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Mathematica Slovaca, Vol. 38 (1988), No. 4, 409--417

Persistent URL: http://dml.cz/dmlcz/129723

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APPROXIMATING THE FIXED POINTS OF SOME NONLINEAR OPERATOR EQUATIONS

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Introduction. Consider the quadratic equation

\[ x = y + B(x, x) \]  \hspace{1cm} (1)

in a Banach space \( X \), where \( y \in X \) is fixed and \( B \) is a bounded symmetric bilinear operator on \( X \) [4]. We choose \( z \in X \) and \( F \) to be a bounded symmetric bilinear operator on \( X \) in such a way that the following auxiliary quadratic equation is satisfied

\[ z = y + F(z, z). \]  \hspace{1cm} (2)

We then use the solutions of (2) to approximate the fixed points of (1).

We make use of the following theorem. The proof can be found in [3].

**Theorem 1.** Let \( P \) be a nonlinear operator defined on \( D \subset X \) such that \( P \) is twice Fréchet differentiable on \( D \). Let \( z \in D \) be such that:

(i) \( \Gamma_0 = (P'(z))^{-1} \) exists and is bounded;
(ii) \( \| P(z) \| \leq \nu; \)
(iii) \( \| P''(z) \| \leq b \) if \( \| x - z \| < r \), \( U(z, r) = \{ x \in X | \| x - z \| < r \} \subset D; \)
(iv) \( h = \| \Gamma_0 \|^2 \nu b \leq \frac{1}{2}; \)
(v) \( r_0 = (1 - \sqrt{1 - 2h}) \nu \| \Gamma_0 \| / h < r. \)

Then there exists \( x \in U(z, r_0) \) such that \( P(x) = 0. \) Furthermore, \( x \) is the only solution of \( P \) contained in \( U(z, r) \cap U(z, r_1) \), where

\[ r_1 = (1 + \sqrt{1 - 2h}) \nu \| \Gamma_0 \| / h. \]

**Definition 1.** Let \( z \in X \) be such that

\[ z = y + F(z, z) \]  \hspace{1cm} (2)

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Key words and phrases. Newton's method, quadratic operator. 1980 A.M.S. classification code(s): 46(B15), 65.
for some auxiliary bounded symmetric bilinear operator $F$ defined on $D$. Define the operator $P$ on $D$ by

$$P(x) = x - z + F(z, z) - B(x, x).$$  \hfill (3)

Then every solution $x$ of (3) is a solution of (1).

Note that

$$P'(x) = I - 2B(x) \quad \text{and} \quad P''(x) = -2B.$$

The following theorem now follows easily from Theorem 1 and the above observations.

**Theorem 2.** Let $P$, $z$ be as in definition and such that:

(i) $(I - 2B(z))^{-1}$ exists and is bounded;

(ii) $\|P(z)\| = \|(F - B)(z, z)\| \leq \|F - B\| \cdot \|z\|^2 = \nu$;

(iii) $\|P''(x)\| \leq 2\|B\| = b$ if $\|x - z\| < r$, $U(z, r) \subset D$;

(iv) $\bar{h} = \|(I - 2B(z))^{-1}\| \nu \cdot b \leq \frac{1}{2}$;

(v) $r_0 = (1 - \sqrt{1 - 2\bar{h}}) \nu \|(I - 2B(z))^{-1}\| \bar{h} < r$.

Then there exists $x \in U(z, r_0)$ such that $x = y + B(x, x)$ and $x$ is unique in $U(z, r) \cap U(z, r_1)$, where

$$r_1 = (1 + \sqrt{1 - 2\bar{h}}) \nu \|(I - 2B(z))^{-1}\| \bar{h}.$$

Note that if $z$ is such that

$$\|z\| < \frac{1}{\|2B\|},$$

then the linear operator $(I - 2B(z))^{-1}$ exists and

$$\|(I - 2B(z))^{-1}\| \leq \frac{1}{1 - 2\|B\| \cdot \|z\|}.$$

In the above case, (iv) can be replaced by

$$\left(\frac{1}{1 - 2\|B\| \cdot \|z\|}\right)^2 \cdot \|F - B\| \cdot \|z\|^2 \cdot \|2B\| \leq \frac{1}{2},$$

or

$$\|z\| \leq \sqrt{\frac{2\|B\| (\sqrt{\|B\|^2 + \sqrt{\|B - F\|^2}}) + \|B - F\|^2}}. \hfill (4)$$

We now state a lemma that will allow us to replace (i) above with the inverability of the linear operator $I - 2F(z)$. The proof can be found in [1].

