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ALMOST FLOQUET LINEAR DIFFERENCE EQUATIONS

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The Floquet theorem for linear differential equations (see, e.g., [2] and [3]) is formulated as follows:

Theorem 1. *Let $\Phi(t)$ be a fundamental matrix of the linear differential equation*

$$\dot{x} = A(t)x \tag{1}$$

where $x \in C^n$, A: $R \to M_c(n)$ ($M_c(n)$ is the set of matrices of type $n \times n$ with complex elements) is a piecewise continuous function which is τ-periodic, i.e. $A(t + \tau) = A(t)$ for all $t \in R$. Then there exists a constant matrix R and a τ-periodic map P: $R \to M_c(n)$ such that

$$\Phi(t) = P(t)e^{Rt} \quad \textit{for all} \quad t \in R. \tag{2}$$

Is it possible to extend the class of matrices $A(t)$ of the system (1) for which a type of the Floquet theorem holds? This problem is solved in the papers [1], [4] and in the book [5]. A theorem analogical to the Floquet theorem (see, e.g., [7]) can also be formulated for linear difference equations as follows:

Theorem 2. *Let Y_n be the normed fundamental matrix of the linear τ-periodic difference equation*

$$y_{n+1} = A_n y_n \tag{3}$$

(i.e. $A_{n+\tau} = A_n$ for all $n \geqq 0$ where τ is a natural number), i.e. $Y_{n+1} = A_n Y_n$, $n \geqq 0$, $Y_0 = I$ — the unit matrix where all matrices A_n, $n \geqq 0$ are supposed to have complex elements. Then there exists a regular τ-periodic, matrix valued function T_n and a regular constant matrix B such that

$$Y_n = T_n B^n \quad \textit{for all} \quad n \geqq 0. \tag{4}$$

In the present paper we introduce a class of linear difference equations of the form (3) (nonperiodic in general) for which the normed fundamental matrix Y_n ($Y_0 = I$) has the form (4) (T_n is not periodic in general). Results of such type are important for solving stability problems of linear as well as nonlinear difference equations (see [7]). As a specimen of the application of our results we will prove a stability theorem concerning a linear perturbation of the difference system (3).

H. I. Freedman [4] has extended the Floquet theorem for the so-called almost Floquet systems (AFS) which are defined as follows: The system (1) is called an almost Floquet system if there exists a $\tau > 0$ such that $[B(t, \tau), \Phi(t)] = 0$ for all $t \in R$ where $[U, V] = UV - VU$, $\Phi(t)$ is a fundamental matrix of the system (1) and $B(t, \tau) = A(t + \tau) - A(t)$. Obviously, if the matrix function $A(t)$ is τ-periodic, then the system (1) is almost a Floquet system.

Definition 1. *Let all matrices A_n, $n \geqq 0$ in the equation* (3) *be regular, Φ_n be the normed fundamental matrix of this equation and let τ be a natural number. We shall say that the system* (3) *is a τ-almost Floquet system* (τ-AFS) *if*

$$[B_n(\tau), \Phi_n] = 0 \tag{5}$$

for all $n \geqq 0$ where $B_n(\tau) = A_n^{-1} A_{n+\tau}$ and $[U, V] = UV - VU$.

Obviously, if $A_{n+\tau} = A_n$ for all $n \geqq 0$, i.e. the equation (3) is τ-periodic, then $B_n(\tau) = I$ for all $n \geqq 0$ and hence the condition (5) is satisfied. This means that every τ-periodic system of difference equations of the form (3) with A_n regular is a τ-AFS.

Theorem 3. *Let the system* (3) *be a τ-AFS, Φ_n be its normed fundamental matrix and let $\Psi_n(\tau)$ be the normed fundamental matrix of the system*

$$y_{n+1} = B_n(\tau) y_n . \tag{6}$$

Then

$$\Phi_{n+\tau} = \Phi_n \Psi_n(\tau) \Phi_\tau \quad \textit{for all} \quad n \geqq 0. \tag{7}$$

