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COMPARING THE NUMBER OF ABELIAN GROUPS AND OF SEMISIMPLE RINGS OF A GIVEN ORDER[1]

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ABSTRACT. In this article, we study the arithmetic function $\frac{a(n)}{S(n)}$, where $a(n)$ denotes the number of non-isomorphic abelian groups of order $n \in \mathbb{N}$, and $S(n)$ the number of non-isomorphic semisimple rings of the same order. We establish an asymptotic formula for the Dirichlet summatory function of $\frac{a(n)}{S(n)}$, up to an order term which is best possible on the basis of the present knowledge about the zeros of the Riemann zeta-function.

1. Introduction

Let $a(n)$ denote the number of non-isomorphic abelian groups of order $n \in \mathbb{N}$. The study of the average order of this arithmetic function has been initiated by Erdös and Szekeres [1]. Subsequently, various authors contributed to the subject; for an enlightening historical survey, see Krätzel [5; ch. 7.2]. The hitherto sharpest result is due to Liu Hong-Quan [4] and reads

$$\sum_{n \le x} a(n) = C_1 x + C_2 x^{\frac{1}{2}} + C_3 x^{\frac{1}{3}} + O\big(x^{\frac{40}{159}+\varepsilon}\big)\,.$$

(Here C_1, C_2, C_3 are computable constants.) Another arithmetic function which shares some properties of the counting function of abelian groups is $S(n)$, the number of non-isomorphic semisimple rings of a given order $n \in \mathbb{N}$. In order to derive the product representation for the generating Dirichlet series of $S(n)$, we note that each semisimple finite ring can be expressed as a direct sum of a finite number of simple finite rings, in a way that is unique up to permutation.

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A simple finite ring R, however, is isomorphic to a full matrix ring $M_n(K)$ over a finite field K. Thus K is a finite Galois field $GF(p^k)$ for some prime power p^k and $\mathrm{card}(R) = p^{kn^2}$. Therefore (see I v i ć [2; p. 38]),

$$\sum_{n=1}^{\infty} \frac{S(n)}{n^s} = \prod_{k\in\mathbb{N}} \prod_{m\in\mathbb{N}} \zeta(km^2 s) \qquad (\,\mathrm{Re}(s) > 1\,).$$

(For the algebraic background, c.f. N o r t h c o t t [7].) Hence, the generating functions of $S(n)$ and of $a(n)$ are identical up to a factor which possesses an absolutely convergent Dirichlet series for $\mathrm{Re}(s) > \frac{1}{4}$. Therefore, there is little hope to obtain any asymptotic result about $\sum_{n\le x} S(n)$ which is not completely analogous (in statement and proof) to the case of $\sum_{n\le x} a(n)$.

2. Subject and result of this paper

One way to establish a result which (in a quantitative sense) compares behaviour of the two arithmetic functions $a(n)$ and $S(n)$ is to investigate the average order of the ratio $\frac{a(n)}{S(n)}$. The aim of this note thus is a proof of an asymptotic formula for the Dirichlet summatory function of $\frac{a(n)}{S(n)}$, up to an order term, which is best possible on the basis of the present knowledge about the zeros of the Riemann zeta-function.

THEOREM. *As $x \to \infty$,*

$$\sum_{n\le x} \frac{a(n)}{S(n)} = Ax + x^{\frac{1}{4}} \sum_{k=0}^{M(x)} A_k (\log x)^{-\frac{7}{6}-k} + O\Big(x^{\frac{1}{4}} \exp\big(-c(\log x)^{\frac{3}{5}} (\log\log x)^{-\frac{1}{5}}\big)\Big)\,,$$

where

$$M(x) = \big[c_0 (\log x)^{\frac{3}{5}} (\log\log x)^{-\frac{6}{5}}\big] \tag{2.1}$$

and $A_k \ll (b_* k)^k$ with a certain positive constant b_*.

3. Preliminaries

Throughout the paper, b and c (also with a subscript or a dash) denote positive constants.

