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SOME SEQUENCE SPACES DEFINED BY A MODULUS

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ABSTRACT. The object of this paper is to introduce some sequence spaces which arise from the notions of strong almost convergence and a modulus function f.

1. Introduction

Let m be the set of all the real or complex bounded sequences with the norm $\|x\| = \sup_k |x_k| < \infty$. A sequence $x = (x_k) \in m$ is said to be almost convergent if all of its Banach limits coincide. Lorentz [5] has proved that x is almost convergent to a number s if and only if

$$t_{kn} = (k+1)^{-1} \sum_{i=n}^{n+k} x_k \to s$$

as $k \to \infty$ uniformly in n. We denote the set of all almost convergent sequences by $\hat{c}$ and we denote the set of all sequences which are almost convergent to zero by $\hat{c}_0$. Maddox [7] has defined that x is strongly almost convergent to a number s if and only if

$$t_{kn}\big(|x-s|\big) = (k+1)^{-1} \sum_{i=0}^{k} |x_{i+n} - s| \to 0$$

as $k \to \infty$ uniformly in n. We denote the space of all strongly almost convergent sequences by $[\hat{c}]$ and we denote the space of all sequences which are strongly almost convergent to zero by $[\hat{c}_0]$. It is obvious that

$$[\hat{c}_0] \subset [\hat{c}] \subset \hat{c} \subset m\,.$$

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Das and Sahoo [2] extended the space $[\hat{c}]$ to the space $[w_1]$, where $[w_1]$ is the space defined recently in [2] as follows:

$$[w_1] = \left\{x:\ (m+1)^{-1}\sum_{k=0}^{m} t_{kn}\big(|x-s|\big) \text{ as } m\to\infty \text{ uniformly in } n \text{ for some } s\right\}.$$

It is obvious that $[\hat{c}] \subset [w_1]$ and $[\hat{c}] - \lim x = [w_1] - \lim x = s$.

The notion of a modulus function was introduced by Nakano [10]. Ruckle [12] has investigated the sequence space defined by a modulus function f. Recently, Maddox has introduced and discussed some properties of three spaces defined using a modulus f, which generalized the well-known spaces w_0, w and w_∞ of strongly summable sequences, (Maddox [8], [9]). It may be noted here that the spaces of strongly summable sequences were discussed by Maddox [6]. In [11], the spaces $[\hat{c}_0]$, $[c]$ and $[c_\infty]$ were extended to $[\hat{c}_0(f)]$, $[\hat{c}(f)]$ and $[\hat{c}_\infty(f)]$.

Now we extend the spaces $[w_1]$ and $[w_0]$ to the spaces $[w_1(f)]$ and $[w_0(f)]$. Then we extend the relationship between the uniform statistical null sequences and the sequence space $[w_0(f)]$.

2. Definitions

We recall that a *modulus* f is a function from $[0,\infty)$ to $[0,\infty)$ such that

(i) $f(x)=0$ if and only if $x=0$,
(ii) $f(x+y)\le f(x)+f(y)$ for $x,y\ge 0$,
(iii) f is increasing,
(iv) f is continuous from the right at 0.

Since $|f(x)-f(y)|\le f(x-y)$, in view of (iv), f is continuous on $[0,\infty)$. A modulus may be bounded or unbounded. For example, $f(x)=x^p$ $(0<p\le 1)$ is unbounded, but $f(x)=x/(1+x)$ is bounded.

Now suppose that we are given a modulus f. We define

$$[\hat{c}(f)] = \left\{x:\ (k+1)^{-1}\sum_{i=0}^{k} f\big(|x_{i+n}-s|\big)\to 0 \text{ as } k\to\infty, \text{ uniformly in } n, \text{ for some } s\right\}.$$

$$[w_1(f)] = \left\{x:\ (m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k} f\big(|x_{i+n}-s|\big)\to 0 \text{ as } m\to\infty, \text{ uniformly in } n, \text{ for some } s\right\}.$$

If we put $s=0$, then we obtain $[w_0(f)]$. Note that, if we put $f(x)=x$, then $[w_1(f)]=[w_1]$ and $[w_0(f)]=[w_0]$.

