \mathcal{M}(M_{X\delta B}) \iff X'L_0 \in \mathcal{M}(B') \iff \exists \{v \in \mathbb{R}^q \mid X'L_0 + B'v = 0 \iff \left(\begin{array}{c} L_0' \\ v \end{array}\right) \in \mathcal{M}(M_{X\delta B}) \iff L_0'(Y - X\beta_0) = L_0'Y + v'\beta_0 = L_0'Y + v'(-b) \}.$$

The following lemma is useful before studying the rule $R_3$ in LMC (2.3).

Lemma 2.4. Let $W$ be an $n \times n$ p.s.d. matrix and let $\mathcal{M}(X) \subset (W)$. Then

(a) $$P_{X \delta B}^W = \left\{ \begin{array}{ll} P_{X'(X'WX) - B'}^W & \text{for } \mathcal{M}(B') \subset \mathcal{M}(X') \\ P_{X'(X'WX + B'VB) - B'}^W & \text{otherwise.} \end{array} \right.$$ where $V$ is any $q \times q$ matrix with the property $\mathcal{M}(B'VB) = \mathcal{M}(B')$.

(b) $$P_{X \delta B}^W P_{X'(X'WX) - B'}^W = P_{X'(X'WX + B'VB) - B'}^W = P_{X'(X'WX + B'VB) - B'}^W \iff \mathcal{M}(B') \subset \mathcal{M}(X')$$

and

(c) $$P_{X \delta B}^W (X'(X'WX + B'VB) - B') = P_{X'(X'WX + B'VB) - B'}^W = 0 \iff \mathcal{M}(B') \subset \mathcal{M}(X').$$

Proof. The first equality in (a) can be proved directly; as

$$\mathcal{M}(K_B) = \mathcal{M}(M_{X\delta B}) \iff P_{X \delta B}^W = P_{X \delta B}^W = P_{X \delta B}^W = X M_{B'} M_{B'} X'' M_{X\delta B} X' W \iff M_{B'} X' W X M_{B'} X' W,$$

Now the equality $M_{B'} X' W X M_{B'} - M_{B'} = (M_{B'} X' W X M_{B'})^+$ and the implication $\mathcal{M}(B') \subset \mathcal{M}(X' W X) \implies \mathcal{M}(X' W X) = \mathcal{M}(B' X' W X) = \mathcal{M}(X' W X)^+ = \mathcal{M}(B' X' W X)^+$ from Lemma 1.4 is to be us.  
\[ P_{XK_B}^W = X(M_B' X'WX M_B) + X'W = X(X'WX)^+ X'W - X(X'WX)^+ B' [B(X'WX)^+ X'W] = P_X - P_{X(X'WX + B'VB)} W' \\
\]

In the case of the second equality in (a), it is sufficient to prove \( R(X) = R(XK_B) + R[X(X'WX + B'VB)^+ X'] = R[X(X'WX + B'VB)^+ B'] \) and \( M(XK_B) = M(X)'M_B = X'WX'WX + B'VB \\
\]

where \( \perp\ ) means orthogonality with respect to \( W \). Let \( M_1 = M(X) \), \( M_2 = M(XK_B) = M'X'M_B = \) and \( M_3 = M_B' \). As \( M_B' X'WX + B'VB + B' = M_B', X'WX + B'VB = M_B' \), \( B' = M_B' X'WX + B'VB + B' \).

To prove \( R(X) = R(XK_B) + R[X(X'WX + B'VB)^+ B'] \) we proceed as follows:

\[ P_{XK_B}^W = P_{XM_B}^W + X'M_B' [M_B' X'WX M_B] + M_B' X'WX = X[M_B' X'WX + B'VB] \cdot \\
M_B' + X'W = X[X'WX + B'VB] + X'W - X[X'WX + B'VB] \cdot B' = B[X'WX + B'VB] + B' \cdot [X'WX + B'VB] + B' \\
\]

where \( M_3 = X(X'WX + B'VB)^+ B' = M_B' X'WX + B'VB + B' \).

Both matrices \( W_{XK_B}^W \) and \( W_{XM_B}^W \) are p.s.d. and \( (W_{XK_B}^W)' W + W_{XM_B} = 0 \) (it is a consequence of \( M_2 \perp W, M_3 \) ); thus with respect to Lemma 1.1, we have

\[ R(WX(X'WX + B'VB)^+ X') = R(W_{XK_B}^W + W_{XM_B}) = R(W_{XK_B}^W) + R(W_{XM_B}) \]

The last three equalities are consequences of the following relations, cf. Lemma 1.5. \( X'WX + B'VB = JJ' \), \( W_{XK_B}^W = W_{XM_B} \) \& \( M(X'WX) \subset M(J) \times \exists \{ F : X'W = JF \} \), thus \( W_{XK_B}^W = W_{XM_B} \).

Similarly \( R(W_{XK_B}^W) = R(W_{XK_B}) \geq R(W_{XK_B}^W) \geq R(W_{XK_B}) \geq R(W_{XK_B}) \).

The statement (b) is a consequence of the equalities \( K_B' X'WX(X'WX)^+ X'W = K_B' B' = 0 \) and \( K_B' X'WX(X'WX + B'VB)^+ X'W = K_B' B' = K_B' B' = 0 \) respectively.

(e) is implied by the equality \( [M_B X'WX + B'VB] M_B^{-1} \) and by the last statement of Lemma 1.4.

**Theorem 2.5.** In LMC (2.3) the rule \( R_3 \) is valid.
Proof. With respect to Lemma 1.2, the rule $R_3$ in LMC (2.2) states: a statistic $L'_1(Y - X\beta_0) + f'\beta_0$ is the $\vartheta_0$-BLUE of the function $f(\beta) = f'\beta$, $\beta \in \mathcal{V}$, where $K'_B f = K'_B X' L_1$ ( $\implies$ $E[L'_1(Y - X\beta_0) + f'\beta_0 | \beta] = L'_1(X\beta - X\beta_0) + f'\beta_0 = L'_1 X K_B \gamma + f'\beta_0 = f'(K_B \gamma + \beta_0) = f'\beta$, $\beta \in \mathcal{V}$) iff

$$M_{X K_B} \Sigma(\vartheta_0) L_1 = O.$$  \hspace{1cm} (A)

