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THE MEASURE EXTENSION THEOREM FOR SUBADDITIVE MEASURES IN σ-CONTINUOUS LOGICS

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The quantum theory requires the study of measures on logics (see [1], [6]). The basic problem of the extension of measures on logics has not been solved so far.

There are some results in [2], [3], but for modular lattices only. B. Riečan proved the extension theorem for subadditive probability measures in [5]. B. Riečan assumes the given measure to be a probability measure defined on an orthocomplemented sublattice of a logic. Every orthocomplemented sublattice of a logic is a ring.

In this paper we prove an extension theorem for subadditive σ-finite measures defined on rings.

Notations and notions

If $\mathscr{H}$ is a lattice, we shall write $x_n \nearrow x$, if $x_n \leqq x_{n+1}$ $(n = 1, 2, \ldots)$ and $x = \bigvee_{n=1}^{\infty} x_n$, similarly for $x_n \searrow x$. A σ-complete lattice will be called σ-continuous if $x_n \nearrow x$, $y_n \nearrow y$ implies $x_n \wedge y_n \nearrow x \wedge y$ and respectively.

By an orthocomplementation of a lattice $\mathscr{H}$ with the least element 0 we mean a mapping $\perp : a \to a^{\perp}$ of $\mathscr{H}$ into itself such that

(i) $a \leqq b$ implies $b^{\perp} \leqq a^{\perp}$,

(ii) $(a^{\perp})^{\perp} = a$ for all a,

(iii) $a \wedge a^{\perp} = 0$ for all a.

A σ-complete lattice $\mathscr{H}$ with an orthocomplementation $\perp$ is said to be a logic in the following case

(iv) if a, $b \in \mathscr{H}$ and $a \leqq b$, there exists an element $d \in \mathscr{H}$ such that $d \leqq a^{\perp}$ and $b = a \vee d$.

The element d in (iv) is unique and is equal to $b \wedge a^{\perp}$ (see e.g. [6]). If a_1, a_2, ... is a sequence of elements of a logic, then

$$\left(\bigvee_n a_n\right)^\perp = \bigwedge_n a_n^\perp \quad \text{and} \quad \left(\bigwedge_n a_n\right)^\perp = \bigvee_n a_n^\perp.$$

Two elements a, b of a logic are called orthogonal ($a \perp b$) if $a \leqq b^\perp$. If $a \perp b$ and $a \leqq c$, then $(a \vee b) \wedge c = a \vee (b \wedge c)$.

A subset $\mathscr{A}$ of a logic is called a ring (Σ-ring) if $a, b \in \mathscr{A}$ ($a_n \in \mathscr{A}$, $n = 1, 2, \ldots$) implies $a \vee b \in \mathscr{A}$ $\left(\bigvee_n a_n \in \mathscr{A}\right)$, $a \wedge b \in \mathscr{A}$, $a \wedge b^\perp \in \mathscr{A}$. A mapping $m: \mathscr{A} \to \langle 0, \infty \rangle$ is called a measure if the following statements are satisfied:

(α) $m(0) = 0$

(β) if $a_n \in \mathscr{A}$ $(n = 1, 2, \ldots)$ and a_n are pairwise orthogonal and $\bigvee_n a_n \in \mathscr{A}$, then

$$m\left(\bigvee_n a_n\right) = \sum_n m(a_n).$$

A measure m is called subadditive if $m(a \vee b) \leqq m(a) + m(b)$ for every $a, b \in \mathscr{A}$.

Preparatory constructions

Let $\mathscr{H}$ be a σ-continuous logic. Let $\mathscr{A} \subset \mathscr{H}$ be a ring, let $m: \mathscr{A} \to \langle 0, \infty)$ be a finite subadditive measure. We want to extend it to the Σ-ring $\Sigma(\mathscr{A})$ generated by $\mathscr{A}$. We shall prove the main theorem in the case of m being σ-finite.