Let $H(s)$ be any analytic function without zeros on a certain simply connected domain S of $\mathbb{C}$ which contains the real line to the right of $s = \sigma_0$, where $\sigma_0 = 1$ or $\frac{1}{2}$. Suppose that $H(s) \in \mathbb{R}^+$ for real $s > \sigma_0$, and let $\alpha \in \mathbb{R}$ arbitrary. Then we define $\big(H(s)\big)^\alpha$ on S by

$$\big(H(s)\big)^\alpha = \exp\left(\alpha\left(\log\big(H(2)\big) + \int\limits_2^s \frac{H'(z)}{H(z)}\,\mathrm{d}z\right)\right),$$

the path of integration being completely contained in S but otherwise arbitrary.

In our analysis, S will usually be a domain symmetric with respect to the real line, with a "cut" along $L = \{s \in \mathbb{R} :\ s \le \sigma_0\}$ (such that $S \cap L = \emptyset$). We shall join in the common abuse of terminology to think of an "upper" and a "lower edge" of $L \cap \partial S$, on which $\big(H(s)\big)^\alpha$ are attributed two different values, depending on whether L is approached from above or from below.

In our first lemma, we summarize the present state of art about zero-free regions of the Riemann zeta-function.

LEMMA 1. *Define for short*

$$\psi(t) = (\log t)^{\frac{2}{3}}(\log\log t)^{\frac{1}{3}} \qquad (t \ge 3)$$

and, for positive constants $b_1 \ge 3$ *and* b_2,

$$\lambda(t) = \begin{cases} 1 - b_0 = 1 - \dfrac{b_2}{\psi(b_1)} & \textit{for } |t| \le b_1\,, \\[2ex] 1 - \dfrac{b_2}{\psi\big(|t|\big)} & \textit{for } |t| \ge b_1\,. \end{cases}$$

Then there exist values of b_1, b_2, b_3 *such that for all* $s = \sigma + \mathrm{i}t$ *with*

$$\sigma \ge \lambda(t)$$

it is true that

$$\zeta(s) \ne 0$$

and

$$\big(\zeta(s)\big)^{-1} \ll \big(\log\big(2 + |t|\big)\big)^{b_3}$$

Proof. This result is contained in the textbook of Walfisz [13]; see also Mitsui [6]. The very last assertion is readily derived on classical lines: see, e.g., Prachar [10; p. 71]. □

Our next auxiliary result provides an asymptotic expansion for a certain contour integral, which is essential in the type of problem under consideration.

LEMMA 2. *Let $H(s)$ be a holomorphic function on the disk*

$$\{s \in \mathbb{C} : \ |s-1| < 2b_0\} \qquad (b_0 > 0 \text{ fixed}),$$

and let $\alpha \in \mathbb{R}\backslash\mathbb{Z}$. Let C_0 denote the circle $|s-1| = b_0$, with positive orientation, starting and ending at $1 - b_0$. For a large real variable w, it follows that

$$\frac{1}{2\pi \mathrm{i}} \int_{C_0} (s-1)^{-\alpha} H(s) w^s \, \mathrm{d}s$$

$$= w \sum_{k=0}^{M(w)} \frac{\beta_k}{\Gamma(\alpha - k)} (\log w)^{\alpha-k-1} + O\Big(w \exp\big(-c''(\log w)^{\frac{3}{5}} (\log\log w)^{-\frac{1}{5}}\big)\Big)$$

$$(c'' > 0),$$

where $M(w)$ is defined as in (2.1), β_k are the coefficients in the Taylor expansion of $H(s)$ at $s = 1$. By Cauchy's estimates and standard results on the Gamma-function, they satisfy

$$\frac{\beta_k}{\Gamma(\alpha - k)} \ll b_0^{-k} \Gamma(1 - \alpha + k) \max_{|s-1|=b_0} |H(s)| \ll (b_0^{-1} k)^k \max_{|s-1|=b_0} |H(s)| .$$

The constant c'' and the $\ll$-constants depend only on α.