The rule $R_3$ in LMC (2.3) states: a statistic $L'_1 Y + L'_2(-b)$ is the $\vartheta_0$-BLUE of the same function $f(\beta) = f'\beta$, $\beta \in \mathcal{V}$, where $f = X' L_1 + B' L_2$ (cf. Theorem 2.2) ( $\implies$ $E[L'_1 Y + L'_2(-b) | \beta] = L'_1 X \beta + L'_2 B \beta = f'\beta$, $\beta \in \mathcal{V}$) iff

$$M_{(X B)} \begin{pmatrix} \Sigma(\vartheta_0), & 0 \\ 0, & 0 \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = \begin{pmatrix} O \\ O \end{pmatrix}.$$ \hspace{1cm} (B)

Let (B) be valid, i.e.

$$\begin{pmatrix} (I - X(X'X + B'B)^{-}X', & -X(X'X + B'B)^{-}B' \\ -B(X'X + B'B)^{-}X', & I - B(X'X + B'B)^{-}B' \end{pmatrix} \begin{pmatrix} \Sigma(\vartheta_0), & 0 \\ 0, & 0 \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = \begin{pmatrix} O \\ O \end{pmatrix}$$

Let (A) be valid, i.e. (cf. Lemma 2.4 (c), where $W = I$, $V = I$) \{ $I - X(X'X + B'B)^{-}X' + X(X'X + B'B)^{-}B'[B(X'X + B'B)^{-}B'] - B(X'X + B'B)^{-}X' \}$ $\Sigma(\vartheta_0) L_1 = O$. Let $\mathbf{1} = I - X(X'X + B'B)^{-}X'$, $\mathbf{2} = X(X'X + B'B)^{-}B'[B(X'X + B'B)^{-}B'] - B(X'X + B'B)^{-}X'$. As the matrices $\mathbf{1}$ and $\mathbf{2}$ are symmetric and p.s.d., we have:

$$\begin{align*}
(A) & \iff \Sigma(\vartheta_0) L_1 \perp M(\mathbf{1} + \mathbf{2}) \implies \Sigma(\vartheta_0) L_1 \perp M(\mathbf{1}, \mathbf{2}) \\
& \iff \Sigma(\vartheta_0) L_1 \perp M(\mathbf{1}) & \Sigma(\vartheta_0) L_1 \perp M(\mathbf{2}) \\
& \iff (B).
\end{align*}$$

The equivalence $\Sigma(\vartheta_0) L_1 \perp M(\mathbf{1} + \mathbf{2}) \iff \Sigma(\vartheta_0) L_1 \perp M(\mathbf{1}, \mathbf{2})$ is a consequence of Lemma 1.1 (The implication (B) $\implies$ (A) is obvious.) $\square$

The rule $R_4$ in LMC (2.2) states: a statistic $L'_1(Y - X\beta_0) + f'\beta_0$, where $K'_B f = K'_B X' L_1$, is the UBLUE of its mean value $f(\beta) = f'\beta$, $\beta \in \mathcal{V}$, iff

$L_1 \in \text{Ker} \left( \sum_{i=1}^{p} V_i M_{X K_B} V_i \right)$. The corresponding statistic in LMC (2.3) is $L'_1 Y + L'_2(-b)$, where $L'_1 X + L'_2 B = f'$. The question is whether

$L_1 \in \text{Ker} \left( \sum_{i=1}^{p} V_i M_{X K_B} V_i \right) \iff \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \in \text{Ker} \left[ \sum_{i=1}^{p} \begin{pmatrix} V_i, & 0 \\ 0, & 0 \end{pmatrix} M_{(X B)} \begin{pmatrix} V_i, & 0 \\ 0, & 0 \end{pmatrix} \right].$  \hspace{1cm} (2.4)
Theorem 2.6.

(a) The equivalence (2.4) is valid; thus the rule $R_4$ holds in LMC (2.3), i.e. $L_1' Y + L_2'(-b)$ is the UBLUE of its mean value iff

$$L_1 \in \text{Ker} \left( \sum_{i=1}^{p} V_i \{ I - X'X + B'B \}^{-1} X' + X'X + B'B \}^{-1} B'(X'X + B'B)\right) \forall i \in \text{Ker} \left( \sum_{i=1}^{p} V_i \{ I - X'X + B'B \}^{-1} X' \right).$$

(b) If $M(B') \subset M(X')$, then $L_1' Y + L_2'(-b)$ is the UBLUE of its mean value iff

$$L_1 \in \text{Ker} \left( \sum_{i=1}^{p} V_i M_X V_i + \sum_{i=1}^{p} V_i P_{X'X} - B'V_i \right) \subset \text{Ker} \left( \sum_{i=1}^{p} V_i M_X V_i \right).$$

(c)

$$\text{Ker} \left( \sum_{i=1}^{p} V_i \{ I - X'X + B'B \}^{-1} X' + X'X + B'B \}^{-1} B'(X'X + B'B)\right) \forall i \in \text{Ker} \left( \sum_{i=1}^{p} V_i \{ I - X'X + B'B \}^{-1} X' \right).$$

Proof. As the matrices $V_i M_X V_i$, $i = 1, \ldots, p$, are p.s.d., we have

$$L_1 \in \text{Ker} \left( \sum_{i=1}^{p} V_i M_X Y V_i \right) \iff L_1 \perp M \left( \sum_{i=1}^{p} V_i M_X V_i \right), \quad i = 1, \ldots, p$$

$$\iff L_1 \perp M \{ V_i \{ I - X'X + B'B \}^{-1} X' + X'X + B'B \}^{-1} B' \}

\cdot \{ [B(X'X + B'B)\}^{-1} B'(X'X + B'B)\}^{-1} X' \} V_i), \quad i = 1, \ldots, p.$$ Similarly

$$\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \in \text{Ker} \left[ \sum_{i=1}^{p} \begin{pmatrix} V_i & 0 \\ O & 0 \end{pmatrix} M_X \begin{pmatrix} V_i & 0 \\ O & 0 \end{pmatrix} \right]$$

$$\iff L_1 \in \text{Ker} \left( \sum_{i=1}^{p} V_i \{ I - X'X + B'B \}^{-1} X' \right) \forall i \in \text{Ker} \left( \sum_{i=1}^{p} V_i \{ I - X'X + B'B \}^{-1} X' \right).$$

$$\iff L_1 \perp M \{ V_i \{ I - X'X + B'B \}^{-1} X' \} V_i), \quad i = 1, \ldots, p.$$
The matrix \( M \left( \begin{array}{c} x \\ b \end{array} \right) = \left( \begin{array}{cc} I - X(X'X + B'B)^{-X'} & -X(X'X + B'B)^{-B'} \\ -B(X'X + B'B)^{-X'} & I - B(X'X + B'B)^{-B'} \end{array} \right) \) is p.s.d., therefore

\[
M[X(X'X + B'B)^{-B'}] \subset M[I - X(X'X + B'B)^{-X'}]
\]

\[\implies M[V_iX(X'X + B'B)^{-B'}] \subset \{ V_i[I - X(X'X + B'B)^{-X'}] \} \in \{ V \}
\]