Let $\mathscr{A}^+ = \{b \in \mathscr{H};\ \exists b_n \in \mathscr{A},\ b_n \nearrow b\}$. It is easy to prove that a mapping m^+: $\mathscr{A}^+ \to \langle 0, \infty \rangle$ is well defined by the formula

$$m^+(b) = \lim_n m(b_n), \quad b_n \nearrow b$$

Now put

$$m^*(x) = \inf\{m^+(b);\ b \in \mathscr{A}^+, x \leqq b\}, \quad x \in \mathscr{H}.$$

Similarly can be defined $\mathscr{A}^-$, m^-, m_*. It is easy to prove that m^+, m^- are non-negative extension of m, m^+ is non-decreasing, subadditive and upper continuous, m^* is non-decreasing and subadditive and m^* is an extension of m^+.

Lemma 1. *Let $a \in \mathscr{A}^-$, $b \in \mathscr{A}^+$, $a \leqq b$. Then $m^-(a) \leqq m^+(b)$.*

Proof. It is sufficient to consider $m^+(b) < \infty$. Let a_n, $b_n \in \mathscr{A}$ $(n = 1, 2, \ldots)$, $a_n \searrow a$, $b_n \nearrow b$. If $K = a_1 \vee b$, then $a^\perp$, K, $K \wedge a^\perp \in \mathscr{A}^+$, $m^+(K) < \infty$,

$$m^+(K) = \lim_n m(a_1 \vee b_n) = \lim_n m(a_n) + \lim_n m((a_1 \vee b_n) \wedge a_n^\perp) =$$
$$= m^-(a) + m^+(K \wedge a^\perp),$$
$$K = a \vee (K \wedge a^\perp) \leqq b \vee (K \wedge a^\perp) \leqq K.$$

If $m^-(a) > m^+(b)$, then $m^+(K) \leqq m^+(b) + m^+(K \wedge a^\perp) < m^-(a) + m^+(K \wedge a^\perp) = m^+(K)$. This is a contradiction.

Corollary. *For every* $x \in \mathcal{H}\, m_*(x) \leqq m^*(x)$.

Lemma 2. *If* $a \in \mathcal{A}^-$, $b \in \mathcal{A}^+$, $a \leqq b$, *then* $m^+(b) = m^-(a) + m^+(b \wedge a^\perp)$.

Proof. Let $b_n \nearrow b$, $a_n \searrow a$, a_n, $b_n \in \mathcal{A}$ $(n = 1, 2, \ldots)$; then

$$m^+(b) = \lim_n m(b_n) \geqq \lim_n m((b_n \wedge a_m) \vee (b_n \wedge a_m^\perp)) =$$
$$= \lim_n m(b_n \wedge a_m) + \lim_n m(b_n \wedge a_m^\perp) =$$
$$= m^+(b \wedge a_m) + m^+(b \wedge a_m^\perp) \geqq m^-(a) + m^+(b \wedge a_m^\perp).$$

Taking $m \to \infty$ we obtain

$$m^+(b) \geqq m^-(a) + m^+(b \wedge a^\perp). \tag{1}$$

Further

$$(a_m \vee (b_n \wedge a_n^\perp)) \nearrow (a_m \vee (b \wedge a^\perp)) \geqq b,$$
$$m^+(b) \leqq m^+(a_m \vee (b \wedge a^\perp)) = \lim_n m(a_m \vee (b_n \wedge a_n^\perp)) \leqq$$
$$\leqq m(a_m) + m^+(b \wedge a^\perp), \quad \text{hence}$$

$$m^+(b) \leqq m^-(a) + m^+(b \wedge a^\perp). \tag{2}$$

The assertion follows from (1) and (2).

Let us denote $L = \{x \in \mathcal{H};\ m_*(x) = m^*(x) < \infty\}$.

Lemma 3. *Let* $y \in \mathcal{H}$, $x \in L$, $x \leqq y$. *Then* $m^*(y) = m^*(x) + m^*(y \wedge x^\perp)$.