If f is a modulus, then $[w_0(f)]$ and $[w_1(f)]$ are linear spaces. We consider only $[w_1(f)]$. Suppose that $x_k \to s$ in $[w_1(f)]$, $y_k \to s'$ in $[w_1(f)]$ and α, μ are in C. Then there exist integers K_α and B_μ such that $|\alpha| \le K_\alpha$ and $|\mu| \le B_\mu$. We therefore have, uniformly in n

$$(m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k} f\big(|\alpha x_{i+n} + \mu y_{i+n} - (\alpha s + \mu s')|\big)$$

$$\le K_\alpha (m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k} f\big(|x_{i+n} - s|\big)$$

$$+ B_\mu (m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k} f\big(|y_{i+n} - s'|\big).$$

This implies that $\alpha x + \mu y \to \alpha s + \mu s'$ in $[w_1(f)]$.

3. Main results

We now establish a number of theorems about the sequence spaces mentioned above.

For the proof of Theorem 1 we will use the following lemma.

Lemma. *Let f be a modulus. Let $0 < \delta < 1$. Then for each $x \ge \delta$ we have $f(x) \le 2f(1)\delta^{-1}x$.*

Proof.

$$\begin{aligned} f(x) &\le f\big(1 + [\delta^{-1}x]\big) \le f(1) + f\big([\delta^{-1}x]\big) \\ &\le f(1) + [\delta^{-1}x] f(1) = f(1)\big(1 + [\delta^{-1}x]\big) \\ &\le f(1)(1 + \delta^{-1}x) \le 2f(1)\delta^{-1}x, \end{aligned}$$

where $[h]$ denotes the integer part of h. □

Theorem 1. *Let f be any modulus. If $\beta = \lim_{t\to\infty} f(t)/t > 0$, then $[w_1(f)] = [w_1]$.*

Proof. We note that the limit exists for any modulus f by [9; Proposition 1] of Maddox. Then $x \in [w_1]$ implies that

$$a(m,n) = (m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k} |x_{i+n} - s| \to 0$$

as $m \to \infty$, uniformly in n for some s. For arbitrary $\varepsilon > 0$, choose δ, with $0 < \delta < 1$, such that $f(u) < \varepsilon$ for every u with $0 \le u \le \delta$. We can write for each n,

$$\begin{aligned}(m+1)^{-1}&\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k}f\big(|x_{i+n}-s|\big)\\ =(m+1)^{-1}&\sum_{k=0}^{m}(k+1)^{-1}\sum_{\substack{i=0\\ |x_{i+n}-s|\le\delta}}^{k}f\big(|x_{i+n}-s|\big)\\ &+(m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{\substack{i=0\\ |x_{i+n}-s|>\delta}}^{k}f\big(|x_{i+n}-s|\big)\\ \le\varepsilon&+2f(1)\delta^{-1}a(m,n)\to 0\,,\end{aligned}$$

by the lemma as $m\to\infty$, uniformly in n. Therefore $x\in\big[w_1(f)\big]$.

Note that in this part of the proof we do not need $\beta>0$.

Now suppose that $\beta>0$ and $x\in\big[w_1(f)\big]$. Since this $\beta>0$, we have $f(t)\ge\beta t$ for all $t\ge 0$. It follows that $x\in\big[w_1(f)\big]$ implies that $x\in[w_1]$. □

We now establish some relations between $\big[\hat{c}(f)\big]$ and $\big[w_1(f)\big]$.

