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Comparing the number of abelian groups and of semisimple rings of a given order


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COMPARING THE NUMBER OF
ABELIAN GROUPS AND OF SEMISIMPLE
RINGS OF A GIVEN ORDER

MANFRED KÜHLEITNER

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ABSTRACT. In this article, we study the arithmetic function \( \frac{a(n)}{S(n)} \), where \( a(n) \) denotes the number of non-isomorphic abelian groups of order \( n \in \mathbb{N} \), and \( S(n) \) the number of non-isomorphic semisimple rings of the same order. We establish an asymptotic formula for the Dirichlet summatory function of \( \frac{a(n)}{S(n)} \), up to an order term which is best possible on the basis of the present knowledge about the zeros of the Riemann zeta-function.

1. Introduction

Let \( a(n) \) denote the number of non-isomorphic abelian groups of order \( n \in \mathbb{N} \). The study of the average order of this arithmetic function has been initiated by Erdős and Szekeres [1]. Subsequently, various authors contributed to the subject; for an enlightening historical survey, see Krätzel [5; ch. 7.2]. The hitherto sharpest result is due to Liu Hong-Quan [4] and reads

\[
\sum_{n \leq x} a(n) = C_1 x + C_2 x^{\frac{1}{2}} + C_3 x^{\frac{1}{3}} + O(x^{\frac{1}{159} + \varepsilon}).
\]

(Here \( C_1, C_2, C_3 \) are computable constants.) Another arithmetic function which shares some properties of the counting function of abelian groups is \( S(n) \), the number of non-isomorphic semisimple rings of a given order \( n \in \mathbb{N} \). In order to derive the product representation for the generating Dirichlet series of \( S(n) \), we note that each semisimple finite ring can be expressed as a direct sum of a finite number of simple finite rings, in a way that is unique up to permutation.
A simple finite ring $R$, however, is isomorphic to a full matrix ring $M_n(K)$ over a finite field $K$. Thus $K$ is a finite Galois field $GF(p^k)$ for some prime power $p^k$ and $\text{card}(R) = p^{kn^2}$. Therefore (see Ivic [2; p. 38]),

$$\sum_{n=1}^{\infty} \frac{S(n)}{n^s} = \prod_{k \in \mathbb{N}} \prod_{m \in \mathbb{N}} \zeta(km^2s) \quad (\text{Re}(s) > 1).$$

(For the algebraic background, c.f. Northcott [7].) Hence, the generating functions of $S(n)$ and of $a(n)$ are identical up to a factor which possesses an absolutely convergent Dirichlet series for $\text{Re}(s) > \frac{1}{4}$. Therefore, there is little hope to obtain any asymptotic result about $\sum_{n \leq x} S(n)$ which is not completely analogous (in statement and proof) to the case of $\sum_{n \leq x} a(n)$.

2. Subject and result of this paper

One way to establish a result which (in a quantitative sense) compares behaviour of the two arithmetic functions $a(n)$ and $S(n)$ is to investigate the average order of the ratio $\frac{a(n)}{S(n)}$. The aim of this note thus is a proof of an asymptotic formula for the Dirichlet summatory function of $\frac{a(n)}{S(n)}$, up to an order term, which is best possible on the basis of the present knowledge about the zeros of the Riemann zeta-function.

**Theorem.** As $x \to \infty$,

$$\sum_{n \leq x} \frac{a(n)}{S(n)} = Ax + x^{\frac{1}{2}} \sum_{k=0}^{M(x)} A_k (\log x)^{-\frac{7}{6}-k} + O \left( x^{\frac{1}{2}} \exp(-c(\log x)^{\frac{3}{5}} \log \log x)^{-\frac{1}{5}} \right),$$

where

$$M(x) = \left[ c_0 (\log x)^{\frac{2}{5}} \log \log x \right]$$

and $A_k \ll (b_*)^{k}$ with a certain positive constant $b_*$. 

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3. Preliminaries

Throughout the paper, \( b \) and \( c \) (also with a subscript or a dash) denote positive constants.

Let \( H(s) \) be any analytic function without zeros on a certain simply connected domain \( S \) of \( \mathbb{C} \) which contains the real line to the right of \( s = \sigma_0 \), where \( \sigma_0 = 1 \) or \( \frac{1}{2} \). Suppose that \( H(s) \in \mathbb{R}^+ \) for real \( s > \sigma_0 \), and let \( \alpha \in \mathbb{R} \) arbitrary. Then we define \( (H(s))^{\alpha} \) on \( S \) by

\[
(H(s))^{\alpha} = \exp \left( \alpha \left( \log(H(2)) + \int_{2}^{s} \frac{H'(z)}{H(z)} \, dz \right) \right),
\]

the path of integration being completely contained in \( S \) but otherwise arbitrary.

