Peter Vrábel
The measure extension theorem for subadditive measures in \( \sigma \)-continuous logics

*Mathematica Slovaca*, Vol. 31 (1981), No. 2, 141--147

Persistent URL: [http://dml.cz/dmlcz/129774](http://dml.cz/dmlcz/129774)

Terms of use:

© Mathematical Institute of the Slovak Academy of Sciences, 1981

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
THE MEASURE EXTENSION THEOREM FOR SUBADDITIVE MEASURES IN \( \sigma \)-CONTINUOUS LOGICS

PETER VRÁBEL

The quantum theory requires the study of measures on logics (see [1], [6]). The basic problem of the extension of measures on logics has not been solved so far. There are some results in [2], [3], but for modular lattices only. B. Riečan proved the extension theorem for subadditive probability measures in [5]. B. Riečan assumes the given measure to be a probability measure defined on an orthocomplemented sublattice of a logic. Every orthocomplemented sublattice of a logic is a ring.

In this paper we prove an extension theorem for subadditive \( \sigma \)-finite measures defined on rings.

Notations and notions

If \( \mathcal{H} \) is a lattice, we shall write \( x_n \uparrow x \), if \( x_n \leq x_{n+1} \) \( (n = 1, 2, \ldots) \) and \( x = \bigvee_{n=1}^{\infty} x_n \), similarly for \( x_n \downarrow x \). A \( \sigma \)-complete lattice will be called \( \sigma \)-continuous if \( x_n \uparrow x, y_n \uparrow y \) implies \( x_n \wedge y_n \uparrow x \wedge y \) and respectively.

By an orthocomplementation of a lattice \( \mathcal{H} \) with the least element 0 we mean a mapping \( \perp : a \rightarrow a^\perp \) of \( \mathcal{H} \) into itself such that

(i) \( a \leq b \) implies \( b^\perp \leq a^\perp \),
(ii) \( (a^\perp)^\perp = a \) for all \( a \),
(iii) \( a \wedge a^\perp = 0 \) for all \( a \).

A \( \sigma \)-complete lattice \( \mathcal{H} \) with an orthocomplementation \( \perp \) is said to be a logic in the following case

(iv) if \( a, b \in \mathcal{H} \) and \( a \leq b \), there exists an element \( d \in \mathcal{H} \) such that \( d \leq a^\perp \) and \( b = a \vee d \).

The element \( d \) in (iv) is unique and is equal to \( b \wedge a^\perp \) (see e.g. [6]). If \( a_1, a_2, \ldots \) is a sequence of elements of a logic, then
Two elements $a$, $b$ of a logic are called orthogonal $(a \perp b)$ if $a \leq b^\perp$. If $a \perp b$ and $a \leq c$, then $(a \lor b) \land c = a \lor (b \land c)$.

A subset $\mathcal{A}$ of a logic is called a ring ($\Sigma$-ring) if $a, b \in \mathcal{A}$ ($a_n \in \mathcal{A}$, $n = 1, 2, \ldots$) implies $a \lor b \in \mathcal{A}$ ($\bigvee a_n \in \mathcal{A}$), $a \land b \in \mathcal{A}$, $a \land b^\perp \in \mathcal{A}$. A mapping $m: \mathcal{A} \to (0, \infty)$ is called a measure if the following statements are satisfied:

(a) $m(0) = 0$

(\beta) if $a_n \in \mathcal{A}$ ($n = 1, 2, \ldots$) and $a_n$ are pairwise orthogonal and $\bigvee a_n \in \mathcal{A}$, then

$$m\left(\bigvee a_n\right) = \sum_n m(a_n).$$

A measure $m$ is called subadditive if $m(a \lor b) \leq m(a) + m(b)$ for every $a, b \in \mathcal{A}$.

**Preparatory constructions**

Let $\mathcal{H}$ be a $\sigma$-continuous logic. Let $\mathcal{A} \subset \mathcal{H}$ be a ring, let $m: \mathcal{A} \to (0, \infty)$ be a finite subadditive measure. We want to extend it to the $\Sigma$-ring $\Sigma(\mathcal{A})$ generated by $\mathcal{A}$. We shall prove the main theorem in the case of $m$ being $\sigma$-finite.