(the matrix \( I - X(X'X + B'B)^{-X'} \) is p.s.d. since it is a diagonal submatrix of a p.s.d. matrix). This inclusion implies the equivalence

\[
L_1 \perp M(V_i[I - X(X'X + B'B)^{-X'}] + X(X'X + B'B)^{-B'}[B(X'X + B'B)^{-B'}]^{-B}(X'X + B'B)^{-X'}) \in \{ V_i[I - X(X'X + B'B)^{-X'}] \} \in \{ V \}
\]

which proves (a) and (c).

If \( M(B') \subset M(X') \), then, with respect to Lemma 2.4. (a),

\[
M_{XK_B} = M_x + P_{X(X'X)^{-B'}}
\]

If the relationship

\[
\text{Ker} \left( \sum_{i=1}^{p} V_i M_x V_i + \sum_{i=1}^{p} V_i P_{X(X'X)^{-B'}} V_i \right)
\]

\[
= \left[ M \left( \sum_{i=1}^{p} V_i M_x V_i + \sum_{i=1}^{p} V_i P_{X(X'X)^{-B'}} V_i \right) \right]^{\perp}
\]

\[
= \left[ M \left( \sum_{i=1}^{p} V_i M_x V_i, \sum_{i=1}^{p} V_i P_{X(X'X)^{-B'}} V_i \right) \right]^{\perp}
\]

\[
\subset \left[ M \left( \sum_{i=1}^{p} V_i M_x V_i \right) \right]^{\perp} = \text{Ker} \left( \sum_{i=1}^{p} V_i M_x V_i \right)
\]

are taken into account, then (b) is proved. \( \square \)

**Lemma 2.7.** In LMC

(a) \( M[I - X(X'X + B'B)^{-X'}] = M(M_X) \oplus M[X(X'X + B'B)^{-B'}] \),

(b) \( M(X) = M(XK_B) \oplus M[X(X'X + B'B)^{-B'}] \).

**Proof.**

(a) \( M = M[I - X(X'X + B'B)^{-X'}] = \{ [I - X(X'X + B'B)^{-X'}](Xu + k_{X'}) : \}

\( u \in \mathbb{R}^k, \ k_{X'} \in \text{Ker}(X') \} \) since \( M(X) \oplus \text{Ker}(X') = \mathbb{R}^n \). Thus \( M = \{ Xu + k_{X'} - \).
\[ X(X'X + B'B)^{-} (X'X + B'B - B'B) u : u \in \mathbb{R}^k, \ k_u \in \text{Ker}(X') = \text{Ker}(X') \oplus \mathcal{M}(X'X + B'B)^{-} B'B \] is implied by the following relations: \[ \mathcal{M}(X'X + B'B)^{-} B'B \subset \mathcal{M}(X'X + B'B)^{-} B'B \subset \mathcal{M}(X'X + B'B)^{-} B'B(X'X + B'B)^{-} X' = \mathcal{M}(X'X + B'B)^{-} B'B \] since for any matrix \( A \) we have \( \mathcal{M}(A) = \mathcal{M}(AA') \).

(b) Let \( T = X(X'X + B'B)^{-} X' \) and \( U = X(X'X + B'B)^{-} B'[B(X'X + B'B)^{-} B']^{-} B'X + B'B)^{-} X' \). Then \( P_{XK_B} = T - U \) (cf. Lemma 2.4 for \( W = I \) and \( V = I \)) and \( T = P_{XK_B} + U \) Both matrices \( P_{XK_B} \), \( U \) are p.s.d and \( M_{B'}X'X + B'B)^{-} B' = O \Longrightarrow \mathcal{M}(P_{XMB'}) \perp \mathcal{M}(X'X + B'B)^{-} B' \) is implied by the following: \( \mathcal{M}(P_{XK_B} + U) = \mathcal{M}(P_{XK_B}, U) = \mathcal{M}(T) \).

Thus \( \mathcal{M}(P_{XK_B} + U) = \mathcal{M}(P_{XK_B}, U) = \mathcal{M}(XK_B) \oplus \mathcal{M}(U) = \mathcal{M}(T) \). The equality \( \mathcal{M}(T) = \mathcal{M}(X) \) is implied by the following: \( \mathcal{M}(X') \subset \mathcal{M}(X'X + B'B) \subset \mathcal{M}([X'X + B'B]^+) \), where \( (X'X + B'B)^+ \) can be expressed as \( JJ' \); thus \( \exists \{F : X' = JF \} \Longrightarrow \mathcal{M}(X) = \mathcal{M}(F'J') = \mathcal{M}(F'J'JF) \subset \mathcal{M}(F'J') = \mathcal{M}(X) \). The equality \( \mathcal{M}(U) \) is implied by an analogous consideration from the inclusion \( \mathcal{M}(B(X'X + B'B)^{-} B') \subset \mathcal{M}(B(X'X + B'B)^{-} B') (\Longleftrightarrow \mathcal{M}(BJJ'X') \subset \mathcal{M}(BJJ'B')) \).

