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APPROXIMATING THE FIXED POINTS OF SOME NONLINEAR OPERATOR EQUATIONS

IOANNIS K. ARGYROS

Introduction. Consider the quadratic equation

$$x = y + B(x, x) \tag{1}$$

in a Banach space X, where $y \in X$ is fixed and B is a bounded symmetric bilinear operator on X [4]. We choose $z \in X$ and F to be a bounded symmetric bilinear operator on X in such a way that the following auxiliary quadratic equation is satisfied

$$z = y + F(z, z). \tag{2}$$

We then use the solutions of (2) to approximate the fixed points of (1).

We make use of the following theorem. The proof can be found in [3].

Theorem 1. *Let P be a nonlinear operator defined on $D \subset X$ such that P is twice Fréchet differentiable on D. Let $z \in D$ be such that:*

(i) $\Gamma_0 = (P'(z))^{-1}$ *exists and is bounded;*

(ii) $\|P(z)\| \leq v$;

(iii) $\|P''(x)\| \leq b$ if $\|x - z\| < r$, $U(z, r) = \{x \in X | \, \|x - z\| < r\} \subset D$;

(iv) $h = \|\Gamma_0\|^2 vb \leq \frac{1}{2}$;

(v) $r_0 = (1 - \sqrt{1 - 2h})\, v \|\Gamma_0\|/h < r$.

Then there exists $x \in U(z, r_0)$ such that $P(x) = 0$. Furthermore, x is the only solution of P contained in $U(z, r) \cap U(z, r_1)$, where

$$r_1 = (1 + \sqrt{1 - 2h})\, \|\Gamma_0\|\, v/h.$$

Definition 1. *Let $z \in X$ be such that*

$$z = y + F(z, z) \tag{2}$$

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for some auxiliary bounded symmetric bilinear operator F defined on D. Define the operator P on D by

$$P(x) = x - z + F(z,z) - B(x,x). \tag{3}$$

Then every solution x of (3) is a solution of (1).

Note that

$$P'(x) = I - 2B(x) \quad \text{and} \quad P''(x) = -2B.$$

The following theorem now follows easily from Theorem 1 and the above observations.

Theorem 2. *Let P, z be as in definition and such that:*

(i) $(I - 2B(z))^{-1}$ *exists and is bounded;*

(ii) $\|P(z)\| = \|(F - B)(z,z)\| \leq \|F - B\| \cdot \|z\|^2 = v$;

(iii) $\|P''(x)\| \leq 2\|B\| = b$ if $\|x - z\| < r$, $U(z,r) \subset D$;

(iv) $\bar{h} = \|(I - 2B(z))^{-1}\|^2 v \cdot b \leq \dfrac{1}{2}$;

(v) $r_0 = (1 - \sqrt{1 - 2\bar{h}})\, v \cdot \|(I - 2B(z))^{-1}\| / \bar{h} < r$.

Then there exists $x \in U(z, r_0)$ such that $x = y + B(x,x)$ and x is unique in $U(z,r) \cap$ $\cap\, U(z, r_1)$, where

$$r_1 = (1 + \sqrt{1 - 2\bar{h}})\, v \|(I - 2B(z))^{-1}\| / \bar{h}.$$

Note that if z is such that

$$\|z\| < \frac{1}{\|2B\|},$$

then the linear operator $(I - 2B(z))^{-1}$ exists and

$$\|(I - 2B(z))^{-1}\| \leq \frac{1}{1 - 2\|B\| \cdot \|z\|}.$$

In the above case, (iv) can be replaced by

$$\left(\frac{1}{1 - 2\|B\| \cdot \|z\|}\right)^2 \|F - B\| \cdot \|z\|^2 2\|B\| \leq \frac{1}{2},$$

or

$$\|z\| \leq [2\sqrt{\|B\|}(\sqrt{\|B\|} + \sqrt{\|B - F\|})]^{-1}. \tag{4}$$

We now state a lemma that will allow us to replace (i) above with the inverability of the linear operator $I - 2F(z)$. The proof can be found in [1].

Lemma. *Let L_1 and L_2 be bounded linear operators on X. Suppose that $(I - L_1)^{-1}$ exists as a bounded linear operator on X and*

$$\|L_1 L_2 - L_2^2\| < \frac{1}{\|(I - L_1)^{-1}\|}.$$

Then $(I - L_2)^{-1}$ *exists and*

$$\|(I - L_2)^{-1}\| \le \frac{1 + \|(I - L_1)^{-1}\| \cdot \|L_2\|}{1 - \|(I - L_1)^{-1}\| \cdot \|L_1 L_2 - L_2^2\|}.$$

If L_2 *is compact, then* $(I - L_2)^{-1}$ *is defined on all of* X.

