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QUASINORMAL CONVERGENCE

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ABSTRACT. Quasinormal convergence of sequences of real-valued functions is investigated.

Showing that every countable set E of reals is a certain type of thin set in the trigonometrical series theory, N. N. Bari ([2] p. 737) constructs a sequence of positive reals $\varepsilon_k \to 0$ and an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of natural numbers such that for every $x \in E$ there exists an index k_x such that $\|n_k x\| < \varepsilon_k$ for every $k > k_x$. That is a new type of convergence which was not investigated in analysis before.

A. Denjoy ([6] p. 183) introduces the following notion: the series $\sum_{n=0}^{\infty} f_n(x)$ of real-valued functions converges pseudonormally on a set E iff there is a convergent series $\sum_{n=0}^{\infty} \varepsilon_n$ of positive reals such that for every $x \in E$ there exists k_x such that $|f_k(x)| < \varepsilon_k$ for every $k > k_x$. J. Arbault ([1] p. 303—304) studies pseudonormal convergence of some trigonometrical series.

That is all we could find in literature about this new type of convergence.

Inspired by A. Denjoy I have called the convergence quasinormal, investigated its properties and tried to use them for the study of thin sets in trigonometrical series [3]. All my results were presented in my thesis in January 1988. When my thesis was almost completed, L. Zajíček kindly informed me that Á. Császár and M. Laczkovich [4], [5] considered the same type of convergence under the name "equal convergence". Many of my results coincided with those of [4], [5]. Anyway, I still hope that not all the results presented in this paper are explicitly contained in [4], [5], or at least, I present them from a different point of view.

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§1. Quasinormal convergence

Definition 1.1. *Let f_n, f, $n = 0, 1, \dots$ be real-valued functions defined on a set X. We shall say that the sequence $\{f_n\}_{n=0}^{\infty}$ converges quasinormally to f on X, written $f_n \xrightarrow{QN} f$ on X, if there is a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ of non-negative reals converging to zero such that for every $x \in X$ there is an index k such that $|f_n(x) - f(x)| \leqslant \varepsilon_n$ for every $n \geqslant k$.*

Let us stress that the index k may generally depend on x. If $\{f_n\}_{n=0}^{\infty}$ converges uniformly to f on X, then it converges also quasinormally. On the other hand the quasinormal convergence implies the pointwise one. If all ε_n are equal zero, then the corresponding convergence is said to be discrete (compare [4]).

Before giving some simple examples distinguishing all those types of convergences we start with a theorem which is contained also in Á. Császár, M. Laczkovich [5].

Theorem 1.2. *Let f_n, f, $n = 0, 1, \dots,$ be real-valued functions defined on a set X. The following conditions are equivalent*

(i) $f_n \xrightarrow{QN} f$ *on X,*

(ii) *there are sets $E_k \subseteq X$ such that $X = \bigcup_{k=0}^{\infty} E_k$ and $f_n \rightrightarrows f$ on E_k for every $k = 0, 1, \dots,$*

(iii) *there are sets $E_k \subseteq X$ such that $X = \bigcup_{k=0}^{\infty} E_k$, $E_0 \subseteq E_1 \subseteq \dots$ and $f_n \rightrightarrows f$ on E for every $k = 0, 1, \dots$.*

Moreover, if X is a topological space and f_n, $n = 0, 1, \dots$ are continuous, then (i), (ii), (iii) *are equivalent to*

(iv) *there are closed sets $E_k \subseteq X$, $k = 0, 1, \dots, X = \bigcup_{k=0}^{\infty} E_k$, $E_0 \subseteq E_1 \subseteq \dots$ and $f_n \rightrightarrows f$ on E_k, $k = 0, 1, \dots$.*

Proof. Assume (i). Let $\varepsilon_n \geqslant 0$, $\varepsilon_n \to 0$ and

$$(\forall x \in X)(\exists k)(\forall n \geqslant k)|f_n(x) - f(x)| \leqslant \varepsilon_n .$$

We put

$$E_k = \{x \in X;\ (\forall n \geqslant k)|f_n(x) - f(x)| \leqslant \varepsilon_n\},$$

then $\bigcup_{k=0}^{\infty} E_k = X$, $E_0 \subseteq E_1 \subseteq \dots$ and $f_n \rightrightarrows f$ on E_k, $k = 0, 1, \dots$. So we have (iii).

Evidently (iii) implies (ii).