In our analysis, \( S \) will usually be a domain symmetric with respect to the real line, with a “cut” along \( L = \{ s \in \mathbb{R} : s \leq \sigma_0 \} \) (such that \( S \cap L = \emptyset \)). We shall join in the common abuse of terminology to think of an “upper” and a “lower edge” of \( L \cap \partial S \), on which \( (H(s))^{\alpha} \) are attributed two different values, depending on whether \( L \) is approached from above or from below.

In our first lemma, we summarize the present state of art about zero-free regions of the Riemann zeta-function.

**Lemma 1.** Define for short

\[
\psi(t) = (\log t)^{\frac{2}{3}} (\log \log t)^{\frac{1}{3}} \quad (t \geq 3)
\]

and, for positive constants \( b_1 \geq 3 \) and \( b_2 \),

\[
\lambda(t) = \begin{cases} 
1 - b_0 = 1 - \frac{b_2}{\psi(b_1)} & \text{for } |t| \leq b_1, \\
1 - \frac{b_2}{\psi(|t|)} & \text{for } |t| \geq b_1.
\end{cases}
\]

Then there exist values of \( b_1, b_2, b_3 \) such that for all \( s = \sigma + it \) with \( \sigma \geq \lambda(t) \)

it is true that

\[ \zeta(s) \neq 0 \]

and

\[ (\zeta(s))^{-1} \ll (\log(2 + |t|))^{b_3} \]

Proof. This result is contained in the textbook of Waldisz [13]; see also Mitsui [6]. The very last assertion is readily derived on classical lines: see, e.g., Prachar [10; p. 71].

Our next auxiliary result provides an asymptotic expansion for a certain contour integral, which is essential in the type of problem under consideration.
LEMMA 2. Let $H(s)$ be a holomorphic function on the disk
\[ \{ s \in \mathbb{C} : |s - 1| < 2b_0 \} \quad (b_0 > 0 \text{ fixed}), \]
and let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. Let $C_0$ denote the circle $|s - 1| = b_0$, with positive orientation, starting and ending at $1 - b_0$. For a large real variable $w$, it follows that
\[
\frac{1}{2\pi i} \int_{C_0} (s - 1)^{-\alpha} H(s) w^s \, ds
\]
\[
= w \sum_{k=0}^{M(w)} \frac{\beta_k}{\Gamma(\alpha - k)} (\log w)^{\alpha - k - 1} + O\left( w \exp\left( -c''(\log w)^{\frac{3}{5}} (\log \log w)^{-\frac{1}{2}} \right) \right)
\]
\[
( c'' > 0 ),
\]
where $M(w)$ is defined as in (2.1), $\beta_k$ are the coefficients in the Taylor expansion of $H(s)$ at $s = 1$. By Cauchy’s estimates and standard results on the Gamma-function, they satisfy
\[
\frac{\beta_k}{\Gamma(\alpha - k)} \ll b_0^{-k} \Gamma(1 - \alpha + k) \max_{|s - 1| = b_0} |H(s)| \ll (b_0^{-1} k)^k \max_{|s - 1| = b_0} |H(s)|.
\]
The constant $c''$ and the $\ll$-constants depend only on $\alpha$.

Proof. This result is derived (in a special context) in [9; formula (3.5) and sequel].

4. Proof of the Theorem

Our analysis is based on the ideas of Selberg [12], De Koninck and Ivić [3], and Nowak [8]. We note that $S(n)$ is multiplicative and prime-independent, and that $a(p^k) = S(p^k)$ for $k \leq 3$, while $a(p^4) = 5$, $S(p^4) = 6$. Consequently, for $\text{Re}(s) > 1$, we have
\[
Z(s) = \sum_{n=1}^{\infty} \frac{a(n)}{S(n)} n^{-s}
\]
\[
= \zeta(s) \prod_{p \in \mathcal{P}} (1 - p^{-s}) \left( 1 + p^{-s} + p^{-2s} + p^{-3s} + \frac{5}{6} p^{-4s} + \sum_{k=5}^{\infty} \frac{a(p^k)}{S(p^k)} p^{-ks} \right)
\]
\[
= \zeta(s) (\zeta(4s))^{-\frac{1}{6}} U(s),
\]
where $U(s)$ has a Dirichlet series absolutely convergent for $\text{Re}(s) > \frac{1}{5}$. We define
\[
F(s) = (\zeta(4s))^{-\frac{1}{6}} U(s) = \sum_{n=1}^{\infty} g(n) n^{-s},
\]
\[
(4.1)
\]
\[
(4.2)
\]
where the last equality holds for \( \text{Re}(s) > \frac{1}{4} \). From this, we infer
\[
\frac{a(n)}{S(n)} = \sum_{m|n} g(m) \tag{4.3}
\]

The idea behind this step is, that we cannot apply complex integration directly to \( \sum \frac{a(n)}{S(n)} \), but only to \( \sum g(n) \), and that we have to combine this technique with an elementary convolution argument.