Proof. This result is derived (in a special context) in [9; formula (3.5) and sequel]. □

4. Proof of the Theorem

Our analysis is based on the ideas of Selberg [12], De Koninck and Ivić [3], and Nowak [8]. We note that $S(n)$ is multiplicative and prime-independent, and that $a(p^k) = S(p^k)$ for $k \le 3$, while $a(p^4) = 5$, $S(p^4) = 6$. Consequently, for $\mathrm{Re}(s) > 1$, we have

$$\begin{aligned} Z(s) &= \sum_{n=1}^{\infty} \frac{a(n)}{S(n)} n^{-s} \\ &= \zeta(s) \prod_{p \in \mathbb{P}} (1 - p^{-s}) \Big(1 + p^{-s} + p^{-2s} + p^{-3s} + \frac{5}{6} p^{-4s} + \sum_{k=5}^{\infty} \frac{a(p^k)}{S(p^k)} p^{-ks}\Big) \\ &= \zeta(s) \big(\zeta(4s)\big)^{-\frac{1}{6}} U(s), \end{aligned} \tag{4.1}$$

where $U(s)$ has a Dirichlet series absolutely convergent for $\mathrm{Re}(s) > \frac{1}{5}$. We define

$$F(s) = \big(\zeta(4s)\big)^{-\frac{1}{6}} U(s) = \sum_{n=1}^{\infty} g(n) n^{-s}, \tag{4.2}$$

where the last equality holds for $\mathrm{Re}(s) > \frac{1}{4}$. From this, we infer

$$\frac{a(n)}{S(n)} = \sum_{m|n} g(m)\,. \tag{4.3}$$

The idea behind this step is, that we cannot apply complex integration directly to $\sum \frac{a(n)}{S(n)}$, but only to $\sum g(n)$, and that we have to combine this technique with an elementary convolution argument.

Lemma 3. *For $u \to \infty$,*

$$G(u) = \sum_{n \le u} g(n) = I(u) + R(u)\,,$$

where

$$I(u) = \frac{1}{2\pi \mathrm{i}} \int_{C_0} F\Big(\frac{s}{4}\Big) u^{\frac{s}{4}} \,\frac{\mathrm{d}s}{s}\,,$$

and

$$R(u) \ll u^{\frac{1}{4}} \delta_1(u)$$

for some $c_1 > 0$. C_0 is the circle $|s-1| = b_0$ (b_0 from Lemma 1), with positive orientation, starting and ending at $1-b_0$. Here and throughout the sequel, we write for short

$$\delta_k(u) = \exp\Big(-c_k \big(\log(3+u)\big)^{\frac{3}{5}} \big(\log\log(3+u)\big)^{-\frac{1}{5}}\Big)$$

for $u \ge 0$ and suitable positive constants c_k.

Proof. By a version of Perron's formula,

$$G_1(u) = \int_1^u G(w^4)\,\mathrm{d}w = \frac{1}{2\pi \mathrm{i}} \int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} F\Big(\frac{s}{4}\Big) \frac{u^{s+1}}{s(s+1)}\,\mathrm{d}s\,.$$

We replace the line of integration $\mathrm{Re}(w) = 2$ by the path $C = C_1 \cup C_0 \cup C_2$, where C_1 denote the path from $1-\mathrm{i}\infty$ to $1-b_0$, C_2 the path from $1-b_0$ to $1+\mathrm{i}\infty$, both along $\sigma = \lambda(t)$. (b_0 and $\lambda(t)$ are defined as in Section 3). Defining

$$T = \frac{1}{\delta_2(u)}$$

(with suitable $c_2 > 0$), a short calculation gives that the contribution from C_1 and C_2 is $\ll u^2 \delta_3(u)$, hence

$$G_1(u) = I_1(u) + O\big(u^2 \delta_3(u)\big)\,, \tag{4.4}$$

where

$$I_1(u) = \frac{1}{2\pi\,\mathrm{i}} \int\limits_{C_0} F\Big(\frac{s}{4}\Big) \frac{u^{s+1}}{s(s+1)}\,\mathrm{d}s\,. \tag{4.5}$$

Employing a technique due to R i e g e r [11], we put, for $w \geq 1$,

$$f(w) = G(w^4) - I(w^4) + \big(I(1) - G(1)\big)\,.$$

Now $f(w)$ fulfils the necessary requirements of [11; Hilfssatz 2] (R i e g e r) since (4.4) implies that

$$\int\limits_1^u f(w)\,\mathrm{d}w \ll u^2\delta_3(u)\,.$$

In order to estimate the difference $f(w_1) - f(w_2)$ for $w_1 > w_2$, we see from (4.3) that $g(n)$ is multiplicative and