THEOREM 2. *Let f be any modulus. Then $\big[\hat{c}(f)\big]\subseteq\big[w_1(f)\big]$.*

Proof. If $(k+1)^{-1}\sum\limits_{i=0}^{k}f\big(|x_{i+n}-s|\big)\to 0$ as $k\to\infty$ uniformly in n, then its arithmetic mean also converges to 0 as $m\to\infty$ uniformly in n. □

Although it seems likely that $\big[\hat{c}(f)\big]$ is strictly contained in $\big[w_1(f)\big]$, we have been unable to prove it. It is therefore an open question.

Recall, see [3], that if x is a sequence of complex numbers, we say that x is *statistically convergent* to s if

$$\lim_{n\to\infty}n^{-1}\big|\{k\le n:\ |x_k-s|\ge\varepsilon\}\big|=0\qquad\text{for each}\quad\varepsilon>0\,,$$

where the larger vertical bars indicate the number of elements in the enclosed set. The set of all statistically convergent sequences is denoted by S. Strong summability and statistical convergence were introduced separately, and until recently, followed independent lines of development by Connor, see [1].

DEFINITION. The number sequence x is *uniformly statistically convergent to* 0 provided that for each $\varepsilon>0$,

$$\lim_{k\to\infty}(k+1)^{-1}\max_{n\ge0}\big|\{0\le i\le k:\ |x_{i+n}|\ge\varepsilon\}\big|=0\,.$$

The set of all uniformly statistically null sequences is denoted by S_{u_0}.

It is easy to see that $S_{u_0}\subset S_0$. In this form, S_{u_0}-convergence is seen to be part of uniform zero density convergence as defined in [4].

THEOREM 3. *$S_{u_0} \subset [w_0(f)]$ if and only if f is bounded.*

Proof. Suppose that f is bounded and that $x \in S_{u_0}$. Since f is bounded, there exists an integer K such that $f(x) < K$ for all $x \geq 0$. Let $\varepsilon > 0$. Then for each n we have

$$\begin{aligned} &(m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k} f(|x_{i+n}|) \\ &= (m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{\substack{i=0 \\ |x_{i+n}|\geq\varepsilon}}^{k} f(|x_{i+n}|) \\ &\qquad + (m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{\substack{i=0 \\ |x_{i+n}|<\varepsilon}}^{k} f(|x_{i+n}|) \\ &\leq (m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1} K \max_{n\geq 0}\big|\{0 \leq i \leq k :\ |x_{i+n}| \geq \varepsilon\}\big| + f(\varepsilon). \end{aligned}$$

We now select N_ε such that

$$(k+1)^{-1}\big|\{0 \leq i \leq k :\ |x_{i+n}| \geq \varepsilon\}\big| < \frac{\varepsilon}{K}$$

for each n and $k > N_\varepsilon$. Now for $k > N_\varepsilon$ we see that

$$\begin{aligned} (m+1)^{-1}\sum_{k=0}^{m}(k+1)^{-1}\sum_{i=0}^{k} f(|x_{i+n}|) &\leq (m+1)^{-1}\sum_{k=0}^{m} K\frac{\varepsilon}{K} + f(\varepsilon) \\ &= \varepsilon + f(\varepsilon), \end{aligned}$$

and so, letting $\varepsilon \to 0$, the result follows.

Conversely, suppose that f is unbounded so that there exists a positive sequence v_p with $f(v_p) = p^2$ for $p = 1, 2, \dots$. Now the sequence x is defined by $x_i = v_p$ if $i = p^2$ for $p = 1, 2, \dots$, and $x_i = 0$ otherwise. Then, we have

$$(k+1)^{-1}\max_{n\geq 0}\big|\{0 \leq i \leq k :\ |x_{i+n}| \geq \varepsilon\}\big| \leq (k+1)^{-1}\sqrt{k+1}$$

as $k \to \infty$. Hence $(x_i) \in S_{u_0}$, but $x \notin [w_0(f)]$, contradicting $S_{u_0} \subset [w_0(f)]$. This completes the proof. □

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