**Lemma 3.** For \( u \to \infty \),
\[
G(u) = \sum_{n \leq u} g(n) = I(u) + R(u),
\]
where
\[
I(u) = \frac{1}{2\pi i} \int_{C_0} F\left(\frac{s}{4}\right) u^{\frac{s}{2}} \frac{ds}{s},
\]
and
\[
R(u) \ll u^{\frac{1}{2}} \delta_1(u)
\]
for some \( c_1 > 0 \). \( C_0 \) is the circle \( |s - 1| = b_0 \) (\( b_0 \) from Lemma 1), with positive orientation, starting and ending at \( 1 - b_0 \). Here and throughout the sequel, we write for short
\[
\delta_k(u) = \exp\left(-c_k (\log(3 + u))^\frac{3}{8} (\log \log(3 + u))^{-\frac{1}{8}}\right)
\]
for \( u \geq 0 \) and suitable positive constants \( c_k \).

**Proof.** By a version of Perron’s formula,
\[
G_1(u) = \int_{1}^{u} G(w^4) \, dw = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F\left(\frac{s}{4}\right) \frac{u^{s+1}}{s(s + 1)} \, ds.
\]
We replace the line of integration \( \text{Re}(w) = 2 \) by the path \( C = C_1 \cup C_0 \cup C_2 \), where \( C_1 \) denote the path from \( 1 - i\infty \) to \( 1 - b_0 \), \( C_2 \) the path from \( 1 - b_0 \) to \( 1 + i\infty \), both along \( \sigma = \lambda(t) \). (\( b_0 \) and \( \lambda(t) \) are defined as in Section 3). Defining
\[
T = \frac{1}{\delta_2(u)}
\]
(with suitable \( c_2 > 0 \)), a short calculation gives that the contribution from \( C_1 \) and \( C_2 \) is \( \ll u^2 \delta_3(u) \), hence
\[
G_1(u) = I_1(u) + O\left(u^2 \delta_3(u)\right), \tag{4.4}
\]
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where
\[ I_1(u) = \frac{1}{2\pi i} \int_{C_0} F\left(\frac{s}{4}\right) \frac{u^{s+1}}{s(s+1)} \, ds. \] (4.5)

Employing a technique due to Rieger [11], we put, for \( w \geq 1 \),
\[ f(w) = G(w^4) - I(w^4) + (I(1) - G(1)). \]

Now \( f(w) \) fulfills the necessary requirements of [11; Hilfssatz 2] (Rieger) since (4.4) implies that
\[ \int_1^u f(w) \, dw \ll u^2 \delta(u). \]

In order to estimate the difference \( f(w_1) - f(w_2) \) for \( w_1 > w_2 \), we see from (4.3) that \( g(n) \) is multiplicative and
\[ g(p^k) = \frac{a(p^k)}{S(p^k)} - \frac{a(p^{k-1})}{S(p^{k-1})} \]
for every prime \( p \) and every integer \( k \). From this, it is clear that \( g(p) = g(p^2) = g(p^3) = 0 \) for every prime \( p \). Furthermore, \( |g(n)| \leq 1 \) for every \( n \in \mathbb{N} \), since \( a(n) \leq S(n) \) is immediate from the respective generating functions. Consequently, if \( Q(v) \) denotes the number of 4-full integers \( \leq v \), we obtain
\[ |G(w_1^4) - G(w_2^4)| \leq Q(w_1^4) - Q(w_2^4) \ll w_1 - w_2 + w_1^3, \]
where the last estimate is an immediate consequence of the asymptotic formula for \( Q(v) \) (see Kratzel [5; ch. 7]). Furthermore,
\[ I(w_1^4) - I(w_2^4) = \int_{w_1}^{w_2} \left( \frac{1}{2\pi i} \int_{C_0} F\left(\frac{s}{4}\right)u^{s-1} \, ds \right) du \ll w_1 - w_2. \]

This follows by replacing \( C_0 \) by \( C_0^*(u) \) which we define as the boundary of
\[ \left\{ s \in \mathbb{C} : |s - 1| \leq b_0, \ Re(s) \leq 1 + \frac{1}{\log(2u)} \right\} \]
with positive orientation, starting and ending at \( 1 - b_0 \). [11; Hilfssatz 2] (Rieger) implies therefore that
\[ G(w^4) = I(w^4) + O(w\delta_4(w)). \]

Putting \( u = w^4 \), we complete the proof of Lemma 3.

