Mathematica Slovaca

Dan Kucerovsky
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Mathematica Slovaca, Vol. 51 (2001), No. 2, 229--230

Persistent URL: http://dml.cz/dmlcz/129730

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Math. Slovaca, **51** (2001), No. 2, 229–230

Mathematica Slovaca
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REMARK ON AN OPERATOR INEQUALITY

DAN KUCEROVSKY

(*Communicated by Michal Zajac*)

ABSTRACT. We show that the operator inequality $\pm i[T,S] \le ST^2S$ holds if and only if $\pm i[T,S] \le TS^2T$.

1. Introduction

It is known ([2]) that in an σ-unital C^*-algebra, given two strictly positive operators ℓ and k, there is a nonzero function f such that the commutator of $T := f(\ell + k)$ and $S := f(k)$ satisfies

$$\pm i[T,S] \le ST^2S\,.$$

The purpose of this note is to show that this inequality is equivalent to

$$\pm i[T,S] \le TS^2T\,.$$

2. Theorem

THEOREM 1. *Suppose that S and T are self-adjoint Hilbert space operators. Then $\pm i[T,S] \le ST^2S$ if and only if $\pm i[T,S] \le TS^2T$.*

Remark. This theorem is trivial if the operators are invertible, but the invertible case does not imply the general case.

Proof. Let $Q := iST$. It is sufficient to prove that

$$(1-Q)(1-Q)^* \ge 1 \qquad \text{and} \qquad (1+Q)(1+Q)^* \ge 1$$

if

$$(1-Q)^*(1-Q) \ge 1 \qquad \text{and} \qquad (1+Q)^*(1+Q) \ge 1\,.$$

2000 Mathematics Subject Classification: Primary 46L89, 47A63.
Key words: operator inequality.

Supported by the Swiss National Fund for Scientific Research, grant 20-46899.96, and by NSERC (Canada).

It can be verified that the following triangle is well-defined and commutes:

$$\begin{array}{ccc} \ker(1 - iTS) & \xrightarrow{iS} & \ker(1 - iST) \\ & \nwarrow & \downarrow T \\ & & \ker(1 - iTS) \end{array}$$

where the diagonal arrow is just the identity map. It follows that S is injective when restricted to the kernel $\ker(1-iTS)$, and therefore that $\dim\ker(1-iST) \geq \dim\ker(1 - iTS)$. Repeating the argument with minor changes we find that, in terms of the operator Q,

$$\dim\ker(1 - Q) = \dim\ker(1 + Q^*) \quad \text{and} \quad \dim\ker(1 + Q) = \dim\ker(1 - Q^*)\,,$$

where the dimensions could be *a priori* infinite. However, the hypothesis implies that the kernel of $1 \pm Q$ is zero. The above equalities then imply that the partial isometries $V_{\pm}$ in the polar decomposition of $1 - Q$ and $1 + Q$ are actually unitaries, which proves the theorem, since if $(1 - Q)^*(1 - Q) \geq 1$, then $(1 - Q)(1 - Q)^* \geq V_- V_-^*$. The same holds for the other choice of sign, so we are done. □

It is also possible to prove a one-sided version of the above theorem:

THEOREM 2. *Suppose that S and T are positive Hilbert space operators. Then $i[T, S] \leq ST^2S$ if and only if $i[T, S] \leq TS^2T$.*

The proof of this is omitted.

This paper was motivated by a technical problem in KK-theory that required finding a pair of unbounded Hilbert module operators S^{-1} and T^{-1} such that their commutator would be densely defined, and bounded on its domain (as well as satisfying certain other conditions).

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Received January 14, 1999

Revised April 24, 2000

Department of Mathematics & Statistics
University of New Brunswick
BOX 4400/FREDERICTON
CANADA E3B 5A3

E-mail: dan@math.unb.ca