We can prove the theorem.

Theorem 3. *Let* B *be defined on* $D \subset X$ *such that* $B(x)$ *is compact for each* $x \in D$. *Let* $F(z)$ *be a linear operator on* D *for some* $z \in X$ *such that*

$$z = y + F(z, z).$$

Assume:

(i) $(I - 2F(z))^{-1}$ *exists and is bounded above by some* $K > 0$;

(ii) $4\|F(z)B(z) - B(z)B(z)\| \le \dfrac{1}{\|(I - 2F(z))^{-1}\|}$;

(iii) $\|P(z)\| \le v$;

(iv) $2\|B\| \le b$ if $\|x - z\| < r$, $U(z, r) \subset D$;

(v) $h = K^2 v \cdot b$, $K = \dfrac{1 + 2\|(I - 2F(z))^{-1}\| \cdot \|B(z)\|}{1 - 4\|(I - 2F(z))^{-1}\| \, \|F(z)B(z) - B(z)B(z)\|}$,

(vi) $r_0 = (1 - \sqrt{1 - 2h}) K \cdot v/h < r$.

Then there exists $x \in U(z, r_0)$ *such that* $x = y + B(x, x)$ *and* x *is unique in* $U(z, r) \cap U(z, r_1)$, *where*

$$r_1 = (1 + \sqrt{1 - 2h}) K \cdot v/h.$$

Proof. We obviously have that $(I - 2B(z))^{-1}$ exists and is bounded above by K according to the lemma, (i), (ii) and the compactness of $B(z)$. The rest follows by applying Theorem 1 to

$$P(x) = x - z + F(z, z) - B(x, x).$$

The natural question arises now, what the best choices for F and z are.

(a) For $F = 0$, (2) gives $z = y$ and (4) requires $4\|B\| \cdot \|y\| \le 1$.

(b) For $F = B$, (4) requires $\|z\| \le \dfrac{1}{2\|B\|}$.

The best choice, however, for F and z must be such that

$$z = y + F(z, z).$$

The difficulties in finding solutions of the above auxiliary equation may be equivalent to those of finding solutions x of (1). However, if Q is the unique symmetric quadratic operator associated with F such that

$$Q(x) = F(x, x) \quad \text{for all} \quad x \in X,$$

then (2) can be written as

$$z = y + Q(z). \tag{5}$$

Now assume that Q is of finite rank $\nu = \dim(\text{span}(\text{Rang}(Q)))$ and set $x = z - y$ to obtain

$$x = Q(x + y).$$

The above equation implies that the problem of solving the auxiliary equation can be translated to a finite dimensional one since x must lie in $\text{rang}(Q)$.

Definition 1. *Let A denote the set of all bounded quadratic operators Q in X such that Q has finite rank. Denote by E the set of all bounded quadratic functionals f on X.*

Let $f \in E$, $d \in X$; the operator $f \otimes d$: $X \to X$ sending $x \in X$ to $f(x) d \in X$ is a bounded quadratic operator of rank one. Thus

$$Q = \sum_{i=1}^{n} f_i \otimes d_i \in A$$

for any $f_i \in E$, $i = 1, 2, \ldots, n$, $d_i \in X$, $i = 1, 2, \ldots, n$.

Note that if $Q = X \to Y$ is a bounded quadratic operator and L: $Y \to Z$ is a bounded linear operator, then $L \circ Q$: $X \to Z$ is a bounded quadratic operator. (Q and L need not be of finite rank.)

Definition 2. *Denote by $E \otimes X$ the vector subspace generated in the space of all bounded quadratic operators by the set $\{Q \in A \mid Q = f \otimes d,\ f \in E,\ d \in X\}$, so $Q \in E \otimes X$ if and only if*

$$Q = \sum_{i=1}^{n} f_i \otimes d_i.$$

Theorem 4. $A = E \otimes X$.