Proof. Let us define $Y_n(\tau) = \Phi_n^{-1} \Phi_{n+\tau}$, $n \geqq 0$. Then

$$Y_{n+1}(\tau) = \Phi_{n+1}^{-1} \Phi_{n+\tau+1} = \Phi_n^{-1} A_n^{-1} A_{n+\tau} \Phi_{n+\tau} = (\Phi_n^{-1} A_n^{-1} A_{n+\tau} \Phi_n) \Phi_n^{-1} \Phi_{n+\tau} .$$

From the equality (5) it follows that $\Phi_n^{-1} A_n^{-1} A_{n+\tau} \Phi_n = A_n^{-1} A_{n+\tau} = B_n(\tau)$ and thus we have $Y_{n+1}(\tau) = B_n(\tau) Y_n(\tau)$. Since $Y_0(\tau) = \Phi_\tau$, $\Psi_0(\tau) = I$ we obtain that $Y_n(\tau) = \Psi_n(\tau) \Phi_\tau$ and thus $\Phi_n^{-1} \Phi_{n+\tau} = \Psi_n(\tau) \Phi_\tau$, or $\Phi_{n+\tau} = \Phi_n \Psi_n(\tau) \Phi_\tau$.

Theorem 4. *Let the system* (3) *be a τ-AFS, Φ_n be its normed fundamental matrix and*

$$[C(\tau), B_n(\tau)] = 0 \quad \textit{for all} \quad n \geqq 0 \tag{8}$$

where $C(\tau) = (\Phi_\tau)^{1/\tau}$. Then there exists a matrix function T_n and a constant matrix B such that

$$\Phi_n = T_n B^n \tag{9}$$

(10) $$T_{n+\tau} = T_n \Psi_n(\tau)$$

for all $n \geqq 0$ *where* $\Psi_n \tau)$ *is the normed fundamental matrix of the system* (6). *Moreover*, $y_n = T_n x_n$ *transforms the system* (3) *into the form*

(11) $$y_{n+1} = By_n.$$

We remark that the matrix Φ_τ is regular and therefore from [6, Theorem 5. 4. 1] it follows that the matrix $C(\tau) = (\Phi_\tau)^{1/\tau}$ is well defined.

If the system (3) is τ-periodic, then the assumption (8) is satisfied, $\Psi_n(\tau) = I$ for all $n \geqq 0$ and thus the assertion of Theorem 4 is in coincidence with the assertion of Theorem 2.

Proof of Theorem 4. If we define $T_n = \Phi_n B^{-n}$ for $n \geqq 0$ where $B = C(\tau)$, then obviously $\Phi_n = T_n B^n$. Let $\omega_n = [B, \Psi_n(\tau)]$ where $\Psi_n(\tau)$ is the normed fundamental matrix of the system (6). The using the equality (8) we obtain that

$$\begin{aligned}\omega_{n+1} &= [B, \Psi_{n+1}(\tau)] = B\Psi_{n+1}(\tau) - \Psi_{n+1}(\tau) B = \\ &= BB_n(\tau) \Psi_n(\tau) - B_n(\tau) \Psi_n(\tau) B = B_n(\tau) B \Psi_n(\tau) - \\ &- B_n(\tau) \Psi_n(\tau) B = B_n(\tau) \omega_n.\end{aligned}$$

Since $\omega_0 = [B, \Psi_0(\tau)] = 0$, we obtain that $[B, \Psi_n(\tau)] = 0$, i.e. $B\Psi_n(\tau) = \Psi_n(\tau) B$ for all $n \geqq 0$. This implies that

(12) $$\Psi_n(\tau) B^{-n} = B^{-n} \Psi_n(\tau) \quad \text{for all} \quad n \geqq 0.$$

Using (7) and (12) we obtain that

$$\begin{aligned}T_{n+\tau} &= \Phi_{n+\tau} B^{-n-\tau} = \Phi_n \Psi_n(\tau) \Phi_\tau B^{-\tau} B^{-n} = \\ &= \Phi_n \Psi_n(\tau) B^\tau B^{-\tau} B^{-n} = \Phi_n B^{-n} \Psi_n(\tau) = T_n \Psi_n(\tau),\end{aligned}$$

i.e. the equality (10) holds. If $y_n = T_n x_n$, then

$$(T_{n+1})^{-1} A_n T_n = (\Phi_{n+1} B^{-n} B^{-1})^{-1} A_n \Phi_n B^{-n} = B \quad \text{for all} \quad n \geqq 0$$

and thus the equality (11) holds.