Remark 2.8. Lemma 2.7 (a) and Theorem 2.6 show that the conditions on UBLUE are stronger in LMC than in LM. It is implied by the inclusion \( \text{Ker}\left\{ \sum_{i=1}^{p} V_i[I - X(X'X + B'B)^{-} X']V_i \right\} \subset \text{Ker}\left( \sum_{i=1}^{p} V_i M_X V_i \right) \). When \( \mathcal{M}(B') \subset \mathcal{M}(X') \), then the statement is obvious directly from Theorem 2.6 (b); further from Lemma 2.4 (a)

\[
M_{XK_B} = M_X + P_{X(X'X + B'B)^{-} B'}
\]

\[
\Rightarrow \mathcal{M}\left( \sum_{i=1}^{p} V_i M_{XK_B} V_i \right) = \mathcal{M}\left( \sum_{i=1}^{p} V_i M_X V_i + V_i P_{X(X'X + B'B)^{-} B'} V_i \right) \subset \mathcal{M}\left( \sum_{i=1}^{p} V_i M_X V_i \right)
\]

\[ \Longleftrightarrow \text{Ker}\left( \sum_{i=1}^{p} V_i M_{XK_B} V_i \right) \subset \text{Ker}\left( \sum_{i=1}^{p} V_i M_X V_i \right). \]

Theorem 2.9. In LMC a function \( f(\beta) = f' \beta ; \ \beta \in V \), can be estimated by UBLUE iff \( f \in \mathcal{M}(X'Ker\left\{ \sum_{i=1}^{p} V_i[I - X(X'X + B'B)^{-} X']V_i \right\}, B') \).
Proof. As \( f(\cdot) \) is unbiasedly estimable \( f \in \mathcal{M}(X', B') \) (cf. Theorem 2.2), i.e. \( \exists \{L_1 \in \mathbb{R}^n, L_2 \in \mathbb{R}^q\} \ f = X'L_1 + B'L_2 \), and \( L_1'Y + L_2'(-b) \) is an unbiased estimator. It is UBLUE with respect to Theorem 2.6 iff \( L_1 \in \text{Ker}\left\{ \sum_{i=1}^p V_i[l - X(X'X + B'B)^{-1}X']V_i \right\} \Rightarrow f = X'L_1 + B'L_2 \in \mathcal{M}\left( X'\text{Ker}\left\{ \sum_{i=1}^p V_i[l - X(X'X + B'B)^{-1}X']V_i \right\}, B' \right) \). □

Remark 2.10. In LMC, the condition \( b + B\beta = 0 \) enlarges the class of the linear functions of \( \beta \) which are uniformly best linearly estimable by adding \( \mathcal{M}(B') \) but simultaneously it reduces this class with respect to Remark 2.8. Compare \( R_5 \) in Lemma 1.2.

Lemma 2.11. Let in \( (Y, X\beta, \beta \in \mathbb{R}^k, \Sigma(\vartheta), \vartheta \in \vartheta) \), \( h(\beta) = h'\beta, \beta \in \mathbb{R}^k \), be a function with the property \( h \in \mathcal{M}(X') \). Then the \( \vartheta_0 \)-LBLUE of it can be calculated by the following equivalent (i.e. the same with probability one) expressions.

1. \( \hat{h}'\beta = h'\left[ X^{(m)}[\Sigma(\vartheta_0)] \right]'Y \),
2. \( \hat{h}'\beta = h'(X'MX)^{-1}X'MY \),

where \( M = (\Sigma(\vartheta_0) + UXU')^+ + K, R(X'MX) = R(X') \), \( U, K \) are arbitrary matrices with properties \( \mathcal{M}(\Sigma(\vartheta_0), X) = \mathcal{M}(\Sigma(\vartheta_0) + UXU') = \mathcal{M}(\Sigma(\vartheta_0), X) = \mathcal{M}(\Sigma(\vartheta_0) + UXU') \), \( \Sigma(\vartheta_0)K'X = 0, X'XX = 0 \),

3. \( \hat{h}'\beta = h'C_3 Y = h'C_2 Y \),

where

\[
\begin{pmatrix}
\Sigma(\vartheta_0), & X \\
X', & 0
\end{pmatrix}^{-1} =
\begin{pmatrix}
C_1, & C_2 \\
C_3, & -C_4
\end{pmatrix},
\]

4. \( \hat{h}'\beta = h'[X'(\Sigma(\vartheta_0) + XX')^{-1}X' \Sigma(\vartheta_0) + XX']^{-1}Y \)

(a special choice of \( \left( X^{(m)}[\Sigma(\vartheta_0)] \right) \)),

5. \( \hat{h}'\beta = h'\left[ (\Sigma(\vartheta_0) + XX')^{-1}X' [XX'(\Sigma(\vartheta_0) + XX')^{-1}X']^{-1}X \right]'Y \)

(a special choice of \( \left( X^{(m)}[\Sigma(\vartheta_0)] \right) \)).

Proof. See in [4] and [1, p. 161].

Theorem 2.12. Let in \( (Y, X\beta, \beta \in \mathcal{V}, \Sigma(\vartheta), \vartheta \in \vartheta) \), \( \mathcal{V} = \{u: b + Bu = O\} \), \( f(\beta) = f'\beta, \beta \in \mathcal{V} \), be a function with the property \( f \in \mathcal{M}(X', B') \). Then the \( \vartheta_0 \)-LBLUE of it can be calculated by the expressions given in Lemma 2.11,
where \( \begin{pmatrix} X \\ B \end{pmatrix} \) is substituted for \( X \), \( \begin{pmatrix} \Sigma(\vartheta_0), & 0 \\ 0, & 0 \end{pmatrix} \) for \( \Sigma(\vartheta_0) \) and \( \begin{pmatrix} Y \\ -b \end{pmatrix} \) for \( Y \).

**Proof.**

(1) is a consequence of Theorem 2.2 and definition of the \( \vartheta_0 \)-LBLUE.

(2) Let \( M = \left[ \begin{pmatrix} \Sigma(\vartheta_0), & 0 \\ 0, & 0 \end{pmatrix} + \begin{pmatrix} X \\ B \end{pmatrix} U(X',B') \right]^{-1} + K \),

where \( R(X',B')M \begin{pmatrix} X \\ B \end{pmatrix} = R(X',B') \).

Then the system \((X',B')M \begin{pmatrix} X \\ B \end{pmatrix} \tilde{\beta} = (X',B')M \begin{pmatrix} Y \\ -b \end{pmatrix} \) is solvable.