Proof. It is sufficient to consider $m^*(y) < \infty$. If $\varepsilon > 0$, then there exist $a \in \mathcal{A}^-$, $b \in \mathcal{A}^+$ such that $a \leqq x$, $y \leqq b$ and

$$m^*(x) = m_*(x) < m^-(a) + \varepsilon,\ m^+(b) - \varepsilon < m^*(y),$$
$$m^*(y \wedge x^\perp) \leqq m^+(b \wedge a^\perp).$$

Further

$$m^*(x) + m^*(y \wedge x^\perp) < m^-(a) + m^+(b \wedge a^\perp) + \varepsilon =$$
$$= m^+(b) + \varepsilon < m^*(y) + 2\varepsilon, \quad \text{hence}$$
$$m^*(x) + m^*(y \wedge x^\perp) \leqq m^*(y).$$

The opposite inequality follows from the subadditivity of m^*.

Proposition 1. *If* x, $y \in L$ *and* $x \leqq y$, *then* $y \wedge x^\perp \in L$.

Proof. To any $\varepsilon > 0$ there exist a, $c \in \mathcal{A}^-$ and b, $d \in \mathcal{A}^+$ such that $a \leqq x \leqq b$, $c \leqq y \leqq d$, $a \leqq c$, $b \leqq d$ and

$$\begin{aligned} m^+(b) - m^-(a) &< \varepsilon \\ m^+(d) - m^-(c) &< \varepsilon. \end{aligned} \tag{3}$$

Obviously $c \wedge b^\perp \leqq y \wedge x^\perp \leqq d \wedge a^\perp$, $c \wedge b^\perp \in \mathcal{A}^-$ and $d \wedge a^\perp \in \mathcal{A}^+$. Further

$$((d \wedge c^\perp) \vee (b \wedge a^\perp))^\perp = (d^\perp \vee c) \wedge (b^\perp \vee a) =$$

$$= a \vee ((d^{\perp} \vee c) \wedge b^{\perp}) = a \vee d^{\perp} \vee (c \wedge b^{\perp}) =$$
$$= (d \wedge a^{\perp})^{\perp} \vee (c \wedge b^{\perp}) = ((d \wedge a^{\perp}) \wedge (c \wedge b^{\perp})^{\perp})^{\perp},$$

hence

$$(d \wedge c^{\perp}) \vee (b \wedge a^{\perp}) = (d \wedge a^{\perp}) \wedge (c \wedge b^{\perp})^{\perp}.$$

We have by Lemma 2 and (3)

$$m^{+}(d \wedge a^{\perp}) - m^{-}(c \wedge b^{\perp}) =$$
$$= m^{+}((d \wedge a^{\perp}) \wedge (c \wedge b^{\perp})^{\perp}) \leqq m^{+}(d \wedge c^{\perp}) + m^{+}(b \wedge a^{\perp}) =$$
$$= m^{+}(d) - m^{-}(c) + m^{+}(b) - m^{-}(a) < 2\varepsilon,$$

hence it follows that $m_{*}(y \wedge x^{\perp}) = m^{*}(y \wedge x^{\perp})$.

Proposition 2. *If $z_n \in L$ $(n = 1, 2, \dots)$, $z_n \nearrow z$ $(z_n \searrow z)$, $z \in H$ and $\lim_n m^{*}(z_n) < \infty$, then $z \in L$ and $m^{*}(z) = \lim_n m^{*}(z_n)$.*

Proof. The first part of the Proposition can be proved analogously as in [5]. Let $z_n \searrow z$; then $z_1 \geqq z_n \geqq z$ $(n = 1, 2, \dots)$, $z_1 \wedge z_n^{\perp} \in L$, $z_1 \wedge z_n^{\perp} \nearrow z_1 \wedge z^{\perp}$. From the first part we have $z_1 \wedge z^{\perp} \in L$ because $m^{*}(z_1 \wedge z^{\perp}) \leqq m^{*}(z_1) < \infty$. Further

$$z = z_1 \wedge (z_1 \wedge z^{\perp})^{\perp} \in L,\ m^{*}(z_1) = m^{*}(z) + m^{*}(z_1 \wedge z^{\perp}),$$
$$m^{*}(z) = m^{*}(z_1) - m^{*}(z_1 \wedge z^{\perp}) = m^{*}(z_1) - \lim_n m^{*}(z_1 \wedge z_n^{\perp}) =$$
$$= \lim_n m^{*}(z_1 \wedge (z_1 \wedge z_n^{\perp})^{\perp}) = \lim_n m^{*}(z_n).$$