Now we prove two theorems giving criteria for the system (3) to be a τ-AFS, which are similar to these formulated by Freedman [4] for almost Floquet systems of differential equations.

Theorem 5. *Let all matrices* A_n, $n \geqq 0$ *be regular*, τ *be a natural number and* $[B_m(\tau), A_n] = 0$ *for all* $m, n \geqq 0$ *where* $B_m(\tau) = A_m^{-1} A_{m+\tau}$. *Then the system* (3) *is a* τ-AFS.

Proof. If we define $\alpha_n(m, \tau) = [B_m(\tau), \Phi_n]$ for $m, n \geqq 0$, then

$$\begin{aligned}\alpha_{n+1}(m, \tau) &= [B_m(\tau), \Phi_{n+1}] = B_m(\tau) A_n \Phi_n - A_n \Phi_n B_m(\tau) = \\ &= A_n B_m(\tau) \Phi_n - A_n \Phi_n B_m(\tau) = A_n \alpha_n(m, \tau).\end{aligned}$$

Since $a_0(m, \tau) = [B_m(\tau), I] = 0$ for all $m \geqq 0$, we obtain that $[B_m(\tau), \Phi_n] = 0$ for all $m, n \geqq 0$ and in particular $[B_n(\tau), \Phi_n] = 0$ for all $n \geqq 0$, i.e. (3) is a τ-AFS.

Theorem 6. *Let all matrices A_n, $n \geqq 0$ be regular, τ be a natural number and let $B_n(\tau) = A_n^{-1}A_{n+\tau}$ be such that $[B_{n+i}(\tau), A_n] = 0$ for all $n \geqq 0$ and $i = 0, 1, \ldots, \ldots, k$. Suppose that for any $n \geqq 0$, $B_n(\tau)$ satisfies the following difference equation:*

$$L_n(Z_n) = C_n^0 Z_{n+k} + C_n^1 Z_{n+k-1} + \ldots + C_n^k Z_n + Z_{n+k} D_n^0 + + Z_{n+k-1} D_n^1 + \ldots + Z_n D_n^k = F_n \tag{13}$$

where the matrices C_n^i, D_n^i, F_n, $i = 0, 1, \ldots, k$ commute with the normed fundamental matrix Φ_n of the system (3). *Then the system* (3) *is a* τ-AFS.

Proof. Let $U_n(\tau) = \Phi_n^{-1} B_n(\tau) \Phi_n$ for $n \geqq 0$. Since $B_{n+1}(\tau) A_n = A_n B_{n+1}(\tau)$, we have that

$$U_{n+1}(\tau) = \Phi_{n+1}^{-1} B_{n+1}(\tau) \Phi_{n+1} = \Phi_n^{-1} A_n^{-1} B_{n+1}(\tau) A_n \Phi_n = = \Phi_n^{-1} B_{n+1} \Phi_n .$$

One can easily show by induction that

$$U_{n+i}(\tau) = \Phi_n^{-1} B_{n+i} \Phi_n \quad \text{for} \quad i = 0, 1, \ldots, k. \tag{14}$$

Therefore from the commutability hypothesis we get

$$L_n(U_n(\tau)) = \Phi_n^{-1} L_n(B_n(\tau)) \Phi_n = \Phi_n^{-1} F_n \Phi_n = F_n ,$$

i.e. $U_n(\tau)$ is a solution of the difference equation (13) with the same initial conditions as $B_n(\tau)$ and hence $U_n(\tau) = B_n(\tau)$, or $[B_n(\tau), \Phi_n] = 0$ for all $n \geqq 0$, i.e. (3) is a τ-AFS.