**Lemma.** Let $L_1$ and $L_2$ be bounded linear operators on $X$. Suppose that $(I - L_1)^{-1}$ exists as a bounded linear operator on $X$ and
\[ \| L_1L_2 - L_2^2 \| < \frac{1}{\| (I - L_1)^{-1} \|}. \]

Then \((I - L_2)^{-1}\) exists and

\[ \| (I - L_2)^{-1} \| \leq \frac{1 + \| (I - L_1)^{-1} \| \cdot \| L_2 \|}{1 - \| (I - L_1)^{-1} \| \cdot \| L_1L_2 - L_2^2 \|}. \]

If \(L_2\) is compact, then \((I - L_2)^{-1}\) is defined on all of \(X\).

We can prove the theorem.

**Theorem 3.** Let \(B\) be defined on \(D \subset X\) such that \(B(x)\) is compact for each \(x \in D\). Let \(F(z)\) be a linear operator on \(D\) for some \(z \in X\) such that

\[ z = y + F(z, z). \]

Assume:

(i) \((I - 2F(z))^{-1}\) exists and is bounded above by some \(K > 0\);

(ii) \(4\|F(z)B(z) - B(z)B(z)\| \leq \frac{1}{\| (I - 2F(z))^{-1} \|} ;

(iii) \(\| P(z) \| \leq \nu ;

(iv) \(2\|B\| \leq b\) if \(\|x - z\| < r, U(z, r) \subset D ;

(v) \(h = K^2 \nu \cdot b, K = \frac{1 + 2\| (I - 2F(z))^{-1} \| \cdot \| B(z) \|}{1 - 4\| (I - 2F(z))^{-1} \| \cdot \| F(z)B(z) - B(z)B(z) \|} ,

(vi) \(r_0 = (1 - \sqrt{1 - 2h}) K \cdot \nu / h < r .

Then there exists \(x \in U(z, r_0)\) such that \(x = y + B(x, x)\) and \(x\) is unique in \(U(z, r) \cap U(z, r_1)\), where

\[ r_1 = (1 + \sqrt{1 - 2h}) K \cdot \nu / h. \]

**Proof.** We obviously have that \((I - 2B(z))^{-1}\) exists and is bounded above by \(K\) according to the lemma, (i), (ii) and the compactness of \(B(z)\). The rest follows by applying Theorem 1 to

\[ P(x) = x - z + F(z, z) - B(x, x). \]

The natural question arises now, what the best choices for \(F\) and \(z\) are.

(a) For \(F = 0\), (2) gives \(z = y\) and (4) requires \(4\| B \| \cdot \| y \| \leq 1 .

(b) For \(F = B\), (4) requires \(\| z \| \leq \frac{1}{2\| B \|} .

The best choice, however, for \(F\) and \(z\) must be such that

\[ z = y + F(z, z). \]
The difficulties in finding solutions of the above auxiliary equation may be equivalent to those of finding solutions $x$ of (1). However, if $Q$ is the unique symmetric quadratic operator associated with $F$ such that

$$Q(x) = F(x, x) \quad \text{for all} \quad x \in X,$$

then (2) can be written as

$$z = y + Q(z). \quad (5)$$

Now assume that $Q$ is of finite rank $v = \dim(\text{span}(\text{Rang}(Q)))$ and set $x = z - y$ to obtain

$$x = Q(x + y).$$

The above equation implies that the problem of solving the auxiliary equation can be translated to a finite dimensional one since $x$ must lie in $\text{rang}(Q)$.

**Definition 1.** Let $A$ denote the set of all bounded quadratic operators $Q$ in $X$ such that $Q$ has finite rank. Denote by $E$ the set of all bounded quadratic functionals $f$ on $X$.

Let $f \in E$, $d \in X$; the operator $f \otimes d : X \to X$ sending $x \in X$ to $f(x)d \in Y$ is a bounded quadratic operator of rank one. Thus

$$Q = \sum_{i=1}^{n} f_i \otimes d_i \in A$$

for any $f_i \in E$, $i = 1, 2, \ldots, n$, $d_i \in X$, $i = 1, 2, \ldots, n$.

Note that if $Q: X \to Y$ is a bounded quadratic operator and $L: Y \to Z$ is a bounded linear operator, then $L \circ Q: X \to Z$ is a bounded quadratic operator. ($Q$ and $L$ need not be of finite rank.)

