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THE INCIDENCE SUBMANIFOLD OF $RP^n \times G_1(RP^n)$ FOR n ODD IS NONORIENTABLE

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The assertion given in the title will be proved.

Notation: $B^n = \{x \in R^n, |x| \leqq 1\}$ n-dimensional closed ball
$\mathring{B}^n = \{x \in R^n, |x| < 1\}$ n-dimensional open ball
$S^n = \{x \in R^{n+1}, |x| = 1\}$ n-dimensional sphere
RP^n = n-dimensional real projective space
$G_1(RP^n)$ = first Grassmannian of RP^n
$F(n)$ = the sumbanifold of the product-manifold
$RP^n \times G_1(RP^n)$ consisting of all couples (b, p) with $b \in p$.

Let us define a continuous map $i_1 : B^n \times RP^{n-1} \to RP^n$ via $b = (b_1, \ldots, b_n) \in B^n$, $c = (c_1, \ldots, c_n) \in RP^{n-1}$; $i_1(b, c) = (1, b_1, \ldots, b_n) \in RP^n$; $b_0 = 1$.

The space $G_1(RP^n)$ will be endowed with generalized Plücker coordinates. Then $G_1(RP^n) \subset RP^m$, $2m = n \,.\, (n + 1) - 2$. The map i_2 is defined as a composition $B^n \times RP^{n-1} \to RP^n \times RP^n - \Delta \to G_1(RP^n)$ of two maps, $b \in B^n$, $c \in RP^{n-1}$, $(b, c) \to ((1, b_1, \ldots, b_n), (0, c_1, \ldots, c_n)) \mapsto (p_{01}, \ldots, p_{0n}, \ldots, p_{n-1,n}) \in \in RP^m$, where Δ is the diagonal and $p_{ij} = b_i c_j - b_j c_i$ for $i < j$ are generalized Plücker coordinates of a line BC, $B = (1, b)$, $C = (0, c)$ in RP^n. The map i_2 and hence $i = i_1 \times i_2$ is continuous. It is obvious $i[B^n \times RP^{n-1}] \subset F(n)$.

1. i is injective. In fact, let $b, b' \in B^n$ and $c, c' \in RP^{n-1}$ and $(b, c) \neq (b', c')$. If $b \neq b'$, then $i_1(b, c) \neq i_1(b', c')$. If $b = b'$ and $c \neq c'$, then there exist $i, j \in \in \{1, \ldots, n\}$ such that $c_i = c_i' \neq 0$, $c_j \neq c_j'$ and $i \neq j$; hence $p_{0i} = c_i = c_i' = p_{0i}'$ and $p_{0j} = c_j \neq c_j' = p_{0j}'$, $i_2(b, c) \neq i_2(b', c')$.

2. i is embedding, because the map i is continuous, injective and both spaces $B^n \times RP^{n-1}$, $F(n)$ are compact and Haussdorf ones.

Theorem. *The manifold $F(n)$ is nonorientable for n odd.*

Proof. Let us suppose that $F(n)$ for $n = 2k + 1$ is orientable. Then the open submanifold $i[\mathring{B}^n \times RP^{n-1}]$ of $F(n)$ is also orientable. The continuous map i^{-1} (which is in fact an embedding) describes an orientation of $\mathring{B}^n \times RP^{n-1}$

and hence the manifold $B^n \times RP^{n-1}$ with the boundary $\partial(B^n \times RP^{n-1}) = S^{n-1} \times RP^{n-1}$ is orientable as well. The orientability of the manifold $\mathring{B}^n \times RP^{n-1}$ yields an orientability of the manifold $S^{n-1} \times RP^{n-1}$, which is a compact manifold without a boundary, thus it follows that $H_{2n-2}(S^{n-1} \times RP^{n-1}) = \mathbf{Z}$.

On the other hand an easy computation shows that $H_{2n-2}(S^{n-1} \times RP^{n-1}) = H_{2n-2}(RP^{n-1}, H_0(S^{n-1})) + H_{n-1}(RP^{n-1}, H_{n-1}(S^{n-1})) = 0$ for $n = 2k + 1$. which contradict our assumption.

Corollary. $\mathrm{H}_{2n-1}(F(n)) = 0$ *for* n *odd.*

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