Assume (ii). Since $f_n \rightrightarrows f$ on E_k, $k = 0, 1, \dots,$ there exists a non increasing sequence of non-negative reals $\delta_n^k \to 0$ such that

$$(\exists m)(\forall n \geqslant m)(\forall x \in E_k)|f_n(x) - f(x)| \leqslant \delta_n^k .$$

We start with the construction of an increasing sequence of natural numbers j_m, $m = 0, 1, \ldots$. Let j_m be the first $l > j_{m-1}$ such that $\delta_n^0, \ldots, \delta_n^m$ are not greater than $\frac{1}{2^m}$ for all $n \geqslant l$.

Now, set

$$\varepsilon_i = 1 \quad \text{for} \quad i < j_0,$$

$$\varepsilon_i = \frac{1}{2^m} \quad \text{for} \quad j_m \leqslant i < j_{m+1}.$$

One can easily show that for every $x \in X$ we have $|f_n(x) - f(x)| \leqslant \varepsilon_n$ starting from some n_0.

In the continuous case evidently (iv) implies (iii). Assume (i). It suffices to set

$$E_k = \{x \in X; (\forall n, m \geqslant k) |f_n(x) - f_m(x)| \leqslant \varepsilon_n + \varepsilon_m\}.$$

Evidently E_k is closed and $f_n \rightrightarrows f$ on E_k, $k = 0, 1, \ldots$. □

Corollary 1.3. *Let* $X = \bigcup_{i=0}^{\infty} X_i$. *If* $\{f_n\}_{n=0}^{\infty}$ *converges quasinormally to* f *on every* X_i, $i = 0, 1, \ldots$, *then it does so on* X.

Example 1.4. Let $\mathbf{Q} = \{r_k; k \in \mathbf{N}\}$ be a one-to-one enumeration of rational numbers. Let

$$f(x) = 0 \quad \text{for} \quad x \in \mathbf{R} - \mathbf{Q},$$

$$f(r_k) = 2^{-k} \quad \text{for} \quad k = 0, 1, \ldots.$$

Evidently, f is not continuous in any interval. For every n choose a positive real $\delta_n \leqslant 2^{-n}$ such that

$$\delta_n \leqslant \frac{1}{2} |r_i - r_j|, \quad i = 0, 1, \ldots, n, \; j = 0, 1, \ldots, n, \; i \neq j.$$

For $x \in \mathbf{R} - \bigcup_{i=0}^{n} (r_i - \delta_i, r_i + \delta_i)$ we set $f_n(x) = 0$, $f_n(r_i) = 2^{-i}$ and f_n is piecewise linear ($f_n(x) \leqslant 2^{-i}$ for $x \in (r_i - \delta_i, r_i + \delta_i)$), $i = 0, 1, \ldots, n$.

One can easily show that $f_n \to f$ pointwise on $\mathbf{R}$. We show that $\{f_n\}_{n=0}^{\infty}$ does not converge quasinormally to f on $\mathbf{R}$. Assume it does. Then by (iv) of Theorem 1.2, $\mathbf{R} = \bigcup_{k=0}^{\infty} E_k$, E_k closed and $f_n \rightrightarrows f$ on every E_k, $k = 0, 1, \ldots$. By the Baire category theorem there is k such that $\operatorname{Int} E_k \neq \emptyset$, i.e. there are $a < b$ such that $\langle a, b\rangle \subseteq E_k$. Since f_n are continuous, $f_n \rightrightarrows f$ on $\langle a, b\rangle$, f is also continuous on $\langle a, b\rangle$, which is a contradiction.

Example 1.5. Now let $f_n(x) = x^n$ for $x \in \langle 0, 1\rangle$ and $f(x) = 0$ for $x \in \langle 0, 1)$, $f(1) = 1$. Then $f_n \xrightarrow{QN} f$ on $\langle 0, 1\rangle$ and not uniformly.

Example 1.6. Let us remember that Denjoy says that the series $\sum_{n=0}^{\infty} f_n(x)$ pseudonormally converges on X iff there is a convergent series $\sum_{n=0}^{\infty} \varepsilon_n$ of non-negative reals ε_n such that

$$(\forall x \in X)(\exists k)(\forall n \geqslant k)\,|f_n(x)| \leqslant \varepsilon_n .$$

If the series converges pseudonormally, then it (the sequence of partial sums) converges quasinormally. However, it suffices to set

$$f_n(x) = (-1)^n \cdot \frac{1}{n} \quad \text{for any} \quad x \in \mathbf{R}$$

and we obtain: $\sum_{n=1}^{\infty} f_n$ converges quasinormally but not pseudonormally.

