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ON THE FUNCTION $a_p, \ p^{a_p(n)} \mid n \ (n > 1)$

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ABSTRACT. Some elementary properties of the arithmetical function $a_p(n)$ ($= \text{ord}_p n$) are studied in this paper.

Introduction

Let $p$ be a prime number. Then the function $a_p$ is defined in the following way: $a_p(1) = 0$ and if $n > 1$, then $p^{a_p(n)} \mid n$, i.e. $p^{a_p(n)} \mid n$, but $p^{a_p(n)+1} \nmid n$. In this paper we shall study some fundamental properties of the arithmetic function $a_p$.

1. Elementary properties of $a_p$ and the average order of $a_p$

The function $a_p$ is obviously completely additive, i.e.

$$a_p(n_1 \cdot n_2) = a_p(n_1) + a_p(n_2)$$

for arbitrary $n_1, n_2 \in \mathbb{N}$.

First of all we shall prove two simple results on $a_p$.

**Proposition 1.1.** Let $p$ be a fixed prime number. Then the series

$$\sum_{n=1}^{\infty} \frac{a_p(n)}{n^t}$$

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converges for \( t > 1 \) and diverges for \( t \leq 1 \).

Proof. Let \( t > 1 \). Since \( p^{a_p(n)} \mid n \ (n > 1) \), we get

\[
a_p(n) \leq \frac{\log n}{\log p} \quad (n = 1, 2, \ldots).
\]

Hence

\[
\sum_{n=1}^{\infty} \frac{a_p(n)}{n^t} \leq \frac{1}{\log p} \sum_{n=1}^{\infty} \frac{\log n}{n^t} < +\infty.
\]

Let \( t \leq 1 \). Then

\[
\sum_{n=1}^{\infty} \frac{a_p(n)}{n^t} \geq \sum_{n: a_p(n) \geq 1} \frac{a_p(n)}{n^t}.
\]

If \( a_p(n) \geq 1 \), then \( n = kp, \ k \geq 1 \). The series on the right-hand side contains each term

\[
\frac{a_p(kp)}{(kp)^t} \quad (k \geq 1).
\]

Therefore

\[
\sum_{n=1}^{\infty} \frac{a_p(n)}{n^t} \geq \sum_{k=1}^{\infty} \frac{a_p(kp)}{(kp)^t} \geq \frac{1}{p^t} \sum_{k=1}^{\infty} \frac{1}{k^t} = +\infty.
\]

\[\square\]

In the following result we shall describe the behaviour of the differences \( a_p(n + 1) - a_p(n) \) \((n = 1, 2, \ldots)\).

**Proposition 1.2.** The set

\[
(a_p(n + 1) - a_p(n))_n^t
\]

of all limit points of the sequence \( (a_p(n + 1) - a_p(n))_{n=1}^{\infty} \) contains \(+\infty\) and all integers if \( p \) is an odd prime number and it contains \(+\infty\) and all non-zero integers if \( p = 2 \).

Proof. First of all observe that, if \( n_k = p^k - 1 \ (k = 1, 2, \ldots) \), then

\[
\lim_{k \to \infty} (a_p(n_{k+1}) - a_p(n_k)) = \lim_{k \to \infty} k = +\infty.
\]

Further, let \( k \) be a fixed positive integer. We put \( n_s = sp^k - 1 \), where \( s \) runs over all positive integers which are not divisible by \( p \). Then we get
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\[ a_p(n_s + 1) - a_p(n_s) = k \]

for each \( s \). The assertion for \(-k < 0\) can be proved by choosing \( n_s = sp^k \).

If \( p > 2 \), then we put \( n_s = sp + 1 \), where \( p \nmid s \). Then \( a_p(n_s + 1) - a_p(n_s) = 0 - 0 = 0 \).

Finally, it can be easily checked that \( a_2(n + 1) - a_2(n) \neq 0 \) for every \( n \in \mathbb{N} \).