**Definition 2.** Denote by $E \otimes X$ the vector subspace generated in the space of all bounded quadratic operators by the set \{\$Q \in A | Q = f \otimes d, f \in E, d \in X\\}, so $Q \in E \otimes X$ if and only if

$$Q = \sum_{i=1}^{n} f_i \otimes d_i.$$

**Theorem 4.** $A = E \otimes X$.

**Proof.** Let $\{d_1, \ldots, d_n\}$ be a basis for $\text{rang}(Q)$ and choose $g$, such that $g_i(d_j) = \delta_{ij}, \quad i, j = 1, 2, \ldots, n$. Since $\text{rang}(Q)$ is finite dimensional, the $\{g_i\}, i = 1, 2, \ldots, n$ functionals are bounded and by the Hahn-Banach theorem they can be extended to bounded linear functionals on $X$ without increasing their norms. Let

$$f_i = g_i \circ Q, \quad i = 1, 2, \ldots, n.$$ 

Then the $f_i, i = 1, 2, \ldots, n$ are bounded quadratic functionals and
\[ Q = \sum_{i=1}^{n} f_i \otimes d_i. \]

**Definition 3.** Let \( f_i^* \), \( i = 1, 2, \ldots, n \) denote the symmetric bilinear functionals associated with the \( f_i \), \( i = 1, 2, \ldots, n \), given by
\[ f_i^*(x, y) = \frac{1}{4}(f_i(x + y) - f_i(x - y)). \]

Denote by \( C' \) the matrix of the linear transformation \( 2B(y)(\circ) \) restricted to \( \text{rang}(Q) \) relative to the basis \( d_1, \ldots, d_n \). Define the \( n \times n \) matrix \( C \), by
\[ C = I - C', \]
\[ l = \begin{bmatrix} l_1 \\ \vdots \\ l_n \end{bmatrix}, \text{ by } l_i = f_i(y), \quad i = 1, 2, \ldots, n, \]
the block of matrices \( C, C = \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix} \) by \( C_i = \{c_i^k\} \), where
\[ c_i^k = f_i^*(d_j, d_k), \quad i, j, k = 1, 2, \ldots, n. \]

Define \( \psi \) by \( \psi = C^{-1}l \) if \( |C| \neq 0 \) and the block of matrices
\[ M = \begin{bmatrix} M_1 \\ \vdots \\ M_n \end{bmatrix}, \text{ with } M_k = |C|^{-1}M'_k, \text{ where each } M'_k, k = 1, 2, \ldots, n \text{ is the } n \times n \text{ matrix which results from the determinant of the matrix } C \text{ if we replace the } k \text{th column by } \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}. \]

Define \( CM \) by
\[ \begin{bmatrix} CM_1 \\ \vdots \\ CM_n \end{bmatrix}. \]
Note that \( M'_k, k = 1, 2, \ldots, n \) is indeed an \( n \times n \) matrix. For the case \( n = 2 \),
\[ M'_1 = \begin{bmatrix} C_1 & c_{12} \\ C_2 & c_{22} \end{bmatrix} = c_{22}C_1 - c_{12}C_2. \]
\[ M'_2 = \begin{bmatrix} c_{11} & C_1 \\ c_{21} & C_2 \end{bmatrix} = c_{11}C_2 - c_{21}C_1. \]

**Theorem 5.** The point \( w \in X \) is a solution of the auxiliary equation (5) if and only if \( w = y + \sum_{i=1}^{n} \xi_i d_i \),
where the vector \( \xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \in \mathbb{R}^n \) (or \( \mathbb{C}^n \)) is a solution of

\[
\chi = l + C' \chi + \chi^+ C \chi \quad \text{in} \quad \mathbb{R}^n \quad \text{(or} \quad \mathbb{C}^n). \tag{6}
\]

Moreover, if \( |C| = |I - C'| \neq 0 \), the Cramer rule transforms the above to

\[
\chi = y + \chi^+ M \chi \quad \text{in} \quad \mathbb{R}^n \quad \text{(or} \quad \mathbb{C}^n). \tag{7}
\]

Proof. Assume that (5) has a solution \( w \in X \). Then

\[
w = y + Q(w)
= y + \sum_{i=1}^{n} f_i(w) d_i.
\]

Apply \( f_1, f_2, \ldots, f_n \) in turn to this vector identity to obtain for \( p = 1, 2, \ldots, n \)

\[
f_p(w) = f_p \left( y + \sum_{k=1}^{n} f_k(w) d_k \right)
= f_p(y) + \sum_{k=1}^{n} f_k^2(w) f_p(d_k) + 2 \sum_{k=1}^{n} f_k(w) f_p^*(y, d_k)
+ 2 \sum_{i \neq j} f_i(w) f_j(w) f_p^*(d_i, d_j).
\]

Letting

\[
f_i(w) = x_i, \quad i = 1, 2, \ldots, n
\]

and writing these equations in vector form, we obtain

\[
\chi = l + C' \chi + \chi^+ C \chi
\]

or

\[
C \chi = l + \chi^+ C \chi.
\]

Since \( |C| \neq 0 \), we obtain (7) by composing both sides of the above equation by \( C^{-1} \).