A natural question arises: if the sequence of continuous functions on $\mathbf{R}$ (or some other reasonable topological space) converges pointwise to a continuous function, does it converge already quasinormally? I have found a counterexample to this question. However, L. Bukovský has found a more convenient example with solves simultaneously several other problems. With his kind permission I will describe it.

Example 1.7. Let $a \in \mathbf{R}$, $\varepsilon > 0$. By an ε-wave at a we mean a sequence of continuous functions $\{f_n\}_{n=0}^{\infty}$ such that:

a) $0 \leqslant f_n(x) \leqslant \varepsilon$ for every $x \in \mathbf{R}$,

b) there is a sequence $x_n \to a$ such that $f_n(x_n) = \varepsilon$,

c) $f_n \to 0$ discretely on $\mathbf{R}$, i.e.

$$(\forall x \in \mathbf{R})(\exists k)(\forall n \geqslant k)\, f_n(x) = 0.$$

For any $a \in \mathbf{R}$ and $\varepsilon > 0$ one can easily construct an ε-wave at a.

Now let $\{r_k;\, k \in \mathbf{N}\}$ be an enumeration of all rational numbers. For any r_k, let $\{f_n^k\}_{n=0}^{\infty}$ be a 2^{-k}-wave at r_k.

Let $\{g_m\}_{m=0}^{\infty}$ be some one-to-one enumeration of all functions f_n^k, $n = 0, 1, \ldots$, $k = 0, 1, \ldots$. More precisely let $\pi\colon \mathbf{N} \times \mathbf{N} \xrightarrow[\text{onto}]{1-1} \mathbf{N}$. We set $g_m = f_n^k$, where $m = \pi(n, k)$.

One can easily show that $g_m \to 0$ pointwise on $\mathbf{R}$. We prove that $\{g_m\}_{m=0}^{\infty}$ does not quasinormally converge to zero on $\mathbf{R}$. Assume it does. Then $\mathbf{R} = \bigcup_{k=0}^{\infty} E_k$, E_k closed and $g_m \rightrightarrows 0$ on E_k, $k = 0, 1, \ldots$. By the Baire category theorem, $\operatorname{Int} E_k \neq \emptyset$ for some k. So there are $a < b$ such that $\langle a, b\rangle \subseteq E_k$. There exists n_0 such that $r_{n_0} \in (a, b)$. Since $g_m \rightrightarrows 0$ on E_k then also $f_m^{n_0} \rightrightarrows 0$ on $\langle a, b\rangle$ — a contradiction with b).

We show that from every subsequence of $\{g_m\}_{m=0}^{\infty}$ one can choose a subsequence quasinormally converging to zero. Since the whole sequence $\{g_m\}_{m=0}^{\infty}$ does not quasinormally converges to zero we obtain the following: the quasinormal convergence on the family of all continuous bounded real-valued functions $C^*(\mathbf{R})$ does not satisfy the Urysohn axiom of convergence (see [9], p. 84, condition 3^0). Moreover, one can easily see that both here and in Example 1.7 the space $\mathbf{R}$ may be replaced by any completely regular separable first-countable topological space which is not of the first category, e.g. by any Polish space with no isolated point.

Let $k_0 < k_1 < \dots$ be a sequence of natural numbers. Denote

$$K = \{k_i;\ i \in \mathbf{N}\}.$$

Consider two cases:

i) there is k such that $\pi(n, k) \in K$ for infinitely many n, i.e. there is an increasing sequence $n_0 < n_1 < \dots$ such that $\pi(n_i, k) \in K$. Then

$$g_{\pi(n_i, k)} = f_{n_i}^k$$

is a 2^{-k}-wave at r_k, hence it converges discretely to zero,

ii) there is no such k. Then for infinitely many n there is l such that $\pi(n, l) \in K$, i.e. there are two sequences $n_0, n_1, \dots, l_0 < l_1 < \dots$ such that $\pi(n_i, l_i) \in K$. Since

$$|g_{\pi(n_i, l_i)}(x)| \leqslant 2^{-l_i}, \quad l_0 < l_1 < \dots$$

we have $g_{\pi(n_i, l_i)} \rightrightarrows 0$ on $\mathbf{R}$.