Proposition 3. *The mapping $\bar{m} = m^{*}|L$ is additive, i.e. $x, y \in L$, $y \leqq x^{\perp}$ implies $m^{*}(x \vee y) = m^{*}(x) + m^{*}(y)$.*

Proof. Let $x, y \in L$, $y \leqq x^{\perp}$; then by Lemma 3 we have

$$m^{*}(x \vee y) = m^{*}(x) + m^{*}((x \vee y) \wedge x^{\perp}) = m^{*}(x) + m^{*}(y).$$

Definition. *Let $\mathcal{H}$ be a σ-continuous logic, $A \subset \mathcal{H}$. By $\Sigma(A)$ ($\mathcal{S}(A)$, $\sigma(A)$, $\mathcal{D}(A)$) we shall denote the Σ-ring generated by A (the smallest monotone system containing A; the smallest ring containing A closed with respect to the least upper bounds of any sequences of elements of $\sigma(A)$ upper bounded in $\sigma(A)$; the smallest system containing A closed with respect to the limits of any decreasing sequences and the limits of any increasing sequences of elements of $\mathcal{D}(A)$ upper bounded in $\mathcal{D}(A)$).*

Lemma 4. *Let $\mathcal{H}$ be a σ-continuous logic and let $\mathcal{A} \subset \mathcal{H}$ be a ring. Then $\mathcal{S}(\mathcal{A})$, $\mathcal{D}(\mathcal{A})$ are rings and $\mathcal{S}(\mathcal{A}) = \Sigma(\mathcal{A})$, $\mathcal{D}(\mathcal{A}) = \sigma(\mathcal{A})$. If $a \in \mathcal{S}(\mathcal{A})$, $b \in \mathcal{D}(\mathcal{A})$ and $a \leqq b$, then $a \in \mathcal{D}(\mathcal{A})$.*

Proof. See [4].

Main theorem

Theorem. *Let $\mathcal{H}$ be a σ-continuous logic. Let $\mathcal{R} \subset \mathcal{H}$ be a ring and let m: $\mathcal{R} \to \langle 0, \infty \rangle$ be a σ-finite, subadditive measure. Then there is exactly one measure m: $\Sigma(\mathcal{R}) \to \langle 0, \infty \rangle$ that is an extension of m. The measure $\bar{m}$ is a σ-finite subadditive measure.*

Proof. First let us suppose that m is a finite measure defined on a ring $\mathcal{A} \subset \mathcal{H}$. From Proposition 2 and the inclusion $\mathcal{A} \subset L$ it follows that $\mathcal{D}(\mathcal{A}) \subset L$. Let us denote

$$\overline{\mathcal{D}(\mathcal{A})} = \{x \in L\,;\, \exists x_n \in \mathcal{D}(\mathcal{A}),\ x_n \nearrow x,\ \lim_n m^*(x_n) < \infty\}.$$

By Lemma 4 and Proposition 2 it can be easily proved that $\overline{\mathcal{D}(\mathcal{A})}$ is a lattice, $\mathcal{D}(\mathcal{A}) \subset \overline{\mathcal{D}(\mathcal{A})} \subset \Sigma(\mathcal{A})$ and $\overline{\mathcal{D}(\mathcal{A})} \subset L$. If $x \in \Sigma(\mathcal{A})$, $y \in \overline{\mathcal{D}(\mathcal{A})}$ and $x \leqq y$, then $x \in \overline{\mathcal{D}(\mathcal{A})}$. Indeed if $y_n \nearrow y, y_n \in \mathcal{D}(\mathcal{A})$ and $\lim_n m^*(y_n) < \infty$, then $y_n \wedge x \nearrow x$, $y_n \wedge x \leqq$ $\leqq y_n$, $y_n \wedge x \in \Sigma(\mathcal{A}) = \mathcal{S}(\mathcal{A})$ and by Lemma 4 we have $y_n \wedge x \in \mathcal{D}(\mathcal{A})$. Evidently $\lim_n m^*(y_n \wedge x) \leqq \lim_n m^*(y_n) < \infty$, consequently $x \in \overline{\mathcal{D}(\mathcal{A})}$.