Furthermore the relationship

\[
\left[ \begin{pmatrix} \Sigma(\vartheta_0), & 0 \\ 0, & 0 \end{pmatrix} + \begin{pmatrix} X \\ B \end{pmatrix} U(X',B') \right] M' \begin{pmatrix} X \\ B \end{pmatrix}
= \left[ \begin{pmatrix} \Sigma(\vartheta_0), & 0 \\ 0, & 0 \end{pmatrix} + \begin{pmatrix} X \\ B \end{pmatrix} U(X',B') \right] 
\cdot \left( \begin{pmatrix} \Sigma(\vartheta_0), & 0 \\ 0, & 0 \end{pmatrix} + \begin{pmatrix} X \\ B \end{pmatrix} U(X',B') \right)^{-1} + K' \begin{pmatrix} X \\ B \end{pmatrix}
= \begin{pmatrix} X \\ B \end{pmatrix}
\]

implies

\[
\begin{pmatrix} \Sigma(\vartheta_0), & 0 \\ 0, & 0 \end{pmatrix} M' \begin{pmatrix} X \\ B \end{pmatrix} = \begin{pmatrix} X \\ B \end{pmatrix} - \begin{pmatrix} X \\ B \end{pmatrix} U(X',B')M' \begin{pmatrix} X \\ B \end{pmatrix} = \begin{pmatrix} X \\ B \end{pmatrix} Q.
\]

Hence, if \( Z \) is an arbitrary matrix such that \( M(Z) \subseteq \text{Ker}(X',B') \), then

\[
(X',B')M \begin{pmatrix} \Sigma(\vartheta_0), & 0 \\ 0, & 0 \end{pmatrix} Z = Q'(X',B')Z = 0,
\]

which means with respect to Theorem 2.3 and Theorem 2.5 that

\[
(X',B')M \begin{pmatrix} Y \\ -b \end{pmatrix}
\]

is the \( \vartheta_0 \)-LBLUE of its mean value \((X',B')M \begin{pmatrix} X \\ B \end{pmatrix} \beta, \beta \in \mathcal{V} \); thus

\[
(L_1',L_2') \begin{pmatrix} X \\ B \end{pmatrix} \left( (X',B')M \begin{pmatrix} X \\ B \end{pmatrix} \right)^{-1} (X',B')M \begin{pmatrix} Y \\ -b \end{pmatrix}
\]

is the \( \vartheta_0 \)-LBLUE of the function

\[
f(\beta) = (L_1',L_2') \begin{pmatrix} X \\ B \end{pmatrix} \left( (X',B')M \begin{pmatrix} X \\ B \end{pmatrix} \right)^{-1} (X',B')M \begin{pmatrix} X \\ B \end{pmatrix} \beta
= (L_1'X + L_2'B) \beta = f' \beta, \quad \beta \in \mathcal{V}.
\]
(Cf. [4] and Theorem 5.3.2 in [1], where an analogous consideration in LM is made.)

(3) is a consequence of the properties of the Pandora-Box matrix

\[
\begin{pmatrix}
\Sigma(\theta_0), & 0 & X \\
0, & 0 & B \\
X', & B' & 0
\end{pmatrix}^{-1}
= \begin{pmatrix}
C_1 & C_2 \\
C_3 & -C_4
\end{pmatrix}
\]

cf. [4] or Theorem 5.5.6 in [1] which states that

\[
C_2 = (X', B')^m \left[ \begin{pmatrix}
\Sigma(\theta_0), & 0 \\
0, & 0
\end{pmatrix} \right] \quad \text{and} \quad C_3 = (X', B')^m \left[ \begin{pmatrix}
\Sigma(\theta_0), & 0 \\
0, & 0
\end{pmatrix} \right].
\]

(4) and (5) can be obtained by a special choice of the matrix \((X', B')^m \left[ \begin{pmatrix}
\Sigma(\theta_0), & 0 \\
0, & 0
\end{pmatrix} \right] \) (in detail see in [1] Lemma 2.1.20).

Remark 2.13. An estimation of the first order parameter \(\beta\) in LMC can be proceed with respect to Theorem 2.12, Lemma 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.5, Theorem 2.6 and Theorem 2.9, in several different ways. When a numerical calculation is large, i.e. the numbers \(n, k, q\) are large, then the possibility to obtain the same results in different ways is welcome from the point of view of checking the numerical stability. An analogous possibility would be welcome in the estimation of the second order parameter.

3. Estimators of the second order parameters

The simplest kind of an estimator of a function \(g(\theta) = g'\theta, \ \theta \in \theta\), is \(Y'UY\), where the matrix \(U\) fulfils conditions for unbiasedness, i.e. \(\forall \{\beta \in \mathcal{V}\} \ \forall \{\theta \in \theta\} E(Y'UY | \beta, \theta) = g'\theta\) and invariance, i.e. \(\forall \{\beta \in \mathcal{V}\} (Y + X\beta)'U(Y + X\beta) = Y'UY\).

In LMC, two problems arise from the point of view of equivalent algorithms. The first one is connected with an existence of the unbiased and invariant estimator. With respect to Lemma 1.7, the matrix \(K^{(l)}\) in LMC is given by the relations

\[
\{K^{(l)}\}_{i,j} = \text{Tr} \left( M_{XKB} V_i M_{XKB} V_j \right), \quad i, j = 1, \ldots, p
\]

and the problem is if

\[
g \in \mathcal{M}(K^{(l)}) \iff g \in \mathcal{M}(\tilde{K}^{(l)}),
\]

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where

\[ \{K(l)\}_{i,j} = \text{Tr} \left[ M \left( \begin{array}{cc} X & B \\ \end{array} \right) \left( \begin{array}{cc} V_i & 0 \\ 0 & 0 \end{array} \right) M \left( \begin{array}{cc} X & B \\ \end{array} \right) \left( \begin{array}{cc} V_j & 0 \\ 0 & 0 \end{array} \right) \right], \quad i, j = 1, \ldots, p. \]

The other problem is connected with the expression for the estimator. With respect to Lemma 1.8, the MINQUE or the \( \vartheta_0 \)-LMVQUE in the case of normality is

\[ \sum_{i=1}^{p} \lambda_i (Y - X_0 \beta_0)' \left( M_{XK_B} \Sigma_0 M_{XK_B} \right)^+ V_i \left( M_{XK_B} \Sigma_0 M_{XK_B} \right)^+ (Y - X_0 \beta_0), \]

where \( \beta_0 \) is any solution to \( b + B \beta_0 = 0 \). The question is whether this estimator can be calculated from the expression

\[ \sum_{i=1}^{p} \lambda_i (Y', -b') \left[ M \left( \begin{array}{cc} \Sigma_0 & 0 \\ 0 & 0 \end{array} \right) M \left( \begin{array}{cc} X & B \\ \end{array} \right) \right]^+ \left( \begin{array}{cc} V_i & 0 \\ 0 & 0 \end{array} \right), \]

\[ \cdot \left[ M \left( \begin{array}{cc} \Sigma_0 & 0 \\ 0 & 0 \end{array} \right) M \left( \begin{array}{cc} X & B \\ \end{array} \right) \right]^+ (Y'). \]