Corollary 1.3 can be strengthened as follows. Let us recall (see [11]) that $\mathfrak{b}$ denotes the smallest cardinal such that for every $\mathscr{H}$ a family of functions from $\mathbf{N}$ into $\mathbf{N}$, $|\mathscr{H}| < \mathfrak{b}$ there is a function $g\colon \mathbf{N} \to \mathbf{N}$ such that

$$(\forall h \in \mathscr{H})(\exists k)(\forall n \geqslant k)\, h(n) \leqslant g(n).$$

It is know that $\aleph_0 < \mathfrak{b} \leqslant \mathfrak{c}$ but not necessarily $\mathfrak{b} = \aleph_1$.

Theorem 1.8. *Let* $X = \bigcup_{s \in S} E_s$, $|S| < \mathfrak{b}$. *If the sequence* $\{f_n\}_{n=0}^{\infty}$ *converges quasinormally to* f *on every* E_s, $s \in S$, *then it does so on the set* X.

Proof. For every $s \in S$ let $\{\varepsilon_n^s\}_{n=0}^{\infty}$ be a decreasing sequence of positive reals witnessing the quasinormal convergence on E_s. We define

$$h_s(k) = \min\left\{n;\ \varepsilon_n^s \leqslant \frac{1}{k+1},\ n > h_s(k-1)\right\}.$$

Since the family $\{h_s,\ s \in S\}$ is of power less than $\mathfrak{b}$, there exists a function $g\colon \mathbf{N} \to \mathbf{N}$ with the above described property. Moreover, we can assume that g is strictly increasing. Now we denote

$$\varepsilon_n = 1 \qquad \text{for} \quad n < g(1),$$

$$\varepsilon_n = \frac{1}{k+1} \quad \text{for} \quad g(k) \leqslant n < g(k+1).$$

If $x \in X$, then $x \in E_s$ for some $s \in S$. Therefore there is a k_x such that

$$|f_n(x) - f(x)| < \varepsilon_n^s \quad \text{for} \quad n \geqslant k_x.$$

Also there is a natural number k such that $h_s(n) \leqslant g(n)$ for $n \geqslant k$.

Let $n \geqslant k_x$, $n \geqslant g(k)$. Then $g(l) \leqslant n < g(l+1)$ for some $l \geqslant k$. Since $g(l) \geqslant h_s(l)$ we have

$$|f_n(x) - f(x)| < \varepsilon_n^s \leqslant \frac{1}{l+1} \leqslant \varepsilon_n. \qquad \square$$

§2. Borel measurability and quasinormal limits

From now on we shall suppose that $(X, \mathscr{O})$ is a perfectly normal topological space (see [7]). We recall some notions. The family of sets of the additive (multiplicative) class $\mathbf{A}_\alpha(X)$ or simply $\mathbf{A}_\alpha$ ($\mathbf{M}_\alpha(X)$ or simply $\mathbf{M}_\alpha$) is defined as follows: $\mathbf{A}_0(X) = \mathscr{O}$, $\mathbf{A}_\alpha(X)$ is the family of all countable unions of sets from $\bigcup_{\xi < \alpha} \mathbf{M}_\xi(X)$, E belongs to $\mathbf{M}_\alpha(X)$ iff $X - E \in \mathbf{A}_\alpha(X)$. The family of Borel sets $\mathbf{B}(X)$ is the union $\bigcup_{\xi < \omega_1} \mathbf{A}_\xi(X) = \bigcup_{\xi < \omega_1} \mathbf{M}_\xi(X)$. The sets from $\mathbf{A}_\alpha(X) \cap \mathbf{M}_\alpha(X)$ are said to be ambiguous. If $Y \subseteq X$ is a subspace, $\mathbf{A}_\alpha(Y) = \{Y \cap A;\, A \in \mathbf{A}_\alpha(X)\}$ and similarly for $\mathbf{M}_\alpha(Y)$.

If $\mathscr{S}$ is a family of subsets of X, then a function $f\colon X \to \mathbf{R}$ is called $\mathscr{S}$-measurable iff $f^{-1}(U) \in \mathscr{S}$ for any open set $U \subseteq \mathbf{R}$. A family $\mathscr{S}$ is called a σ-topology on X (see [10], p. 90) iff $\emptyset$, $X \in \mathscr{S}$ and $\mathscr{S}$ is closed under finite intersections and countable unions. It is easy to see that assuming $\mathscr{S}$ to be a σ-topology a function $f\colon X \to \mathbf{R}$ is $\mathscr{S}$-measurable iff $f^{-1}((a, +\infty))$, $f^{-1}((-\infty, a)) \in \mathscr{S}$ for any $a \in \mathbf{R}$.