Proof. Let $\{d_1, \ldots, d_n\}$ be a basis for $\text{rang}(Q)$ and choose g_i such that $g_i(d_j) = \delta_{ij}$, $i, j = 1, 2, \ldots, n$. Since $\text{rang}(Q)$ is finite dimensional, the $\{g_i\}$, $i = 1, 2, \ldots, n$ functionals are bounded and by the Hahn-Banach theorem they can be extended to bounded linear functionals on X without increasing their norms. Let

$$f_i = g_i \circ Q, \quad i = 1, 2, \ldots, n.$$

Then the f_i, $i = 1, 2, \ldots, n$ are bounded quadratic functionals and

$$Q = \sum_{i=1}^{n} f_i \otimes d_i.$$

Definition 3. *Let f_i^*, $i = 1, 2, \ldots, n$ denote the symmetric bilinear functionals associated with the f_i, $i = 1, 2, \ldots, n$,* given by

$$f_i^*(x, y) = \frac{1}{4}(f_i(x + y) - f_i(x - y)).$$

Denote by C' the matrix of the linear transformation $2B(y)(\circ)$ restricted to rang(Q) *relative to the basis $d_1, \ldots, d_n$. Define the $n \times n$ matrix C, by*

$$C = I - C',$$

$$\underline{l} = \begin{bmatrix} l_1 \\ \vdots \\ l_n \end{bmatrix}, \text{ by } l_i = f_i(y), \quad i = 1, 2, \ldots, n,$$

the block of matrices $\underline{\underline{C}}$, $\underline{\underline{C}} = \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}$ by $C_i = \{c_i^{jk}\}$, where

$$c_i^{jk} = f_i^*(d_j, d_k), \; i, j, k = 1, 2, \ldots, n.$$

*Define $\underline{v}$ by $\underline{v} = C^{-1}\underline{l}$ if $|C| \neq 0$ and the block of matrices $\underline{\underline{M}} = \begin{bmatrix} M_1 \\ \vdots \\ M_n \end{bmatrix}$ with $M_k = |C|^{-1}M_k'$, where each M_k', $k = 1, 2, \ldots, n$ is the $n \times n$ matrix which results from the determinant of the matrix C if we replace the k*th *column by $\begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}$.*

Define $C\underline{\underline{M}}$ by $\begin{bmatrix} CM_1 \\ \vdots \\ CM_n \end{bmatrix}$.

Note that M_k', $k = 1, 2, \ldots, n$ is indeed an $n \times n$ matrix. For the case $n = 2$,

$$M_1' = \begin{vmatrix} C_1 & c_{12} \\ C_2 & c_{22} \end{vmatrix} = c_{22}C_1 - c_{12}C_2.$$

$$M_2' = \begin{vmatrix} c_{11} & C_1 \\ c_{21} & C_2 \end{vmatrix} = c_{11}C_2 - c_{21}C_1.$$

Theorem 5. *The point $w \in X$ is a solution of the auxiliary equation* (5) *if and only if*

$$w = y + \sum_{i=1}^{n} \xi_i d_i,$$

where the vector $\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \in \mathbb{R}^n$ *(or* $\mathbb{C}^n$*) is a solution of*

$$\underline{x} = \underline{l} + C'\underline{x} + \underline{x}^{+r}\underline{\underline{C}}\underline{x} \text{ in } \mathbb{R}^n \text{ (or } \mathbb{C}^n\text{)}. \tag{6}$$

Moreover, if $|C| = |I - C'| \neq 0$, *the Cramer rule transforms the above to*

$$\underline{x} = \underline{v} + \underline{x}^{+r}\underline{\underline{M}}\underline{x} \text{ in } \mathbb{R}^n \text{ (or } \mathbb{C}^n\text{)}. \tag{7}$$

Proof. Assume that (5) has a solution $w \in X$. Then

$$w = y + Q(w)$$
$$= y + \sum_{i=1}^{n} f_i(w)\, d_i.$$

Apply $f_1, f_2, \ldots, f_n$ in turn to this vector identity to obtain for $p = 1, 2, \ldots, n$

$$f_p(w) = f_p\left(y + \sum_{k=1}^{n} f_i(w)\, d_i\right)$$
$$= f_p(y) + \sum_{k=1}^{n} f_k^2(w) f_p(d_k) + 2\sum_{k=1}^{n} f_k(w) f_p^*(y, d_k)$$
$$+ 2\sum_{i \neq j}^{n} f_i(w) f_j(w) f_p^*(d_i, d_j).$$

Letting

$$f_i(w) = x_i, \quad i = 1, 2, \ldots, n$$

and writing these equations in vector form, we obtain

$$\underline{x} = \underline{l} + C'\underline{x} + \underline{x}^{+r}\underline{\underline{C}}\underline{x}$$

or

$$C\underline{x} = \underline{l} + \underline{x}^{+r}\underline{\underline{C}}\underline{x}.$$

Since $|C| \neq 0$, we obtain (7) by composing both sides of the above equation by C^{-1}.