**Theorem 3.1.** Let the matrix \( K(l) \) be given by the relations \( \{K(l)\}_{i,j} = \text{Tr}(M_{XK_B} V_i M_{XK_B} V_j) \), \( i, j = 1, \ldots, p \), where \( X, B, V_1, \ldots, V_p \) are matrices from LMC. Let the matrix \( \tilde{K}(l) \) be given by the relations

\[ \{\tilde{K}(l)\}_{i,j} = \text{Tr} \left[ M \left( \begin{array}{cc} X & B \\ \end{array} \right) \left( \begin{array}{cc} V_i & 0 \\ 0 & 0 \end{array} \right) M \left( \begin{array}{cc} X & B \\ \end{array} \right) \left( \begin{array}{cc} V_j & 0 \\ 0 & 0 \end{array} \right) \right], \quad i, j = 1, \ldots, p. \]

Then \( M(K(l)) = M(\tilde{K}(l)) \).

**Proof.** The \( (i,j) \)th element of the matrix \( \tilde{K}l \) can be rewritten as

\[ \{K(l)\}_{i,j} = \text{Tr} \{|I - X(X'X + B'B)^{-1}X'|V_i[I - X(X'X + B'B)^{-1}X']V_j\} \]

and with respect to Lemma 2.4 (c) with \( W = I \) and \( V = I \),

\[ \{K(l)\}_{i,j} = \text{Tr} \{|I - X(X'X + B'B)^{-1}X' + X(X'X + B'B)^{-1}B'[B(X'X + B'B)^{-1}B']^{-1} \cdot B(X'X + B'B)^{-1}X'|V_i[I - X(X'X + B'B)^{-1}X' + X(X'X + B'B)^{-1}B'[B(X'X + B'B)^{-1}B']^{-1} \cdot B(X'X + B'B)^{-1}X']V_j\}. \]

Let us denote \( S_1 = I - X(X'X + B'B)^{-1}X' \) and \( S_2 = S_1 + X(X'X + B'B)^{-1}B'[B(X'X + B'B)^{-1}B']^{-1}B(X'X + B'B)^{-1}X' \).
As $\mathcal{M}\{X(X'X + B'B) - B'[B(X'X + B'B) - B'] - B(X'X + B'B) - X'\} \subset M(S_1)$ (cf. Lemma 2.7 (a)) and $S_2 - S_1$ is obviously p.s.d., we have with respect to Lemma 1.1: $M(S_2) = M(S_1 + (S_2 - S_1)) = M(S_1, S_2 - S_1) = M(S_1)$. Now we use the spectral decomposition of the matrices $S_1$ and $S_2$, respectively.

$$S_1 = Q_1 \lambda_1 Q_1', \quad Q_1' Q_1 = I_{r,r}, \quad r = R(S_1) = R(S_2), \quad \lambda_1 \text{ is p.d.,}$$

$$S_2 = Q_2 \lambda_2 Q_2', \quad Q_2' Q_2 = I_{r,r}, \quad \lambda_2 \text{ is p.d.}$$

Thus the matrix $\tilde{K}$ can be expressed as $\tilde{V}'(S_1 \otimes S_1)\tilde{V} = \tilde{V}'(Q_1 \otimes Q_1)(\lambda_1 \otimes \lambda_1)(Q_1' \otimes Q_1')\tilde{V}$, where $\tilde{V} = (\text{vec}(V_1), \ldots, \text{vec}(V_p))$ and the matrix $K^{(l)}$ as $\tilde{V}'(S_2 \otimes S_2)\tilde{V} = \tilde{V}'(Q_2 \otimes Q_2)(\lambda_2 \otimes \lambda_2)(Q_2' \otimes Q_2')\tilde{V}$. As $M(S_1) = M(Q_1) = M(S_2) = M(Q_2)$ and the matrices $Q_1$ and $Q_2$ are of the full rank in columns, there exists a regular $r \times r$ matrix $R$ such that $Q_2 = Q_1 R$. Thus

$$K^{(l)} = \tilde{V}'(Q_1 \otimes Q_1)(R \otimes R)(\lambda_2 \otimes \lambda_2)(R' \otimes R')(Q_1' \otimes Q_1')\tilde{V}.$$ 

As the matrices $\lambda_1 \otimes \lambda_1$ and $(R \otimes R)(\lambda_2 \otimes \lambda_2)(R' \otimes R')$ are p.d., we have $M(\tilde{K}) = M(\tilde{V}'(Q_1 \otimes Q_1)) = M(K^{(l)})$. □

**Theorem 3.2.** Let $g \in M(K^{(l)})$ in LMC and let

$$\tau_1(Y - X \beta_0)$$

$$= \sum_{i=1}^{p}(Y - X \beta_0)' \lambda_i (M_{XK_B} \Sigma_0 M_{XK_B})^+ V_i (M_{XK_B} \Sigma_0 M_{XK_B})^+ (Y - X \beta_0),$$

where

$$\left((M_{XK_B} \Sigma_0 M_{XK_B})^+\right) \lambda = g. \quad (3.1)$$

Let

$$\tau_2(Y, -b) = \sum_{i=1}^{p}(Y', -b') \lambda_i \left[M_{X_B} \left(\Sigma_0, O\right) M_{X_B}\right]^+ \left(V_i, O\right).$$

$$\left[M_{X_B} \left(\Sigma_0, O\right) M_{X_B}\right]^+ \left(Y, -b\right)$$

where

$$S_{(+)} \lambda = g, \quad (+)^+ = \left[M_{X_B} \left(\Sigma_0, O\right) M_{X_B}\right]^+ \quad (3.2)$$

Then

(a) $\tau_1(Y - X \beta_0)$ does not depend on the choice of $\beta_0 \in V$.