$$g(p^k) = \frac{a(p^k)}{S(p^k)} - \frac{a(p^{k-1})}{S(p^{k-1})}$$

for every prime p and every integer k. From this, it is clear that $g(p) = g(p^2) = g(p^3) = 0$ for every prime p. Furthermore, $|g(n)| \leq 1$ for every $n \in \mathbb{N}$, since $a(n) \leq S(n)$ is immediate from the respective generating functions. Consequently, if $Q(v)$ denotes the number of 4-*full* integers $\leq v$, we obtain

$$|G({w_1}^4) - G({w_2}^4)| \leq Q({w_1}^4) - Q({w_2}^4) \ll w_1 - w_2 + w_1^{\frac{4}{5}}\,,$$

where the last estimate is an immediate consequence of the asymptotic formula for $Q(v)$ (see K r ä t z e l [5; ch. 7]). Furthermore,

$$I(w_1^4) - I(w_2^4) = \int\limits_{w_1}^{w_2} \Bigg(\frac{1}{2\pi\,\mathrm{i}} \int\limits_{C_0} F\Big(\frac{s}{4}\Big) u^{s-1}\,\mathrm{d}s \Bigg)\,\mathrm{d}u \ll w_1 - w_2\,.$$

This follows by replacing C_0 by $C_0^*(u)$ which we define as the boundary of

$$\Big\{ s \in \mathbb{C}:\ |s-1| \leq b_0\,,\ \mathrm{Re}(s) \leq 1 + \frac{1}{\log(2u)} \Big\}$$

with positive orientation, starting and ending at $1 - b_0$. [11; Hilfssatz 2] (R i e - g e r) implies therefore that

$$G(w^4) = I(w^4) + O\big(w\delta_4(w)\big).$$

Putting $u = w^4$, we complete the proof of Lemma 3.

We now define

$$y = y(x) = x\delta_5(x)\,,$$

with a positive constant c_5 remaining at our disposition. We recall (4.3) to conclude that

$$\sum_{n\le x}\frac{a(n)}{S(n)}=\sum_{m\le y}g(m)\Big[\frac{x}{m}\Big]+\sum_{k\le\frac{x}{y}}G\Big(\frac{x}{y}\Big)-G(y)\Big[\frac{x}{y}\Big].$$

Writing $\{\cdot\}$ for the fractional part, we see that

$$\sum_{m\le y}g(m)\Big[\frac{x}{m}\Big]=\sum_{m\le y}g(m)\frac{x}{m}-\sum_{m\le y}g(m)\Big\{\frac{x}{m}\Big\}.$$

We note that

$$\Big|\sum_{m\le y}g(m)\Big\{\frac{x}{m}\Big\}\Big|\le Q(y)\ll y^{\frac14}.$$

Furthermore,

$$\sum_{m\le y}g(m)\frac{x}{m}=x\sum_{m=1}^{\infty}\frac{g(m)}{m}-x\sum_{m>y}\frac{g(m)}{m}.$$

The second part yields

$$\begin{aligned}\sum_{m>y}\frac{g(m)}{m}&=\int_y^\infty\frac1u\,\mathrm{d}G(u)\\&=\int_y^\infty\frac1u I'(u)\,\mathrm{d}u+\int_y^\infty\frac1u\,\mathrm{d}R(u)\\&=\int_y^\infty\frac1u I'(u)\,\mathrm{d}u-\frac1y R(y)+\int_y^\infty\frac{1}{u^2}R(u)\,\mathrm{d}u\\&=\int_y^\infty\frac1u I'(u)\,\mathrm{d}u+O\big(y^{-\frac34}\delta_1(y)\big).\end{aligned}$$

Thus we obtain

$$\sum_{n\le x}\frac{a(n)}{S(n)}=Ax-x\int_y^\infty\frac1u I'(u)\,\mathrm{d}u+\sum_{k\le\frac{x}{y}}G\Big(\frac{x}{k}\Big)-G(y)\Big[\frac{x}{y}\Big]+O\big(x^{\frac14}\delta_6(x)\big)$$

with

$$A=\sum_{m=1}^{\infty}\frac{g(m)}{m}$$

by a suitable choice of c_5 and c_6. (Note that $A>0$ by the Euler product representation.)