The class of all $\mathbf{A}_\alpha(X)$-measurable real functions is denoted by $\mathscr{M}_\alpha(X)$. Similarly, $\mathscr{M}\mathscr{A}_\alpha(X)$ is the class of all $\mathbf{A}_\alpha(X) \cap \mathbf{M}_\alpha(X)$-measurable functions. Since $\mathbf{A}_\alpha(X) \cap \mathbf{M}_\alpha(X)$ need not be a σ-topology we introduce the class $\mathscr{M}\mathscr{A}_\alpha^0(X)$ of weakly α-ambiguous functions: $f \in \mathscr{M}\mathscr{A}_\alpha^0(X)$ iff $f^{-1}((a, \infty))$, $f^{-1}((-\infty, a)) \in$ $\in \mathbf{A}_\alpha(X) \cap \mathbf{M}_\alpha(X)$ for any $a \in \mathbf{R}$.

Next we shall need a simple auxilliary result.

Lemma 2.1. *Let $f_n \rightrightarrows f$ on E, U being an open subset of* $\mathbf{R}$. *Then there are open sets U_n, $n = 0, 1, \ldots$ such that*

a) $U = \bigcup_{n=0}^{\infty} U_n$

b) $f^{-1}(U) = \bigcup_{n=0}^{\infty} f_n^{-1}(U_n)$.

Proof. Let $\varepsilon_n = \sup_{x \in E} |f_n(x) - f(x)|$. Then $\lim \varepsilon_n = 0$. Let us set

$$U_n = \{x \in \mathbf{R};\ (\exists \delta > \varepsilon_n)(x - \delta, x + \delta) \subseteq U\}.$$

One can easily show that a) and b) hold true. □

If $\mathscr{E}$ is a class of real-valued functions on a set X, we denote by p.c.($\mathscr{E}$) the pointwise convergence closure of $\mathscr{E}$, i.e. the set of all functions which are pointwise limits of sequences from $\mathscr{E}$. Similarly u.c.($\mathscr{E}$), d.c.($\mathscr{E}$), q.n.c.($\mathscr{E}$) for uniform, discrete and quasinormal convergence, respectively.

Lemma 2.2. *If $\mathscr{E}$ is a set of real-valued functions defined on a set X, then*
a) *q.n.c.($\mathscr{E}$) = q.n.c.(u.c.($\mathscr{E}$)),*
b) *p.c.($\mathscr{E}$) = p.c.(u.c.($\mathscr{E}$)).*

Proof. Let $f_n \in$ u.c.($\mathscr{E}$), $f_n \xrightarrow{QN} f$ on X. Then for every $n \in \mathbf{N}$, there exists $g_n \in \mathscr{E}$ such that

$$|f_n(x) - g_n(x)| < \frac{1}{n+1}$$

for every $x \in X$. Then f is the quasinormal limit of the sequence g_n. Indeed, for a given x there exists a k such that

$$|f_n(x) - f(x)| < \varepsilon_n$$

for any $n \geqslant k$. Then

$$|g_n(x) - f(x)| < \varepsilon_n + \frac{1}{1+n}$$

for every $n \geqslant k$.

The assertion b) can be proved in a similar way (see also [5], Lemma 2.8). □

The classes $\mathscr{M}_\alpha(X)$ are closed for the uniform convergence and the following holds true (see e.g. [5], p. 61):

1) $\mathscr{M}_0(X) = C(X)$,

2) $\mathscr{M}_{\alpha+1}(X)$ is the class of all pointwise limits of elements of $\mathscr{M}_\alpha(X)$.

Theorem 1.2 can be generalized as follows (compare [5]):

Theorem 2.3. *Let $f_n \xrightarrow{QN} f$ on X. If f_n are $\mathbf{A}_\alpha(X)$-measurable, then there is an increasing sequence $E_0 \subseteq E_1 \subseteq \dots$ of sets such that*

a) $\bigcup_{k=0}^{\infty} E_k = X$,

b) $f_n \rightrightarrows f$ *on every* E_k, $k = 0, 1, \ldots$,

c) $E_k \in \mathbf{M}_\alpha(X)$, $k = 0, 1, \ldots$.