Let $\mathcal{A}^+ = \{ b \in \mathcal{H}; \exists b_n \in \mathcal{A}, b_n \not\perp b \}$. It is easy to prove that a mapping $m^*: \mathcal{A}^+ \to (0, \infty)$ is well defined by the formula

$$m^*(b) = \lim_n m(b_n), \quad b_n \not\perp b$$

Now put

$$m^*(x) = \inf \{ m^*(b); b \in \mathcal{A}^+, x \leq b \}, \quad x \in \mathcal{H}.$$  

Similarly can be defined $\mathcal{A}^-, m^-, m^-$. It is easy to prove that $m^+$, $m^-$ are non-negative extension of $m$, $m^+$ is non-decreasing, subadditive and upper continuous, $m^*$ is non-decreasing and subadditive and $m^*$ is an extension of $m^+$.

**Lemma 1.** Let $a \in \mathcal{A}^+$, $b \in \mathcal{A}^+$, $a \leq b$. Then $m^-(a) \leq m^+(b)$.

**Proof.** It is sufficient to consider $m^+(b) < \infty$. Let $a_n, b_n \in \mathcal{A}$ ($n = 1, 2, \ldots$), $a_n \not\perp a$, $b_n \not\perp b$. If $K = a \lor b$, then $a^\perp, K, K \land a^\perp \in \mathcal{A}$, $m^+(K) < \infty$,

$$m^+(K) = \lim_n m(a_n \lor b_n) = \lim_n m(a_n) + \lim m((a_n \lor b_n) \land a_n^\perp) = m^-(a) + m^+(K \land a^\perp),$$

$$K = a \lor (K \land a^\perp) \leq b \lor (K \land a^\perp) \leq K.$$

If $m^-(a) > m^+(b)$, then $m^+(K) \leq m^+(b) + m^+(K \land a^\perp) < m^-(a) + m^+(K \land a^\perp) = m^+(K)$. This is a contradiction.

142
Corollary. For every $x \in \mathcal{H}$, $m^*(x) \leq m^*(x)$.

Lemma 2. If $a \in \mathcal{A}^-$, $b \in \mathcal{A}^+$, $a \preceq b$, then $m^+(b) = m^-(a) + m^+(b \land a^\perp)$.

Proof. Let $b_n \searrow b$, $a_n \searrow a$, $b_n \in \mathcal{A}$ ($n = 1, 2, \ldots$); then

$$m^+(b) = \lim_{n} m(b_n) \leq \lim_{n} m((b_n \land a_n) \lor (b_n \land a_n^\perp)) =$$

$$= \lim_{n} \{m(b_n \land a_n) + \lim_{n} m(b \land a_n^\perp) =$$

$$= m^+(b \land a_n) + m^+(b \land a_n^\perp) \leq m^-(a) + m^+(b \land a_n^\perp).$$

Taking $m \to \infty$ we obtain

$$m^+(b) \geq m^-(a) + m^+(b \land a^\perp).$$

Further

$$m^+(b) \leq m^-(a) + m^+(b \land a^\perp).$$

The assertion follows from (1) and (2).

Let us denote $L = \{x \in \mathcal{H}; m^*(x) = m^*(x) < \infty\}$.

Lemma 3. Let $y \in \mathcal{H}$, $x \in L$, $x \preceq y$. Then $m^*(y) = m^*(x) + m^*(y \land x^\perp)$.