Put

\[ S(a_p, n) = \frac{a_p(1) + a_p(2) + \cdots + a_p(n)}{n} \quad (n = 1, 2, \ldots). \]

**Theorem 1.3.** We have

\[ \lim_{n \to \infty} S(a_p, n) = \frac{1}{p - 1}. \]

**Proof.** On account of the complete additivity of \( a_p \) we get

\[ S(a_p, n) = \frac{1}{n} \sum_{k=1}^{n} a_p(k) = \frac{1}{n} a_p(n!). \]

But for \( a_p(n!) \) we have

\[ a_p(n!) = \sum_{k=1}^{b_n} \left\lfloor \frac{n}{p^k} \right\rfloor, \]

where \( b_n = \left\lfloor \frac{\log n}{\log p} \right\rfloor \) (cf. [3; p. 342, Theorem 416]).

Using this fact a simple estimation yields

\[ \frac{1 - \left( \frac{1}{p} \right)^{b_n}}{1 - \frac{1}{p}} - \frac{b_n}{n} < S(a_p, n) \leq \frac{1}{p} \frac{1 - \left( \frac{1}{p} \right)^{b_n}}{1 - \frac{1}{p}}. \]

From this the assertion follows at once.
2. Level sets of the function $a_p$

For $k \geq 0$ we put

$$T_k = \{n : a_p(n) = k\} = a^{-1}_p(\{k\}).$$

**Theorem 2.1.** We have

$$d(T_k) = \lim_{x \to \infty} \frac{T_k(x)}{x} = \frac{p-1}{p^{k+1}} \quad (k = 0, 1, 2, \ldots)$$

$d(T_k)$ denotes the asymptotic density of $T_k$.

**Proof.** Let $T_k(x)$ ($x > 0$) denote the number of elements of $T_k$ which are not greater than $x$. A positive integer $n$ belongs to $T_k$ if and only if it has the form $bp^k$, where $p \nmid b$. From this we get

$$T_k(x) = \left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x}{p} \right\rfloor.$$

A simple estimation gives

$$x \frac{p-1}{p^{k+1}} - 2 \leq T_k(x) \leq x \frac{p-1}{p^{k+1}} + 2.$$

The theorem follows. \hfill \Box

**Remark 2.1.** In [2] (see also [4]), the concept of statistical convergence is introduced. A sequence $(x_n)_{n=1}^\infty$ of real numbers is said to be statistically convergent to $x \in \mathbb{R}$ (shortly: $\lim \text{stat } x_n = x$) provided that for each $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{n : |x_n - x| \geq \varepsilon\}$, $d$ being the asymptotic density. Theorem 2.1 says that

$$d(T_k) = \frac{p-1}{p^{k+1}} > 0 \quad (k = 0, 1, \ldots).$$

From this it easily follows that $(a_p(n))_{n=1}^\infty$ is not a statistically convergent sequence.
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3. Sets \( \{ n : a_p(n) \| n \} \)

In the paper [1], the sets of the form \( M_f = \{ n : f(n) \| n \} \) are investigated, where \( f \) is an arithmetical function with integer values. In [1], the density of \( M_f \) is determined for various functions \( f \) (e.g. for \( \omega(n) \) – the number of distinct primes that divide \( n \), \( s(n) \) – the digital sum of \( n \) a.s.o.). In connection with these results we prove the following theorem.

**Theorem 3.1.** For each prime number \( p \) we have

\[
d(M_{a_p}) = (p - 1) \sum_{(k,p)=1} \frac{1}{kp^{k+1}} + (p - 1) \sum_{(k,p)>1} \frac{1}{kp^{k-s_k+1}},
\]

where \( p^{s_k} \| k \).

**Proof.** Obviously we have

\[
M_{a_p} = \bigcup_{k=1}^{\infty} B_k,
\]

where
\[
B_k = \{ n : a_p(n) = k \land k \| n \} \quad (k = 1, 2, \ldots).
\]

Let \( x > 0 \). We shall try to calculate the number \( B_k(x) \) of all \( n \in B_k \) not exceeding \( x \).

For \( k \) we have two possibilities: 1. \( p \nmid k \), 2. \( p \mid k \).