Example. Let $B_n(\tau) = B_0$, i.e. $A_{n+\tau} = B_0 A_n$ for all $n \geqq 0$, where τ is a natural number, B_0 is a constant matrix and assume that

$$[B_0, A_n] = 0 \quad \text{for all} \quad n \geqq 0. \tag{15}$$

Then by Theorem 5 the system (3) is a τ-AFS. Since the normed fundamental matrix of the system (6) with $B_n(\tau) = B_0$ is $\Psi_n(\tau) = B_0^n$, Theorem 3 implies the equality

$$\Phi_{n+\tau} = \Phi_n B_0^n \Phi_\tau \quad \text{for all} \quad n \geqq 0 \tag{16}$$

where Φ_n is the normed fundamental matrix of the system (3). Using Theorem 3 and the equality (16) one can show by induction that

$$\Phi_{n+m\tau} = \Phi_n B_0^{n+\frac{1}{2}m(m-1)\tau} B^{m\tau} \quad \text{for all} \quad m, n \geqq 0 \tag{17}$$

where $B = (\Phi_\tau)^{1/\tau}$. This formula implies that the stability properties of the system

(3) substantially depend on whether the matrices B_0 and B have eigenvalues inside or outside the unit circle.

As a specimen of application of the previous results we prove a theorem concerning a linear perturbation of the system (3). To state the theorem and give its proof we need to introduce one notion and then to prove a lemma.

Definition 2. *A norm $\|\cdot\|$ on R^n is called adapted to given continuous linear maps $P_i: R^n \to R^n$, $i = 1, 2$, if $\|P_i\| \leqq \max(\sigma, \varrho)$, $i = 1, 2$ where $\sigma = \max(|\lambda_1|, |\lambda_2|, \ldots, \ldots, |\lambda_n|)$, $\varrho = \max(|\nu_1|, |\nu_2|, \ldots, |\nu_n|)$, λ_i, ν_i, $i = 1, 2, \ldots, n$, are eigenvalues of P_1 and P_2, respectively and $\|S\| = \sup_{\|x\| \leqq 1} \|Sx\|$.*

Lemma 1. *Let two linear and continuous maps $P_i: R^n \to R^n$, $i = 1, 2$, be given. Then there exists a norm on R^n adapted to these maps.*

Proof. By [8, p. 312] there exist norms $\|\cdot\|_1$, $\|\cdot\|_2$ on R^n such that $\|P_1\|_1 = \sigma$ and $\|P_2\|_2 = \varrho$ where $\|S\|_i = \sup_{\|x\| \leqq 1} \|Sx\|$, $i = 1, 2$. The function $x \to \max(\|x\|_1, \|x\|_2)$ is the wanted norm on R^n.

Theorem 7. *Let the system (3) be a τ-AFS with $B_n = A_n^{-1} A_{n+\tau} = B_0$ for all $n \geqq 0$ where B_0 is a constant matrix. Assume that the matrices $B = (\Phi_\tau)^{1/\tau}$ and B_0 have all their eigenvalues inside the open unit circle where Φ_n is the normed fundamental matrix of the system (3) and let $[B, B_0] = 0$. Then the system*

$$x_{n+1} = (A_n + D_n)\, x_n \tag{18}$$

is asymptotically stable provided

$$\sum_{n=0}^{\infty} k^{\gamma(n) - \gamma(n+1)} \|D_n\| < \infty \tag{19}$$

where $k = \max(\sigma, \varrho)$, $\sigma = \max(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|)$, $\varrho = \max(|\nu_1|, |\nu_2|, \ldots, |\nu_n|)$, λ_i, ν_i, $i = 1, 2, \ldots, n$ are eigenvalues of B and B_0, respectively, $\|\cdot\|$ is a norm on R^n adapted to the maps B and B_0, $\gamma(n) = \beta(n)\,\tau + \frac{1}{2}\beta(n)[\beta(n) - 1]\,\tau$ where $\alpha: N \to N \cap [0, \tau)$ and $\beta: N \to N$ are such functions defined on the set N of natural numbers that any $n \in N$ can be written as $n = \alpha(n) + \beta(n)\,\tau$.