Proof. It is sufficient to consider $m^*(y) < \infty$. If $\varepsilon > 0$, then there exist $a \in \mathcal{A}^-$, $b \in \mathcal{A}^+$ such that $a \preceq x$, $y \preceq b$ and

$$m^*(x) = m^*(x) < m^-(a) + \varepsilon, m^+(b) - \varepsilon < m^*(y),$$

$$m^*(y \land x^\perp) \leq m^+(b \land a^\perp).$$

Further

$$m^*(x) + m^*(y \land x^\perp) < m^-(a) + m^+(b \land a^\perp) + \varepsilon =$$

$$= m^+(b) + \varepsilon < m^*(y) + 2\varepsilon,$$

hence

$$m^*(x) + m^*(y \land x^\perp) \leq m^*(y).$$

The opposite inequality follows from the subadditivity of $m^*$.

Proposition 1. If $x$, $y \in L$ and $x \preceq y$, then $y \land x^\perp \in L$.

Proof. To any $\varepsilon > 0$ there exist $a, c \in \mathcal{A}^-$ and $b, d \in \mathcal{A}^+$ such that $a \preceq x \preceq b$, $c \preceq y \preceq d$, $a \preceq c$, $b \preceq d$ and

$$m^+(b) - m^-(a) < \varepsilon,$$

$$m^+(d) - m^-(c) < \varepsilon.$$

Obviously $c \land b^\perp \preceq y \land x^\perp \preceq d \land a^\perp$, $c \land b^\perp \in \mathcal{A}^-$ and $d \land a^\perp \in \mathcal{A}^+$. Further

$$((d \land c^\perp) \lor (b \land a^\perp))^\perp = (d^\perp \land c) \land (b^\perp \land a) =$$

143
= a \lor ((d \lor c) \land b) = a \lor d \lor (c \land b) =
= (d \land a) \lor (c \land b) = ((d \land a) \land (c \land b)) ^{\lor},

hence

(d \land c) \lor (b \land a) = (d \land a) \land (c \land b).}

We have by Lemma 2 and (3)

\[ m^+(d \land a) - m^-(c \land b) = \]
\[ = m^+((d \land a) \land (c \land b)) \leq m^+(d \land c) + m^+(b \land a) = \]
\[ = m^+(d) - m^-(c) + m^+(b) - m^-(a) < 2\varepsilon, \]

hence it follows that \( m^*(y \lor x) = m^*(y \land x) \).

**Proposition 2.** If \( z_n \in L \) (\( n = 1, 2, \ldots \)), \( z_n \not\subseteq z \) \((n = 1, 2, \ldots)\), \( z_1 \land z_n \in L, z_1 \land z_n \not\subseteq z \land z_1 \). From the first part we have \( z_1 \land z \subseteq L \) because \( m^*(z_1 \land z) \leq m^*(z_1) < \infty \). Further

\[ z = z_1 \land (z_1 \land z)^{\lor} \subseteq L, m^*(z_1) = m^*(z) + m^*(z_1 \land z), \]
\[ m^*(z) = m^*(z_1) - m^*(z_1 \land z) = m^*(z_1) - \lim_n m^*(z_1 \land z_n) = \]
\[ = \lim_n m^*(z_1 \land (z_1 \land z_n)^{\lor}) = \lim_n m^*(z_n). \]

**Proposition 3.** The mapping \( m = m^*|L \) is additive, i.e. \( x, y \in L, y \subseteq x^{\lor} \) implies \( m^*(x \lor y) = m^*(x) + m^*(y) \).

**Proof.** Let \( x, y \in L, y \subseteq x^{\lor} \); then by Lemma 3 we have

\[ m^*(x \lor y) = m^*(x) + m^*((x \lor y) \land x) = m^*(x) + m^*(y). \]

**Definition.** Let \( L \) be a \( \sigma \)-continuous logic, \( A \subseteq L \). By \( \Sigma(A) (\mathcal{F}(A), \sigma(A), \mathcal{D}(A)) \) we shall denote the \( \Sigma \)-ring generated by \( A \) (the smallest monotone system containing \( A \); the smallest ring containing \( A \) closed with respect to the least upper bounds of any sequences of elements of \( \sigma(A) \) upper bounded in \( \sigma(A) \); the smallest system containing \( A \) closed with respect to the limits of any decreasing sequences and the limits of any increasing sequences of elements of \( \mathcal{D}(A) \) upper bounded in \( \mathcal{D}(A) \)).