1. Let \( p \nmid k \). A positive integer \( n \) belongs to \( B_k \) if and only if it has the form \( n = kp^k n_1 \), where \( p \nmid n_1 \). From this we get

\[
B_k(x) = \left[ \frac{x}{kp^k} \right] - \left[ \frac{\left[\frac{x}{kp^k}\right]}{p} \right] = c_k(x).
\]

2. Let \( p \mid k \). Then there is an \( s_k \), \( 1 \leq s_k \leq \left\lfloor \frac{\log k}{\log p} \right\rfloor \), such that \( p^{s_k} \| k \).

A positive integer belongs to \( B_k \) if and only if it has the form \( n = kp^{k-s_k} n_1 \), where \( p \nmid n_1 \).

From this we get

\[
B_k(x) = \left[ \frac{x}{kp^{k-s_k}} \right] - \left[ \frac{\left[\frac{x}{kp^{k-s_k}}\right]}{p} \right] = d_k(x).
\]
Since the sets on the right-hand side of (1) are pairwise disjoint, we get

$$M_{a_p}(x) = \sum_{(k,p)=1} c_k(x) + \sum_{(k,p)>1} d_k(x) = S_1(x) + S_2(x).$$

(2)

The summands corresponding to $k$’s greater than $m_x = \left\lfloor \frac{1}{2} \log \frac{x}{\log p} \right\rfloor$ are zero. This is evident for $S_1(x)$ and for $S_2(x)$ it can be seen as follows. If $\frac{x}{p^{k-s_k}} < 1$, then $d_k(x) = 0$. Since $s_k \leq \left[ \frac{\log k}{\log p} \right] \leq \frac{k}{2}$, we have $\frac{x}{p^{k-s_k}} \leq \frac{x}{p^{k/2}}$. Hence, if $\frac{x}{p^{k/2}} < 1$, i.e. if $k > 2 \frac{\log x}{\log p}$, then $d_k(x) = 0$. So we can suppose that $k \leq m_x$.

So we get

$$S_1(x) = \sum_{k \leq m_x, (k,p)=1} c_k(x),$$

(3)

$$S_2(x) = \sum_{k \leq m_x, (k,p)>1} d_k(x).$$

(4)

Simple estimations give

$$x \frac{p-1}{kp^{k+1}} - 2 < c_k(x) < x \frac{p-1}{kp^{k+1}} + 2,$$

$$x \frac{p-1}{kp^{k-s_k+1}} - 2 < d_k(x) < x \frac{p-1}{kp^{k-s_k+1}} + 2.$$

So we get

$$c_k(x) = x \frac{p-1}{kp^{k+1}} + O(1), \quad d_k(x) = x \frac{p-1}{kp^{k-s_k+1}} + O(1).$$

(5)

From (3), (4), (5) we obtain

$$S_1(x) = x(p-1) \sum_{k \leq m_x, (k,p)=1} \frac{1}{kp^{k+1}} + O(m_x),$$

$$S_2(x) = x(p-1) \sum_{k \leq m_x, (k,p)>1} \frac{1}{kp^{k-s_k+1}} + O(m_x).$$
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Hence, according to the definition of \( m_x \),

\[
x^{-1}M_{a_p}(x) = (p-1) \sum_{k \leq m_x, (k,p)=1} \frac{1}{kp^{k+1}} + (p-1) \sum_{k \leq m_x, (k,p)>1} \frac{1}{kp^{k-s_k+1}} + o(1).
\]

By \( x \to \infty \), we get from this

\[
d(M_{a_p}) = \lim_{x \to \infty} \frac{M_{a_p}(x)}{x} = (p-1) \sum_{(k,p)=1} \frac{1}{kp^{k+1}} + (p-1) \sum_{(k,p)>1} \frac{1}{kp^{k-s_k+1}},
\]

where \( p^s_k \parallel k \).

The following result on the behaviour of the sequence \( (d(M_{a_p}))_p \ (p \text{ runs over all primes}) \) is a simple consequence of Theorem 3.1.