Conversely, given (7), assume (6) has a solution vector $\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$. Let $w \in X$ be defined as

$$w = y + \sum_{i=1}^{n} \xi_i d_i.$$

Apply $f_1, f_2, \ldots, f_n$ in turn to this vector identity to obtain for $p = 1, 2, \ldots, n$,

$$f_p(w) = f_p(y) + \sum_{k=1}^{n} \xi_k^2 f_p(d_k) + 2 \sum_{k=1}^{n} \xi_k f_p^*(y, d_k)$$

$$+ 2 \sum_{i \neq j}^{n} \xi_i \xi_j f_p^*(d_i, d_j),$$

or in matrix notation,

$$\underline{f(w)} = \underline{l} + C'\xi + \xi^{+\prime}\underline{\underline{C}}\xi.$$

Now since ξ satisfies (6) we have $\xi = \underline{l} + C'\xi + \xi^{+\prime}\underline{\underline{C}}\xi$.

Now since ξ satisfies (6) we have

$$\xi = \underline{l} + C'\xi + \xi^{+\prime}\underline{\underline{C}}\xi.$$

Comparing the last two equations, we get

$$\xi_i = f_i(w), \quad i = 1, 2, \ldots, n,$$

so

$$w = y + \sum_{i=1}^{n} f_i(w)\, d_i,$$

or

$$w = y + Q(w).$$

Therefore, w is a solution of (5) and the theorem is proved.
Example. Let $X = C[0, 1]$ and consider the equation

$$x(s) = s + s \int_0^1 x^2(t)\, dt,$$

where $s \in [0, 1]$. This equation is of the form (5), with rank $(Q) = 1$,

$$y(s) = s$$

$$d = s, \quad \text{and}$$

$$d(s) = \int_0^1 x^2(t)\, dt.$$

Using the formula,

$$f^*(v, w) = \frac{1}{4}(f(v + w) - f(v - w)),$$

we have

$$C = 1 - 2f^*(y, d) = 1 - 2\frac{1}{4}\int_0^1 4s^2\, ds = \frac{1}{3}$$

$$\underline{l} = f(y) = f(s) = \int_0^1 s^2\, ds = \frac{1}{3}$$

$$\underline{\underline{C}} = f(d) = f(s) = \int_0^1 s^2 \, ds = \frac{1}{3}$$

$$\underline{v} = 3 \cdot \frac{1}{3} = 1$$

$$\underline{\underline{M}} = 3 \cdot \frac{1}{3} = 1.$$

Therefore, (6) becomes

$$\xi = 1 + \xi^2 \text{ in } \mathbb{C} \text{ with solutions } \frac{1 \pm i\sqrt{3}}{2};$$

since $x = y + \xi d$, we finally have

$$x(s) = \left(\frac{3 \pm i\sqrt{3}}{2}\right) s.$$

Now note that if the linear operator $F(z)$ is of finite rank n, then the linear operator $I - 2F(z)$ is invertible if and only if for every fixed $v \in X$ there exists $w \in X$ such that

$$w - 2F(z, w) = v.$$

Since $F(z)$ is of finite rank n, the above equation can be translated exactly as in Theorem 5 for the quadratic case to a linear system in $\mathbb{C}^n$, or ^{n}, similar to system (7).

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АППРОКСИМАЦИЯ НЕПОДВИЖНЫХ ТОЧЕК НЕКОТОРЫХ НЕЛИНЕЙНЫХ ОПЕРАТОРНЫХ УРАВНЕНИЙ

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Резюме

Рассмотрим пару квадратных уравнений

$$x = y + B(x, x)$$

$$z = y + F(z, z)$$

в банаховом пространстве X, где $y \in X$ есть фиксированная точка, а B, F — ограниченные симетрические билинейные операторы на X. Предположим, что решение z второго уравнения известно, и используем ето на апроксимацию решения первого уравнения. В частном случае, когда F есть оператор конечного ранга, показывается, что проблема нахождения решения z второго уравнения эквивалентна задаче решения системы квадратных уравнений в $\mathbb{R}^n$ или ^{n}.