(b) $\tau_1(Y - X \beta_0) = \tau_2(Y, -b)$ with probability one.
Proof.
(a) Let $\beta_{01}, \beta_{02} \in \mathcal{V}$, $\beta_{01} \neq \beta_{02}$. Then $\beta_{02} - \beta_{01} \in M(K_B)$ and $X(\beta_{02} - \beta_{01}) \in M(XK_B)$. As $\left( M_{XK_B} \Sigma_0 M_{XK_B} \right)^+ = \left( M_{XK_B} \Sigma_0 M_{XK_B} \right)^+ M_{XK_B}$, we have

$$\left( M_{XK_B} \Sigma_0 M_{XK_B} \right)^+ [Y - X\beta_{01} - (Y - X\beta_{02})] = \left( M_{XK_B} \Sigma_0 M_{XK_B} \right)^+ M_{XK_B} X(\beta_{02} - \beta_{01}) = 0.$$ 

(b) With respect to (a) $\beta_0$ can be chosen as $B^-(\mathbf{b})$ with an arbitrary g-inverse $B^{-}$. Thus $Y - X\beta_0 = (1, -XB^-) \begin{pmatrix} Y \\ -b \end{pmatrix}$ and

$$t_1 = (Y - X\beta_0)^\prime \left( M_{XK_B} \Sigma_0 M_{XK_B} \right)^+ V_i \left( M_{XK_B} \Sigma_0 M_{XK_B} \right)^+ (Y - X\beta_0)$$

$$= (Y', -\mathbf{b}') \begin{pmatrix} 1 \\ -(B^-)^\prime X^\prime \end{pmatrix} \left( M_{XK_B} \Sigma_0 M_{XK_B} \right)^+ V_i \left( M_{XK_B} \Sigma_0 M_{XK_B} \right)^+. 

\cdot (1, -XB^-) \begin{pmatrix} Y \\ -b \end{pmatrix}.$$ 

The corresponding term of $\tau_2(Y, -\mathbf{b})$ is

$$t_2 = (Y', -\mathbf{b}') \left[ M(X_B) \begin{pmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{pmatrix} M(X_B) \right]^+ \left( V_i, 0 \right) \cdot \left[ M(X_B) \begin{pmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{pmatrix} M(X_B) \right]^+ \begin{pmatrix} Y \\ -b \end{pmatrix}.$$ 

Let us denote $\mathbf{1} = I - X(X'X + B'B)^{-}X'$, $\mathbf{2} = -X(X'X + B'B)^{-}B'$, $\mathbf{S} = X(X'X + B'B)^{-}X'$ and $\mathbf{(*)} = M_{XK_B} \Sigma_0 M_{XK_B}$.

Then

$$M(X_B) \begin{pmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{pmatrix} M(X_B) = \begin{pmatrix} \mathbf{1} \\ \mathbf{2}' \end{pmatrix} \Sigma_0(\mathbf{1}, \mathbf{2}).$$

Let us denote

$$\left[ M(X_B) \begin{pmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{pmatrix} M(X_B) \right]^+ = \begin{pmatrix} \mathbf{A}' \\ \mathbf{B}' \end{pmatrix}.\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}.$$ 

Then

$$t_2 = (Y', -\mathbf{b}') \begin{pmatrix} \mathbf{A}' \\ \mathbf{B}' \end{pmatrix} V_i(\mathbf{1}, \mathbf{2}) \begin{pmatrix} Y \\ -b \end{pmatrix}.$$
and\[ t_1 = (Y', -b') \left( \begin{array}{c} (*)_+ \\ -(B^-)'X'(*)^+ \end{array} \right) V_i((*)_+, -(*_+XB^-) \left( \begin{array}{c} Y \\ -b \end{array} \right). \]

If there exists $\mathcal{G}$ such that the equality
\[
\left( \begin{array}{c} (*)_+, \\ -(B^-)'X'(*)^+ \\ \mathcal{G} \end{array} \right) = \left[ \left( \begin{array}{c} 1 \\ \mathcal{G} \end{array} \right) \Sigma_0(1, 2) \right]^+ 
\]
is valid, then (b) will be proved.

Let $\mathcal{G} = (B^-)'X'(*)^+XB^-$. Let
\[
E = \left( \begin{array}{c} 1 \\ \mathcal{G} \end{array} \right) \Sigma_0(1, 2) \quad \text{and} \quad G = \left( \begin{array}{c} (*)_+, \\ -(B^-)'X'(*)^+, \\ \mathcal{G} \end{array} \right)+(B^-)'X'(*)^+XB^-). 
\]

Then it must be proved:
1. $E = EGE$,  
2. $G = GE^\mathcal{G}$,  
3. $EG = G'E'$,  
4. $GE = E'G'$.  

As $E$ and $G$ are symmetric, the property (4) is implied by the property (3).

(1) \[
E = EGE = \left( \begin{array}{c} 1 \\ \mathcal{G} \end{array} \right) \Sigma_0(1, 2) \left( \begin{array}{c} 1 \\ \mathcal{G} \end{array} \right) \Sigma_0(1, 2). 
\]

The term
\[
\Sigma_0(1, 2) \left( \begin{array}{c} 1 \\ \mathcal{G} \end{array} \right) \Sigma_0 
\]
can be expressed as
\[
\Sigma_0(1)(*)_+^1 \Sigma_0 - \Sigma_0(1)(*)_+^1XB^-\mathcal{G}^2 \Sigma_0 - \Sigma_0(2)(B^-)'X'(*)^+ \Sigma_0^2 \\
+ \Sigma_0(2)(B^-)'X'(*)^+XB^-\mathcal{G}^2 \Sigma_0. 
\]

As $M_{XK}X'B = M_{XK}X$, since $\mathcal{M}(K_B) = M(I - B^-B)$ and $M_{XK}X(I - B^-B) = \mathcal{O}$, we have
\[
(*_+^1XB^-\mathcal{G}^2 = -(*)^1M_{XK}XB^-B(X'X + B'B)'X' = \\
-(*)^1X(X'X + B'B)'X'. 
\]

Thus
\[
\Sigma_0(1, 2) \left( \begin{array}{c} 1 \\ \mathcal{G} \end{array} \right) \Sigma_0 
\]
\[
= \Sigma_0 [(I - S)(*)_+^1(I - S) + (I - S)(*)_+^1 + S(*)_+^1(I - S) + S(*)_+^1] \Sigma_0 
\]
\[
= \Sigma_0(*)_+^1 \Sigma_0 
\]
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and

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \\ \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

Further

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

and

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

Further

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

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As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]