Now let us define $\bar{m}$ on $\Sigma(\mathcal{A})$ in the following way:

If $x \in \overline{\mathcal{D}(\mathcal{A})}$, then $\bar{m}(x) = m^*(x)$, if $x \notin \overline{\mathcal{D}(\mathcal{A})}$, then $\bar{m}(x) = \infty$. The mapping $\bar{m}$ is non-decreasing. Namely, if $x \leqq y$ and $y \in \overline{\mathcal{D}(\mathcal{A})}$, then $x \in \overline{\mathcal{D}(\mathcal{A})}$ and $\bar{m}(x) = m^*(x) \leqq$ $\leqq m^*(y) = \bar{m}(y)$. The mapping $\bar{m}$ is upper continuous. Let x_n, $x \in \Sigma(\mathcal{A})$ $(n = 1, 2, \ldots)$, $x_n \nearrow x$. Evidently $\lim_n \bar{m}(x_n) \leqq \bar{m}(x)$. If $\lim_n \bar{m}(x_n) < \infty$, then $x_n \in \overline{\mathcal{D}(\mathcal{A})}$. Let $x_{nm} \nearrow x_n$, $x_{nm} \in \mathcal{D}(\mathcal{A})$, $\lim_m \bar{m}(x_{nm}) < \infty$ $(n, m = 1, 2, \ldots)$. The sequences are chosen already so that $x_{nm} \leqq x_{rm}$ for any integers n, r, m, $n < r$. If $y_n = x_{nn}$, then $\bigvee_n y_n = \bigvee_n x_n = x$, $y_n \nearrow x$, $\lim_n \bar{m}(y_n) \leqq \lim_n \bar{m}(x_n) < \infty$ hence $x \in \overline{\mathcal{D}(\mathcal{A})} \subset L$. Thus $\lim_n \bar{m}(x_n) = \lim_n m^*(x_n) = m^*(x) = \bar{m}(x)$.

The mapping $\bar{m}$ is additive. Let x, $y \in \Sigma(\mathcal{A})$, $x \perp y$, x, $y \in \overline{\mathcal{D}(\mathcal{A})}$; then $x \vee y \in \overline{\mathcal{D}(\mathcal{A})}$ and $\bar{m}(x \vee y) = m^*(x \vee y) = m^*(x) + m^*(y) = \bar{m}(x) + \bar{m}(y)$. If $x \notin \overline{\mathcal{D}(\mathcal{A})}$ or $y \notin \overline{\mathcal{D}(\mathcal{A})}$, then $x \vee y \notin \overline{\mathcal{D}(\mathcal{A})}$ and the additivity of $\bar{m}$ is evident. The subadditivity of $\bar{m}$ is proved analogously. The mapping $\bar{m}$ is non-decreasing, upper continuous, subadditive, additive hence $\bar{m}$ is a subadditive measure on $\Sigma(\mathcal{A})$.

Now let m be a σ-finite subadditive measure defined on a ring $\mathcal{R} \subset \mathcal{H}$. If $\mathcal{A} = \{x \in \mathcal{R};\ m(x) < \infty\}$, then $\mathcal{A}$ is a ring. According to the preceding part of the proof we can extend m to $\Sigma(\mathcal{A})$, but $\Sigma(\mathcal{A}) = \Sigma(\mathcal{R})$, because if $x \in \mathcal{R}$, then there exist $x_n \in \mathcal{A}$ $(n = 1, 2, \ldots)$, $x_n \nearrow x$. The system $T = \{c \in \Sigma(\mathcal{R});\ c \leqq \bigvee_n a_n,\ a_n \in \mathcal{A},\ n = 1, 2, \ldots\}$ is monotone and it contains $\mathcal{R}$, hence $T = \Sigma(\mathcal{R})$ and $\bar{m}$ is σ-finite.