**Theorem 3.2.** We have \( \lim_{p \to \infty} d(M_{a_p}) = 0 \).

**Proof.** Simple estimations yield

\[
d(M_{a_p}) \leq (p-1) \left( \frac{1}{p^2} + \sum_{k \geq 2, (k,p)=1} \frac{1}{kp^{k+1}} \right)
+ (p-1) \left( \frac{1}{pp^p} + \sum_{k>p, (k,p)>1} \frac{1}{kp^{k-s_k+1}} \right) = S_1 + S_2.
\]

Further,

\[
S_1 < \frac{p-1}{p^2} + (p-1) \int_2^\infty \frac{dt}{t^{p+1}} = \frac{p-1}{p^2} + \frac{p-1}{p^2 \log p} \to 0 \quad \text{by} \quad p \to \infty,
\]

\[
S_2 = \frac{p-1}{pp^{p+1}} + (p-1) \sum_{k>p, (k,p)>1} \frac{1}{kp^{k-s_k+1}}.
\]

But \( k - s_k = k - \left[ \frac{\log k}{\log p} \right] > \frac{k}{2} \) for \( k > p \). Thus

\[
S_2 < \frac{p-1}{pp^{p+1}} + (p-1) \int_p^\infty \frac{dt}{t^{p+1}} = \frac{p-1}{pp^{p+1}} + 2 \frac{p-1}{p^{p+1} \log p} \to 0 \quad \text{by} \quad p \to \infty.
\]
4. Density and statistical convergence of the sequence \( \left( \log p \frac{a_p(n)}{\log n} \right)_{n=2}^{\infty} \)

In [5] O. Strauch has proved the following result:

**Theorem 4.1.** The sequence \( \left( \log p \frac{a_p(n)}{\log n} \right)_{n=2}^{\infty} \) is dense in the interval \((0,1)\).

**Proof.** We shall outline the proof of O. Strauch.

Let \( n \) runs over all numbers of the form \( p^\alpha q^\beta \), where \( q \) is a fixed prime number different from \( p \) and \( \alpha, \beta \) are positive integers.

Let \( x \in (0,1) \). Then \( x = \frac{1}{1+y} \), where \( y > 0 \). Let \( \varepsilon > 0 \). The density of rational numbers in \( \mathbb{R} \) implies the existence of positive integers \( \alpha, \beta \) such that

\[
\left| y - \frac{\beta \log q}{\alpha \log p} \right| < \varepsilon.
\]

If \( n = p^\alpha q^\beta \), then we have

\[
\log \frac{a_p(n)}{\log n} = \log p \frac{\alpha}{\alpha \log p + \beta \log q} = \left( 1 + \frac{\beta \log q}{\alpha \log p} \right)^{-1}.
\]

From (5'), (5'') we get

\[
\left| x - \log p \frac{a_p(n)}{\log n} \right| = \left| \frac{1}{1+y} - \frac{1}{1 + \frac{\beta \log q}{\alpha \log p}} \right| < \left| y - \frac{\beta \log q}{\alpha \log p} \right| < \varepsilon.
\]

The theorem follows. \(\square\)

**Theorem 4.2.** We have

\[
\lim \text{ stat } \log p \frac{a_p(n)}{\log n} = 0.
\]

**Proof.** Let \( \varepsilon > 0 \), put \( A(\varepsilon) = \left\{ n > 1 : \log p \frac{a_p(n)}{\log n} \geq \varepsilon \right\} \).

Let \( \eta > 0 \). Choose an integer \( K > 0 \) such that

\[
p^{-K} < \eta.
\]
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Then there exists an $n_0$ such that for $n > n_0$ we have

$$n^\varepsilon > p^K. \quad (7)$$

Let $n \in A(\varepsilon), n > n_0$. Then, according to (6), (7), we have $\varepsilon \log n > K \log p$ and $a_p(n) \geq \frac{\varepsilon \log n}{\log p} > K$. Therefore

$$A(\varepsilon) \subseteq \{2, 3, \ldots, n_0\} \cup \{n > n_0 : p^K \mid n\} . \quad (8)$$

It follows from (8) and (6) that

$$\limsup_{n \to \infty} \frac{A(\varepsilon)(n)}{n} \leq \frac{1}{p^K} < \eta .$$

Since $\eta > 0$ is an arbitrary positive number, we get $d(A(\varepsilon)) = 0$. □

REFERENCES


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