In view of Lemma 3, one has

$$\sum_{k\le\frac{x}{y}} R\Big(\frac{x}{k}\Big) \ll x^{\frac14}\delta_7(x)$$

and

$$\begin{aligned}\sum_{k\le\frac{x}{y}} I\Big(\frac{x}{k}\Big) &= \int_{\frac12}^{\frac{x}{y}} I\Big(\frac{x}{u}\Big)\,\mathrm{d}[u] \\ &= I(y)\Big[\frac{x}{y}\Big] + x\int_1^{\frac{x}{y}} \frac{[u]}{u^2} I'\Big(\frac{x}{u}\Big)\,\mathrm{d}u \\ &= I(y)\Big[\frac{x}{y}\Big] + x\int_y^{x} I'(v)\,\frac{\mathrm{d}v}{v} - x\int_1^{\frac{x}{y}} I'\Big(\frac{x}{u}\Big)\frac{\{u\}}{u^2}\,\mathrm{d}u\end{aligned}$$

by the substitution $v = \dfrac{x}{u}$ in the last but one integral. Using this, we arrive at

$$\sum_{n\le x}\frac{a(n)}{S(n)} = Ax - x\int_x^{\infty} I'(u)\frac{\mathrm{d}u}{u} - x\int_1^{\frac{x}{y}} I'\Big(\frac{x}{u}\Big)\frac{\{u\}}{u^2}\,\mathrm{d}u + O\big(x^{\frac14}\delta_8(x)\big)\,,$$

where

$$I'(u) = \frac{1}{2\pi\,\mathrm{i}}\int_{\frac14 C_0} F(s)u^{s-1}\,\mathrm{d}s\,.$$

It remains to evaluate these two integrals. We consider first

$$\begin{aligned}\int_x^{\infty} I'(u)\,\frac{\mathrm{d}u}{u} &= \int_x^{\infty}\bigg(\frac{1}{2\pi\,\mathrm{i}}\int_{\frac14 C_0} F(s)u^{s-1}\,\mathrm{d}s\bigg)\frac{\mathrm{d}u}{u} \\ &= \frac{1}{2\pi\,\mathrm{i}}\int_{\frac14 C_0} F(s)\bigg(\int_x^{\infty} u^{s-2}\,\mathrm{d}u\bigg)\,\mathrm{d}s \\ &= -\frac{1}{2\pi\,\mathrm{i}}\int_{\frac14 C_0}\frac{F(s)}{s-1}\,x^{s-1}\,\mathrm{d}s\,.\end{aligned}$$

Similarly,

$$\int_1^{\frac{x}{y}} I'\Big(\frac{x}{u}\Big)\frac{\{u\}}{u^2}\,\mathrm{d}u = \frac{1}{2\pi\,\mathrm{i}}\int_{\frac14 C_0} F(s)x^{s-1}\bigg(\int_1^{\frac{x}{y}}\frac{\{u\}}{u^{s+1}}\,\mathrm{d}u\bigg)\,\mathrm{d}s\,.$$

In view of the well-known identity

$$\int_1^{\infty} \{u\} u^{-s-1}\, \mathrm{d}u = \frac{1}{s-1} - \frac{\zeta(s)}{s}$$

(valid for $\operatorname{Re}(s) > 0$), we obtain

$$\sum_{n \le x} \frac{a(n)}{S(n)} = Ax + I^*(x) + \frac{1}{2\pi \mathrm{i}} \int_{\frac{1}{4}C_0} F(s) \left(\int_{\frac{x}{y}}^{\infty} \frac{\{u\}}{u^{s+1}} x^s\, \mathrm{d}u \right) \mathrm{d}s + O\big(x^{\frac{1}{4}} \delta_8(x)\big)\,, \tag{4.6}$$

where

$$I^*(x) = \frac{1}{2\pi \mathrm{i}} \int_{\frac{1}{4}C_0} \zeta(s) F(s) x^s \, \frac{\mathrm{d}s}{s}\,. \tag{4.7}$$

Our penultimate step is thus to estimate the remaining integral in (4.6).

$$\frac{1}{2\pi \mathrm{i}} \int_{\frac{1}{4}C_0} F(s) \left(\int_{\frac{x}{y}}^{\infty} \frac{\{u\}}{u^{s+1}} x^s\, \mathrm{d}u \right) \mathrm{d}s \ll y^{\frac{1}{4}} \ll x^{\frac{1}{4}} \delta_9(x)\,.$$

This follows by replacing $\frac{1}{4} C_0$ by $\frac{1}{4} {C_0}^*(x)$ defined as in Lemma 3, and by the fact that $F(s)$ is bounded on $\frac{1}{4} {C_0}^*(x)$.