**Lemma 4.** Let \( L \) be a \( \sigma \)-continuous logic and let \( \mathcal{A} \subseteq L \) be a ring. Then \( \mathcal{F}(\mathcal{A}), \mathcal{D}(\mathcal{A}) \) are rings and \( \mathcal{F}(\mathcal{A}) = \Sigma(\mathcal{A}), \mathcal{D}(\mathcal{A}) = \sigma(\mathcal{A}) \). If \( a \in \mathcal{F}(\mathcal{A}), b \in \mathcal{D}(\mathcal{A}) \) and \( a \subseteq b \), then \( a \in \mathcal{D}(\mathcal{A}) \).

**Proof.** See [4].

144
Main theorem

Theorem. Let $H$ be a $\sigma$-continuous logic. Let $R \subseteq H$ be a ring and let $m: R \to (0, \infty)$ be a $\sigma$-finite, subadditive measure. Then there is exactly one measure $m: \Sigma(R) \to (0, \infty)$ that is an extension of $m$. The measure $\bar{m}$ is a $\sigma$-finite subadditive measure.

Proof. First let us suppose that $m$ is a finite measure defined on a ring $A \subseteq H$. From Proposition 2 and the inclusion $A \subseteq L$ it follows that $D(A) \subseteq L$. Let us denote

$$D(A) = \{ x \in L; \exists x_n \in D(A), x_n \not\succ x, \lim_n m^*(x_n) < \infty \}.$$

By Lemma 4 and Proposition 2 it can be easily proved that $D(A)$ is a lattice, $D(A) \subseteq D(A) \subseteq \Sigma(A)$ and $D(A) \subseteq L$. If $x \in \Sigma(A)$, $y \in D(A)$ and $x \subseteq y$, then $x \in D(A)$. Indeed if $y_n \not\succ x$, $y_n \in D(A)$ and $\lim_n m^*(y_n) < \infty$, then $y_n \not\succ x$, $y_n \subseteq x \in \Sigma(A) = D(A)$ and by Lemma 4 we have $y_n \subseteq x \in D(A)$. Evidently

$$\lim_n m^*(y_n) \leq \lim_n m^*(y_n) < \infty,$$

consequently $x \in D(A)$.

Now let us define $\bar{m}$ on $\Sigma(A)$ in the following way:

If $x \in D(A)$, then $\bar{m}(x) = m^*(x)$, if $x \notin D(A)$, then $\bar{m}(x) = \infty$. The mapping $\bar{m}$ is non-decreasing. Namely, if $x \subseteq y$ and $y \in D(A)$, then $x \in D(A)$ and $\bar{m}(x) = m^*(x) \leq m^*(y) = \bar{m}(y)$. The mapping $\bar{m}$ is upper continuous. Let $x_n, x \in \Sigma(A)$ ($n = 1, 2, ...$), $x_n \not\succ x$. Evidently $\lim_n \bar{m}(x_n) \leq \bar{m}(x)$. If $\lim_n \bar{m}(x_n) < \infty$, then $x_n \in D(A)$. Let $x_{nm} \not\succ x_n$, $x_{nm} \in D(A)$, $\lim_n \bar{m}(x_{nm}) < \infty$ ($n, m = 1, 2, ...$). The sequences are chosen already so that $x_{nm} \subseteq x_m$ for any integers $n, r, m, n < r$. If $y_n = x_{nm}$, then $\bigvee_n y_n = \bigvee_n x_n = x$, $y_n \not\succ x$, $\lim_n \bar{m}(y_n) \leq \lim_n \bar{m}(x_n) < \infty$ hence $x \in D(A) \subseteq L$. Thus