Proof. By [7, p. 36] the solution of the nonhomogeneous difference equation

$$x_{n+1} = A_n x_n + f_n \tag{20}$$

satisfying the initial condition $x_{n_0} = \xi$ has the form

$$x_n = \Phi_n \Phi_{n_0}^{-1} \xi + \sum_{\nu=0}^{n-1} \Phi_n \Phi_{\nu+1}^{-1} f_\nu. \tag{21}$$

By this variation of the constant formula we can write the solution x_n of (18) satisfying the condition $x_{n_0} = \xi$ as

$$x_n = \Phi_n \Phi_{n_0}^{-1}\xi + \sum_{\nu=0}^{n-1} \Phi_n \Phi_{\nu+1}^{-1} D_\nu x_\nu. \tag{22}$$

Let $M_1 = \max_{0 \leqq n \leqq \tau} \|\Phi_n\|$ and $M_2 = \max_{0 \leqq n \leqq \tau} \|\Phi_n^{-1}\|$ where $\|\cdot\|$ is a norm on R^n adapted to the maps B and B_0. By Lemma 1 such an adapted norm on R^n exists. Using the equality (17) and the assumption $[B, B_0] = BB_0 - B_0 B = 0$ we obtain that for any m, $n \geqq 0$

$$\Phi_n \Phi_m^{-1} = \Phi_{\alpha(n)} B_0^{\alpha(n) + \frac{1}{2}\beta(n)[\beta(n)-1]\tau} \cdot B^{\beta(n)\tau} \cdot B^{-\beta(m)\tau} \cdot B_0^{-\alpha(m) - \frac{1}{2}\beta(m)[\beta(m)-1]\tau} \cdot \Phi_{\alpha(m)}^{-1},$$

i.e.

$$\Phi_n \Phi_m^{-1} = \Phi_{\alpha(n)} B_0^{\alpha(n)-\alpha(m)} \cdot B^{[\beta(n)-\beta(m)]\tau} \cdot B_0^{\delta(n)-\delta(m)} \Phi_{\alpha(m)}^{-1} \tag{23}$$

where the functions α, β are as in theorem and $\delta(i) = \frac{1}{2}\beta(i)[\beta(i)-1]\tau$. Since $\|B_0\| \leqq k < 1$ and $\|B\| \leqq k < 1$, we obtain from (23) that

$$\|\Phi_n \Phi_m^{-1}\| \leqq M_1 M_2 \cdot k^{\gamma(n)-\gamma(m)} \quad \text{for all} \quad m, n \geqq 0,\ n \geqq m \tag{24}$$

where the function γ is as in the theorem. Substituting (24) in (22) gives

$$\|x_n\| \leqq M_1 M_2 \cdot k^{\gamma(n)-\gamma(n_0)} + \sum_{\nu=0}^{n-1} M_1 M_2 \cdot k^{\gamma(n)-\gamma(\nu+1)} \|D_\nu\| \, \|x_\nu\|$$

for all $n \geqq n_0$ and this implies that

$$k^{-\gamma(n)}\|x_n\| \leqq M_1 M_2 \cdot k^{-\gamma(n_0)} + \sum_{\nu=0}^{n-1} M_1 M_2 \cdot k^{\gamma(\nu)-\gamma(\nu+1)} \|D_\nu\| (k^{-\gamma(\nu)}\|x_\nu\|).$$

From [7, Corollary 1], which is an analogy of the Gronwall inequality, it follows that

$$k^{-\gamma(n)}\|x_n\| \leqq M_1 M_2 \cdot k^{-\gamma(n_0)} \cdot \exp\left[M_1 M_2 \left(\sum_{\nu=0}^{n-1} k^{\gamma(\nu)-\gamma(\nu+1)} \|D_\nu\|\right)\right]$$

for all $n \geqq n_0$ and thus we have

$$\|x_n\| \leqq M \cdot k^{\gamma(n)} \quad \text{for all} \quad n \geqq n_0 \tag{25}$$

where $M = M_1 M_2 \cdot k^{-\gamma(n_0)} \cdot \exp\left[M_1 M_2 \left(\sum_{\nu=0}^{\infty} k^{\gamma(\nu)-\gamma(\nu+1)} \|D_\nu\|\right)\right]$. The assumption (19) implies that $M < \infty$. Therefore the theorem follows from the inequality (25).

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СИСТЕМЫ ЛИНЕЙНЫХ РАЗНОСТНЫХ УРАВНЕНИИ ПОЧТИ ФЛОКЕ

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Резюме

В статье введен класс разностных линейных систем почти Флоке и доказано обобщение теоремы Флоке для линейных разностных систем. Использованием этой теоремы доказана одна теорема об устойчивости, которая касается линейного возмущения данной разностной системы почти Флоке.