We now define
\[ y = y(x) = x\delta_5(x), \]

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with a positive constant $c_5$ remaining at our disposition. We recall (4.3) to conclude that
\[ \sum_{n \leq x} \frac{a(n)}{S(n)} = \sum_{m \leq y} \frac{g(m)}{m} \left\lfloor \frac{x}{m} \right\rfloor + \sum_{k \leq \frac{x}{y}} G \left( \frac{x}{y} \right) - G(y) \left\lfloor \frac{x}{y} \right\rfloor. \]

Writing \{\cdot\} for the fractional part, we see that
\[
\sum_{m \leq y} g(m) \left\lfloor \frac{x}{m} \right\rfloor = \sum_{m \leq y} g(m) \frac{x}{m} - \sum_{m \leq y} g(m) \{\frac{x}{m}\}. 
\]

We note that
\[
\left| \sum_{m \leq y} g(m) \{\frac{x}{m}\} \right| \leq Q(y) \ll y^{\frac{1}{4}}. 
\]

Furthermore,
\[
\sum_{m \leq y} g(m) \frac{x}{m} = x \sum_{m=1}^{\infty} \frac{g(m)}{m} - x \sum_{m > y} g(m) \frac{x}{m}.
\]

The second part yields
\[
\sum_{m > y} \frac{g(m)}{m} = \int_{y}^{\infty} \frac{1}{u} \, dG(u)
\]
\[= \int_{y}^{\infty} \frac{1}{u} I'(u) \, du + \int_{y}^{\infty} \frac{1}{u} \, dR(u)
\]
\[= \int_{y}^{\infty} \frac{1}{u} I'(u) \, du - \frac{1}{y} R(y) + \int_{y}^{\infty} \frac{1}{u^2} R(u) \, du
\]
\[= \int_{y}^{\infty} \frac{1}{u} I'(u) \, du + O(y^{-\frac{3}{2}} \delta_1(y)).
\]

Thus we obtain
\[
\sum_{n \leq x} \frac{a(n)}{S(n)} = Ax - x \int_{y}^{\infty} \frac{1}{u} I'(u) \, du + \sum_{k \leq \frac{x}{y}} G \left( \frac{x}{k} \right) - G(y) \left\lfloor \frac{x}{y} \right\rfloor + O(x^{\frac{1}{4}} \delta_6(x))
\]

with
\[A = \sum_{m=1}^{\infty} \frac{g(m)}{m}
\]