Proof. Similarly as in proof of Theorem 1.2 we set

$$E_k = \{x \in X;\ (\forall m, n \geqslant k)\, |f_m(x) - f_n(x)| \leqslant \varepsilon_m + \varepsilon_n\},$$

i.e.

$$E_k = \bigcap_{n \geqslant k} \bigcap_{m \geqslant k} (f_n - f_m)^{-1}(\langle -\varepsilon_n - \varepsilon_m, \varepsilon_n + \varepsilon_m \rangle).$$

Since the difference $f_n - f_m$ is also $\mathbf{A}_\alpha(X)$-measurable the theorem follows. □

Now we show the main result of this paragraph.

Theorem 2.4. *Let* $f_n \xrightarrow{QN} f$ *on* X, f_n *being* $\mathbf{A}_\alpha(X)$*-measurable. Then* $f \in \mathcal{MA}_{\alpha+1}(X)$.

Proof. By Theorem 2.3 $X = \bigcup_{k=0}^{\infty} E_k$, where $E_k \in \mathbf{M}_\alpha(X)$, $k = 0, 1, \ldots$. Since $f_n \rightrightarrows f$ on E_k, $k = 0, 1, \ldots$, by Lemma 2.1 for any open $U \subseteq \mathbf{R}$,

$$f^{-1}(U) \cap E_k \in \mathbf{A}_\alpha(E_k),$$

i.e.

$$f^{-1}(U) \cap E_k = E_k \cap F_k,$$

where $F_k \in \mathbf{A}_\alpha(X)$, $k = 0, 1, \ldots$. Therefore

$$f^{-1}(U) = \bigcup_{k=0}^{\infty} f^{-1}(U) \cap E_k = \bigcup_{k=0}^{\infty} E_k \cap F_k \in \mathbf{A}_{\alpha+1}(X).$$

For $V \subseteq \mathbf{R}$ closed, by lemma 2.1 we have

$$f^{-1}(V) \cap E_k \in \mathbf{M}_\alpha(E_k), \quad k = 0, 1, \ldots,$$

i.e.

$$f^{-1}(V) \cap E_k = E_k \cap G_k, \quad k = 0, 1, \ldots,$$

where $G_k \in \mathbf{M}_\alpha(X)$, $k = 0, 1, \ldots$. Thus

$$f^{-1}(V) = \bigcup_{k=0}^{\infty} f^{-1}(V) \cap E_k = \bigcup_{k=0}^{\infty} E_k \cap G_k \in \mathbf{A}_{\alpha+1}(X).$$ □

Theorem 2.4 can be also deduced from Corollary 1.2 of [5].

Example 2.5. We show that $\mathcal{MA}_1(\mathbf{R})$ is different from $\mathcal{MA}_1^0(\mathbf{R})$.

Let $\{r_n;\ n \in \mathbf{N}\}$ be a one-to-one enumeration of the set $\mathbf{Q}$. We set

$$f(x) = \sum_{\{k;\, r_k \leqslant x\}} 2^{-k} \quad \text{for} \quad x \in \mathbf{R} - \mathbf{Q},$$

$$f(r_k) = \lim_{x \to r_k^-} f(x) + 2^{-k-1}.$$

So we have

$$2f(r_k) = \lim_{x \to r_k^-} f(x) + \lim_{x \to r_k^+} f(x).$$

For every real $a \in \mathbf{R}$, the sets $f^{-1}((a, \infty)), f^{-1}((-\infty, a))$ are intervals, hence both $\mathbf{G}_\delta$ and $\mathbf{F}_\delta$-sets. However, if we put

$$U = \bigcup_{k=0}^{\infty} (f(r_k) - 2^{-k-1}, f(r_k) + 2^{-k-1}),$$

then $f^{-1}(U) = \mathbf{Q}$, which is not a $\mathbf{G}_\delta$-set.

We close with some open problems. In connection with the example it is natural to ask:

1) for which α and X is $\mathscr{M}\mathscr{A}_\alpha(X) \neq \mathscr{M}\mathscr{A}_\alpha^0(X)$? J. E. Jayne and C. A. Rogers [8] have shown that for a Polish space X the class $\mathscr{M}\mathscr{A}_1(X)$ consists exactly of quasinormal limits of continuous functions (Theorem 5, p. 178),

2) can $\mathscr{M}\mathscr{A}_\alpha(X)$ be characterized by $\mathscr{M}_{\alpha-1}(X)$ and a quasinormal convergence?

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