$$\lim_n \bar{m}(x_n) = \lim_n m^*(x_n) = m^*(x) = \bar{m}(x).$$

The mapping $\bar{m}$ is additive. Let $x, y \in \Sigma(A)$, $x \not\perp y$, $x, y \in D(A)$; then $x \lor y \in D(A)$ and $\bar{m}(x \lor y) = m^*(x \lor y) = m^*(x) + m^*(y) = \bar{m}(x) + \bar{m}(y)$. If $x \notin D(A)$ or $y \notin D(A)$, then $x \lor y \notin D(A)$ and the additivity of $\bar{m}$ is evident. The subadditivity of $\bar{m}$ is proved analogously. The mapping $\bar{m}$ is non-decreasing, upper continuous, subadditive, additive hence $\bar{m}$ is a subadditive measure on $\Sigma(A)$.
Now let $m$ be a $\sigma$-finite subadditive measure defined on a ring $\mathcal{R} \subset \mathcal{A}$. If $\mathcal{A} = \{ x \in \mathcal{R} ; m(x) < \infty \}$, then $\mathcal{A}$ is a ring. According to the preceding part of the proof we can extend $m$ to $\Sigma(\mathcal{A})$, but $\Sigma(\mathcal{A}) = \Sigma(\mathcal{R})$, because if $x \in \mathcal{R}$, then there exist $x_n \in \mathcal{A}$ $(n = 1, 2, \ldots)$, $x_n \not\in x$. The system $T = \{ c \in \Sigma(\mathcal{R}) ; c \subseteq \bigvee a_n, a_n \in \mathcal{A}, n = 1, 2, \ldots \}$ is monotone and it contains $\mathcal{R}$, hence $T = \Sigma(\mathcal{R})$ and $\bar{m}$ is $\sigma$-finite.

Now we prove the uniqueness of the extension. Let $p$ be a measure defined on $\Sigma(\mathcal{A})$ and $p(x) = m(x)$ for every $x \in \mathcal{A}$. Let $Q = \{ x \in \Sigma(\mathcal{A}) ; p(x) = \bar{m}(x) < \infty \}$. Evidently $\mathcal{A} \subset Q$ If $x_n \not\in x, y \in Q, x \leq y, x_n \in Q$ $(n = 1, 2, \ldots)$, then $\bar{m}(x) = \lim_n \bar{m}(x_n) = \lim_n p(x_n) = p(x) \leq p(y) < \infty$, hence $x \in Q$. If $x_n \in Q, x_n \setminus x$, then also $x \in Q$ and $\bar{D}(\mathcal{A}) \subset Q$. If $x \in \bar{D}(\mathcal{A}), x_n \not\in x, x_n \in \bar{D}(\mathcal{A})$ $(n = 1, 2, \ldots)$, then $p(x)$ $= \lim_n p(x_n) = \lim_n \bar{m}(x_n) = \bar{m}(x) < \infty$, hence $\bar{D}(\mathcal{A}) \subset Q$. Let $x \in \Sigma(\mathcal{A})$; then there exists a non-decreasing sequence $\{ a_n \}_{n=1}^\infty$ of elements of $\mathcal{A}$ such that $x \subseteq \bigvee a_n$.

Then $x = \bigvee (x \land a_n), x \land a_n \leq a_n \in \bar{D}(\mathcal{A})$, hence $x \land a_n \in \bar{D}(\mathcal{A})$ and

$$\bar{m}(x) = \lim_n \bar{m}(x \land a_n) = \lim_n p(x \land a_n) = p(x).$$

The proof of Theorem is complete.

REFERENCES


Katedra matematiky
Pedagogickej fakulty
Saratovska 19
949 74 Nitra

146
ПУСТЬ \( \mathcal{H} \) — \( \sigma \)-непрерывная логика, \( m \) — \( \sigma \)-конечная субаддитивная мера на кольце \( \mathcal{R} \subseteq \mathcal{H} \).
Пусть \( \Sigma(\mathcal{R}) \) наименьшее \( \sigma \)-полное кольцо, содержащее \( \mathcal{R} \). Тогда существует единственная мера \( \hat{m} : \Sigma(\mathcal{R}) \rightarrow (0, \infty) \), являющаяся продолжением меры \( m \). Мера \( \hat{m} \) \( \sigma \)-конечна и субаддитивна.