by a suitable choice of $c_5$ and $c_6$. (Note that $A > 0$ by the Euler product representation.)
In view of Lemma 3, one has
\[ \sum_{k \leq \frac{x}{y}} R\left( \frac{x}{k} \right) \ll x^{\frac{3}{4}} \delta_{\gamma}(x) \]
and
\[
\sum_{k \leq \frac{x}{y}} I\left( \frac{x}{k} \right) = \int_{\frac{1}{2}}^{1} I\left( \frac{x}{u} \right) \, du
\]
\[ = I(y) \left[ \frac{x}{y} \right] + x \int_{1}^{\frac{x}{y}} \frac{1}{u} \, I'\left( \frac{x}{u} \right) \, du \]
\[ = I(y) \left[ \frac{x}{y} \right] + x \int_{y}^{x} I'(v) \frac{dv}{v} - x \int_{1}^{\frac{x}{y}} I'\left( \frac{x}{u} \right) \frac{\{u\}}{u^2} \, du \]
by the substitution \( v = \frac{x}{u} \) in the last but one integral. Using this, we arrive at
\[ \sum_{n \leq x} \frac{a(n)}{S(n)} = Ax - x \int_{x}^{\infty} I'(u) \, du - x \int_{1}^{\frac{x}{y}} I'\left( \frac{x}{u} \right) \frac{\{u\}}{u^2} \, du + O(x^{\frac{1}{4}} \delta_{\gamma}(x)) , \]
where
\[ I'(u) = \frac{1}{2\pi i} \int_{\frac{1}{4} C_0} F(s)u^{s-1} \, ds . \]
It remains to evaluate these two integrals. We consider first
\[
\int_{x}^{\infty} I'(u) \, du = \int_{x}^{\infty} \left( \frac{1}{2\pi i} \int_{\frac{1}{4} C_0} F(s)u^{s-1} \, ds \right) \frac{du}{u} \]
\[ = \frac{1}{2\pi i} \int_{\frac{1}{4} C_0} F(s) \left( \int_{x}^{\infty} u^{s-2} \, du \right) \, ds \]
\[ = -\frac{1}{2\pi i} \int_{\frac{1}{4} C_0} \frac{F(s)}{s-1} x^{s-1} \, ds . \]
Similarly,
\[
\int_{1}^{\frac{x}{y}} I'\left( \frac{x}{u} \right) \frac{\{u\}}{u^2} \, du = \frac{1}{2\pi i} \int_{\frac{1}{4} C_0} F(s)x^{s-1} \left( \int_{1}^{\frac{x}{u}} \frac{\{u\}}{u^{s+1}} \, du \right) \, ds . \]
In view of the well-known identity
\[ \int_{1}^{\infty} \{u\} u^{-s-1} \, du = \frac{1}{s-1} - \frac{\zeta(s)}{s} \]
(valid for \( \text{Re}(s) > 0 \)), we obtain
\[
\sum_{n \leq x} \frac{a(n)}{S(n)} = A x + I^*(x) + \frac{1}{2\pi i} \int_{\frac{1}{4}C_0} F(s) \left( \int_{\frac{1}{2}}^{\infty} \frac{\{u\}}{u^{s+1}} x^u \, du \right) \, ds + O(x^{\frac{3}{4}} \delta_{0}(x)),
\]
where
\[ I^*(x) = \frac{1}{2\pi i} \int_{\frac{1}{4}C_0} \frac{\zeta(s) F(s) x^s}{s} \, ds. \]  

Our penultimate step is thus to estimate the remaining integral in (4.6).
\[
\frac{1}{2\pi i} \int_{\frac{1}{4}C_0} F(s) \left( \int_{\frac{1}{2}}^{\infty} \frac{\{u\}}{u^{s+1}} x^u \, du \right) \, ds \ll y^{\frac{1}{4}} \ll x^{\frac{1}{4}} \delta_{0}(x).
\]

This follows by replacing \( \frac{1}{4}C_0 \) by \( \frac{1}{4} C_0^*(x) \) defined as in Lemma 3, and by the fact that \( F(s) \) is bounded on \( \frac{1}{4} C_0^*(x) \).

Applying Lemma 2, we obtain for the integral \( I^*(x) \) (defined in (4.7)) the asymptotic expansion (as \( x \to \infty \))
\[
I^*(x) = x^{\frac{1}{4}} \sum_{k=0}^{M(x)} A_k (\log x)^{-\frac{k}{6}-k} + O(x^{\frac{1}{4}} \delta_{10}(x)).
\]

This completes the proof of our Theorem. \( \square \)

Remark. By the same proof, we can generalize this result to an arbitrary \( r \)-th power moment of \( \frac{a(n)}{S(n)} \) \( (r \) any fixed positive real number). Instead of (4.1), we now have (for \( \text{Re}(s) > 1 \))
\[
Z_r(s) = \sum_{n=1}^{\infty} \left( \frac{a(n)}{S(n)} \right)^r n^{-s} = \zeta(s) \prod_{p \in \mathbb{P}} (1 - p^{-s}) \left( 1 + p^{-s} + p^{-2s} + p^{-3s} + \left( \frac{5}{6} \right)^r p^{-4s} + \sum_{k=5}^{\infty} \left( \frac{a(p^k)}{S(p^k)} \right)^r p^{-ks} \right) = \zeta(s) (\zeta(4s))^{-\alpha} U_r(s),
\]
where
\[ \alpha = 1 - \left( \frac{5}{6} \right)^r, \]
and \( U_r(s) \) has a Dirichlet series absolutely convergent for \( \text{Re}(s) > \frac{1}{5} \). Repeating our argument, we readily obtain
\[
\sum_{n \leq x} \left( \frac{a(n)}{S(n)} \right)^r \sum_{k=0}^{M(x)} A_k^{(r)}(\log x)^{-\alpha - 1 - k} + O \left( x^{\frac{1}{4}} \exp(-c(\log x)^{\frac{3}{2}}(\log \log x)^{-\frac{1}{3}}) \right)
\]
with \( M(x) \) given as in (2.1).

REFERENCES


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