As

\[ \text{EGE} = \left( \begin{array}{cc} \Sigma_0(*)^+ + \Sigma_0(1), & \Sigma_0(*)^+ + \Sigma_0(2) \end{array} \right). \]
\[
\left( \begin{array}{c} 
1 \\
2
\end{array} \right) \Sigma_0 \left( \begin{array}{c} 
1 \\
2
\end{array} \right) \left( \begin{array}{c} 
(*)^+, \\
-(*)^+XB^-
\end{array} \right) = \left( \begin{array}{c} 
\Theta \\
\Theta
\end{array} \right).
\]

\(\Theta = \Sigma_0(1)(*)^+ - \Sigma_0(2)(B^-)'X'^+ + (I - S)\Sigma_0(2)S(*)^+ = (I - S)\Sigma_0(*)^+
\]

\(\Theta = [1 - X(X'X + B'B)'X'][\Sigma_0 \Sigma_0^+ - \Sigma_0 \Sigma_0^+ XK_B(K_B X' \Sigma_0^+ X K_B)^+ K_B X' \Sigma_0^+]
\]

\(\Theta = \Sigma_0 \Sigma_0^+ - X(X'X + B'B)'X',
\]

since \(X' \Sigma_0 \Sigma_0^+ = X'\) and

\([I - X(X'X + B'B)'X']\Sigma_0 \Sigma_0^+ X K_B
\]

\([X - X(X'X + B'B)'X'X + B'B - B'B)]K_B = 0.
\]

Thus \(\Theta = \Theta'\).

Further \(\Theta = -(1) \Sigma_0(1)(*)^+ XB^+ + \Sigma_0(2)(B^-)'X'^+ XB^-
\]

\((-1 - S)\Sigma_0(1 - S)(*)^+ XB^+ - \Sigma_0(2)(B^-)'X'^+ XB^- = -(1 - S)\Sigma_0(*)^+ XB^-;
\]

\(\Theta = \Theta' + \Sigma_0(1)(*)^+ - \Sigma_0(2)(B^-)'X'^+ XB^-
\]

\(= \Theta' + \Sigma_0(1)(*)^+ - \Sigma_0(2)(B^-)'X'^+ XB^-;
\]

\(\Theta = -B(X'X + B'B)'X' \Sigma_0 \Sigma_0^+ X K_B(K_B X' \Sigma_0^+ X K_B)^+ K_B X' \Sigma_0^+ ]
\]

\(= -B(X'X + B'B)'X'X; \Theta = -B(X'X + B'B)'X'X,
\]

\(\Theta' = -(B^-)'X'^+ XB^+ - \Sigma_0(1- S)
\]

\((-B^-)'X'[1 - X(X'X + B'B)'X'] = -(B^-)'B'B(X'X + B'B)'X'.
\]

Thus \(\Theta = \Theta'.
\]

\(\Theta = -2 \Sigma_0(1)(*)^+ XB^+ + \Sigma_0(2)(B^-)'X'^+ XB^+
\]

\((-2) \Sigma_0(1) + \Sigma_0(2)(B^-)'X'^+ XB^+
\]

\(= B(X'X + B'B)'X'X \Sigma_0 \Sigma_0^+ X K_B(K_B X' \Sigma_0^+ X K_B)^+ K_B X' \Sigma_0^+] \Sigma_0 [I - X(X'X + B'B)'X']
\]

\(= B(X'X + B'B)'X'X \Sigma_0 \Sigma_0^+ X K_B(K_B X' \Sigma_0^+ X K_B)^+ K_B X' \Sigma_0^+] \Sigma_0 [I - X(X'X + B'B)'X']
\]

\(= B(X'X + B'B)'X'X \Sigma_0 \Sigma_0^+ X K_B(K_B X' \Sigma_0^+ X K_B)^+ K_B X' \Sigma_0^+] \Sigma_0 [I - X(X'X + B'B)'X']
\]

\(= B(X'X + B'B)'X'X \Sigma_0 \Sigma_0^+ X K_B(K_B X' \Sigma_0^+ X K_B)^+ K_B X' \Sigma_0^+] \Sigma_0 [I - X(X'X + B'B)'X']
\]

\(= B(X'X + B'B)'X'X \Sigma_0 \Sigma_0^+ X K_B(K_B X' \Sigma_0^+ X K_B)^+ K_B X' \Sigma_0^+] \Sigma_0 [I - X(X'X + B'B)'X']
\]

\(= B(X'X + B'B)'X'X; \Theta = -B(X'X + B'B)'X';
\]

\(\Theta = -B(X'X + B'B)'X';
\]

\(\Theta = B(X'X + B'B)'X'X \Sigma_0 \Sigma_0^+ X K_B(K_B X' \Sigma_0^+ X K_B)^+ K_B X' \Sigma_0^+] \Sigma_0 [I - X(X'X + B'B)'X']
\]

\(= B(X'X + B'B)'X'X \Sigma_0 \Sigma_0^+ X K_B(K_B X' \Sigma_0^+ X K_B)^+ K_B X' \Sigma_0^+] \Sigma_0 [I - X(X'X + B'B)'X']
\]

\(= B(X'X + B'B)'X'X \Sigma_0 \Sigma_0^+ X K_B(K_B X' \Sigma_0^+ X K_B)^+ K_B X' \Sigma_0^+] \Sigma_0 [I - X(X'X + B'B)'X']
\]

\(= B(X'X + B'B)'X'X; \Theta = -B(X'X + B'B)'X';
\]

\(\Theta = -B(X'X + B'B)'X';
\]

\(\Theta = B(X'X + B'B)'X'X; \Theta = -B(X'X + B'B)'X';
\]

\(\Theta = B(X'X + B'B)'X'X; \Theta = -B(X'X + B'B)'X';
\]

\(\Theta = B(X'X + B'B)'X'X; \Theta = -B(X'X + B'B)'X';
\]

\(\Theta = B(X'X + B'B)'X'X; \Theta = -B(X'X + B'B)'X';
\]

\(\Theta = B(X'X + B'B)'X'X; \Theta = -B(X'X + B'B)'X';
\]

Thus \(\Theta = \Theta'.
\]

The choice of the vector \(\lambda\) in \(\tau_1(Y - X\beta_0)\) from (3.1) and in \(\tau_2(Y, -B)\) from (3.2), respectively, is a consequence of the unbiasedness of the estimators, cf. Lemma 1.8. □

Remark 3.3. With respect to Theorem 3.2 there exist different expressions for the \(\varphi_0\)-MINQUE in LMC which gave the same estimator of the function \(g(\cdot)\). This is a welcome possibility to check the numerical stability of the calculation.

REFERENCES


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