Now we prove the uniqueness of the extension. Let p be a measure defined on $\Sigma(\mathcal{A})$ and $p(x) = m(x)$ for every $x \in \mathcal{A}$. Let $Q = \{x \in \Sigma(\mathcal{A});\ p(x) = \bar{m}(x) < \infty\}$. Evidently $\mathcal{A} \subset Q$ If $x_n \nearrow x$, $y \in Q$, $x \leqq y$, $x_n \in Q$ $(n = 1, 2, \ldots)$, then $\bar{m}(x) = \lim_n \bar{m}(x_n) = \lim_n p(x_n) = p(x) \leqq p(y) < \infty$, hence $x \in Q$. If $x_n \in Q$, $x_n \searrow x$, then also $x \in Q$ and $\mathcal{D}(\mathcal{A}) \subset Q$. If $x \in \overline{\mathcal{D}(\mathcal{A})}$, $x_n \nearrow x$, $x_n \in \mathcal{D}(\mathcal{A})$ $(n = 1, 2, \ldots)$, then $p(x) = \lim_n p(x_n) = \lim_n \bar{m}(x_n) = \bar{m}(x) < \infty$, hence $\overline{\mathcal{D}(\mathcal{A})} \subset Q$. Let $x \in \Sigma(\mathcal{A})$; then there exists a non-decreasing sequence $\{a_n\}_{n=1}^{\infty}$ of elements of $\mathcal{A}$ such that $x \leqq \bigvee_n a_n$. Then $x = \bigvee_n (x \wedge a_n)$, $x \wedge a_n \leqq a_n \in \overline{\mathcal{D}(\mathcal{A})}$, hence $x \wedge a_n \in \overline{\mathcal{D}(\mathcal{A})}$ and

$$\bar{m}(x) = \lim_n \bar{m}(x \wedge a_n) = \lim_n p(x \wedge a_n) = p(x).$$

The proof of Theorem is complete.

REFERENCES

[1] KAPPOS, D. A.: Measure theory on orthocomplemented posets and lattices. In: Measure theory, Oberwolfach 1975, Proceedings, Springer, Berlin, Lecture notes in math., vol. 541, 323—343.

[2] RIEČAN, B.: On the extension of measures on lattices, Mat. Čas., 19, 1969, 44—49

[3] RIEČAN, B.: A note on the extension of measures on lattices, Mat. Čas., 20, 1970, 239—244.

[4] RIEČAN, B.: Абстрактное построение меры Лебега из меры Бореля, Mat. Čas., 25, 1975, 49—58.

[5] RIEČAN, B.: The measure extension theorem for subadditive probability measures in orthomodular σ-continuous lattices, CMUC, 20, 1979, 309—315.

[6] VARADARAJAN, V. S.: Geometry of quantum theory. Van Nostrand, New York 1968.

[7] VOLAUF, P.: The measure extension problem on ortholattices, Acta F.R.N. Univ. Comen. Mathematica, 36, 1980, 171—177.

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ПРЕДЛОЖЕНИЕ О ПРОДОЛЖЕНИИ МЕРЫ ДЛЯ СУБАДДИТИВНЫХ МЕР В σ-НЕПРЕРЫВНЫХ ЛОГИКАХ

Петр Врабел

Резюме

Пусть $\mathcal{H}$ — σ-непрерывная логика, m — σ-конечная субаддитивная мера на кольце $\mathcal{R} \subset \mathcal{H}$. Пусть $\Sigma(\mathcal{R})$ наименьшее σ-полное кольцо, содержащее $\mathcal{R}$. Тогда существует единственная мера $\bar{m}$: $\Sigma(\mathcal{R}) \to \langle 0, \infty \rangle$, являющаяся продолжением меры m. Мера $\bar{m}$ σ-конечна и субаддитивна.