Applying Lemma 2, we obtain for the integral $I^*(x)$ (defined in (4.7)) the asymptotic expansion (as $x \to \infty$)

$$I^*(x) = x^{\frac{1}{4}} \sum_{k=0}^{M(x)} A_k (\log x)^{-\frac{7}{6}-k} + O\big(x^{\frac{1}{4}} \delta_{10}(x)\big)\,.$$

This completes the proof of our Theorem. □

R e m a r k . By the same proof, we can generalize this result to an arbitrary r-th power moment of $\frac{a(n)}{S(n)}$ (r any fixed positive real number). Instead of (4.1), we now have (for $\operatorname{Re}(s) > 1$)

$$\begin{aligned} Z_r(s) &= \sum_{n=1}^{\infty} \left(\frac{a(n)}{S(n)} \right)^r n^{-s} \\ &= \zeta(s) \prod_{p \in \mathbb{P}} (1 - p^{-s}) \left(1 + p^{-s} + p^{-2s} + p^{-3s} + \left(\frac{5}{6}\right)^r p^{-4s} + \sum_{k=5}^{\infty} \left(\frac{a(p^k)}{S(p^k)} \right)^r p^{-ks} \right) \\ &= \zeta(s) \big(\zeta(4s)\big)^{-\alpha} U_r(s)\,, \end{aligned}$$

where

$$\alpha = 1 - \left(\frac{5}{6}\right)^r ,$$

and $U_r(s)$ has a Dirichlet series absolutely convergent for $\mathrm{Re}(s) > \frac{1}{5}$. Repeating our argument, we readily obtain

$$\sum_{n \le x} \left(\frac{a(n)}{S(n)}\right)^r$$

$$= A^{(r)} x + x^{\frac{1}{4}} \sum_{k=0}^{M(x)} A_k^{(r)} (\log x)^{-\alpha-1-k} + O\Big(x^{\frac{1}{4}} \exp\big(-c(\log x)^{\frac{3}{5}} (\log\log x)^{-\frac{1}{5}}\big)\Big)$$

with $M(x)$ given as in (2.1).

REFERENCES

[1] ERDÖS, P.—SZEKERES, G.: *Über die Anzahl der Abelschen Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem*, Acta Sci. Math. (Szeged) **VII** (1934), 95–102.

[2] IVIĆ, A.: *The Riemann Zeta-Function*, J. Wiley & Sons, New York-Chichester-Brisbane-Toronto-Singapore, 1985.

[3] De KONINCK, J. M.—IVIĆ, A.: *Topics in Arithmetical Functions*, North Holland Publ. Co., Amsterdam-New York-Oxford, 1980.

[4] HONG-QUAN LIU: *On the number of abelian groups of a given order*, Acta Arith. **LIX** (1991), 261–277.

[5] KRÄTZEL, E.: *Lattice Points*, Kluwer Acad. Publ., Dordrecht-Boston-London, 1988.

[6] MITSUI, T.: *On the prime ideal theorem*, J. Math. Soc. Japan **20** (1968), 233–247.

[7] NORTHCOTT, D. G.: *Lessons on Rings, Modules and Multiplicities*, Cambridge University Press, London, 1968.

[8] NOWAK, W. G.: *On the average number of finite abelian groups of a given order*, Ann. Sci. Math. Québec **15** (1991), 193–202.

[9] NOWAK, W. G.: *Sums of reciprocals of general divisor functions and the Selberg divisor problem*, Abh. Math. Sem. Univ. Hamburg **61** (1991), 163–173.

[10] PRACHAR, K.: *Primzahlverteilung*, Springer, Berlin-Heidelberg-New York, 1957.

[11] RIEGER, G. J.: *Zum Teilerproblem von Atle Selberg*, Math. Nachr. **30** (1965), 181–192.

[12] SELBERG, A.: *Note on a paper by L. G. Sathe*, J. Indian Math. Soc. **18** (1954), 83–87.

[13] WALFISZ, A.: *Weylsche Exponentialsummen in der neueren Zahlentheorie*, Dt. Verl. d. Wiss., Berlin, 1963.

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