Conversely, given (7), assume (6) has a solution vector \( \xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \). Let \( w \in X \)

be defined as

\[
w = y + \sum_{i=1}^{n} \xi_i d_i.
\]

Apply \( f_1, f_2, \ldots, f_n \) in turn to this vector identity to obtain for \( p = 1, 2, \ldots, n \),
\[
f_p(w) = f_p(y) + \sum_{k=1}^{n} \xi_k f_p(d_k) + 2 \sum_{k=1}^{n} \xi_k f^*(y, d_k) + 2 \sum_{i \neq j} \xi_i \xi_j f^*(d_i, d_j),
\]

or in matrix notation,

\[
f(w) = l + C' \xi + \xi'^r C \xi.
\]

Now since \(\xi\) satisfies (6) we have

\[
\xi = l + C' \xi + \xi'^r C \xi.
\]

Comparing the last two equations, we get

\[
\xi_i = f_i(w), \quad i = 1, 2, \ldots, n,
\]

so

\[
w = y + \sum_{i=1}^{n} f_i(w) d_i,
\]
or

\[
w = y + Q(w).
\]

Therefore, \(w\) is a solution of (5) and the theorem is proved.

Example. Let \(X = C[0, 1]\) and consider the equation

\[
x(s) = s + s \int_{0}^{1} x^2(t) \, dt,
\]

where \(s \in [0, 1]\). This equation is of the form (5), with \(\text{rank}(Q) = 1\),

\[
y(s) = s
\]
\[
d = s, \quad \text{and}
\]
\[
d(s) = \int_{0}^{1} x^2(t) \, dt.
\]

Using the formula,

\[
f^*(v, w) = \frac{1}{4} (f(v + w) - f(v - w)),
\]

we have

\[
C = 1 - 2f^*(y, d) = 1 - 2 \frac{1}{4} \int_{0}^{1} 4s^2 \, ds = \frac{1}{3}
\]
\[
l = f(y) = f(s) = \int_{0}^{1} s^2 \, ds = \frac{1}{3}
\]
\[ C = f(d) = f(s) = \int_0^1 s^2 \, ds = \frac{1}{3} \]

\[ v = 3 \cdot \frac{1}{3} = 1 \]

\[ M = 3 \cdot \frac{1}{3} = 1. \]

Therefore, (6) becomes

\[ \xi = 1 + \xi^2 \text{ in } \mathcal{C} \text{ with solutions } \frac{1 \pm i \sqrt{3}}{2}; \]

since \( x = y + \xi d \), we finally have

\[ x(s) = \left( \frac{3 \pm i \sqrt{3}}{2} \right) s. \]

Now note that if the linear operator \( F(z) \) is of finite rank \( n \), then the linear operator \( I - 2F(z) \) is invertible if and only if for every fixed \( v \in X \) there exists \( w \in X \) such that

\[ w - 2F(z, w) = v. \]

Since \( F(z) \) is of finite rank \( n \), the above equation can be translated exactly as in Theorem 5 for the quadratic case to a linear system in \( ^n \), or \( ^n \), similar to system (7).

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Received October 31, 1986

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АППРОКСИМАЦИЯ НЕПОДВИЖНЫХ ТОЧЕК НЕКОТОРЫХ НЕЛИНЕЙНЫХ ОПЕРАТОРНЫХ УРАВНЕНИЙ

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Резюме

Рассмотрим пару квадратных уравнений

\[ x = y + B(x, x) \]
\[ z = y + F(z, z) \]

в банаховом пространстве \( X \), где \( y \in X \) есть фиксированная точка, а \( B, F \) — ограниченные симметрические билинейные операторы на \( X \). Предположим, что решение \( z \) второго уравнения известно, и используем его на аппроксимацию решения первого уравнения. В частном случае, когда \( F \) есть оператор конечного ранга, показывается, что проблема нахождения решения \( z \) второго уравнения эквивалента задаче решения системы квадратных уравнений в \( \ell^2 \) или \( \ell^\infty \).