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ON THE DECOMPOSITION OF COMPLETE DIRECTED GRAPHS INTO FACTORS WITH GIVEN RADII

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The author of paper [4] studied the problem of the existence of decompositions of the complete directed graphs into factors with given diameters. In the present paper we study a similar problem for the radii. Some of the results are concerned with q-partite graphs. The main aim of this paper is to determine the necessary and sufficients conditions for the existence of a decomposition of complete directed graphs into two factors with given radii.

1. The general case

All graphs in the present paper are directed, without loops and between two vertices of the graph there exist at most two edges with opposite direction. The complete directed graph G with n vertices will be denoted by $\lll n \ggg$ and we mean by it the graph with n vertices, two arbitrary different vertices of which are connected with just two edges with opposite directions. By a factor of a directed graph G we mean an arbitrary subgraph of G containing all vertieces of G. By a decomposition of a graph G into factors we mean such a system $\mathcal{S}$ of factors of the graph G that every edge of G is contained in exactly one factor of $\mathcal{S}$. The eccentricity $e(v)$ of a vertex v is sup $\varrho_G(u, v)$, for all $u \in V_G$ where $\varrho_G(u, v)$ denotes the distance between two vertices $u, v \in V_G$ in G. The radius $r(G)$ of G is defined as $r(G) = \min e(v)$ and the diameter $d(G)$ of a graph G as a $d(G) = \max e(v)$. A vertex v is a center of G if $e(v) = r(G)$. The radius $r(G)$ is ∞ if G is a disconnected graph or if G is connected but $e(v)$ is infinite for all v. The remaining terms are used in the usual sense [1, 2, 3, 4, 5].

Let a cardinal number $n > 1$ and a natural number m be given. Our aim is to determine the conditions for the existence of a decomposition of the graph $\lll n \ggg$ into m factors with given radii $r_1, r_2, \ldots, r_m$, where each r_i $(i = 1, 2, \ldots, m)$ is a natural number or the symbol ∞.

Obviously, in the graph $\lll n \ggg$, where n is natural, the cannot exist a factor with a finite radius $r \geq n$.

Lemma 1. *Let r and n be positive integers. Then for a graph $\ll n \gg$ with n vertices and radius r we have:*

$$2r \leq n .$$

Proof. For $n>1$ in the graph $\ll n \gg$ there exists a factor with an arbitrary finite r satysfying the inequalities $1 \leq r \leq \frac{n}{2}$ (for $d \neq 1$ there exists even such a spanning tree) as well as a factor with an infinite radius ∞.

2. The case $m=2$

Denote by $D(r_1, r_2)$ the smallest cardinal number n such that the graph $\ll n \gg$ can be decomposed into two factors with the radii r_1 and r_2. If such a cardinal number does not exist, we shall write $D(r_1, r_2)=\infty$.

The importance of the function $D(r_1, r_2)$ can be seen from the next theorem.

Theorem 1. *Let $n \geq 2$ be a cardinal number and r_1, r_2 natural numbers or symbols ∞. Then we have: if the complete directed graph with n vertices $\ll n \gg$ can be decomposed into two factors with given radii r_1 and r_2, then for any cardinal number $N>n$ the complete directed graph $\ll N \gg$ can also be decomposed into two factors with radii r_1 and r_2.*

Proof. It is the same as the proof of Theorem 1 of [5].

It is easy to see:

Lemma 2. *If $r(G)=1$, then the complement $\bar{G}$ of G is a disconnected graph. If G is a disconnected graph, then $r(\bar{G})$ is 1 or 2.*

Proof: It is the same as the proof of Lemma 2 in [1].

Lemma 3. *If a graph has radius $r(G) \geq 3$, then $r(\bar{G}) \leq 2$.*

Proof. According to Lemma 2 we may suppose that G is connected. We shall distinguish two cases:

a) $d(G) \geq 4$, then due to Lemma 3 of [1] we get $r(\bar{G}) \leq d(\bar{G})=2$.

b) $r(G)=d(G)=3$. Then for every vertex x there exists a vertex x' with $\varrho_G(x, x')=3$. We shall prove indirectly: suppose there exist two vertices u, v for which $\varrho_{\bar{G}}(u, v)=3$ (then the edge uv belongs to G). Let v' be a vertex for which $\varrho_G(v, v')=3$ (then the edge vv' belongs to $\bar{G}$). Consider the edge uv'. If uv' belongs to G, then vuv' is a path of the length 2 in the G between the vertices v and v', which is a contradiction. If uv'belongs to $\bar{G}$, then the path vuv' is in $\bar{G}$ (the length is 2) — a contradiction.

Theorem 2.

$$D(r_1, r_2) = \begin{cases} r_2+1 & \text{if } 2=r_1\leqslant r_2<\infty, \\ \infty & \text{if } 1=r_1\leqslant r_2<\infty, \\ \infty & \text{if } 3\leqslant r_1\leqslant r_2\leqslant\infty, \\ 2 & \text{if } 1=r_1<r_2=\infty, \\ 4 & \text{if } 2=r_1<r_2=\infty, \\ 2 & \text{if } r_1=r_2=\infty. \end{cases}$$

Proof. Denote the vertices of the graph $\ll n \gg$ by symbols v_i for $i=0, 1, 2, \ldots, n-1$. To prove the first relation it is sufficient to decompose the graph $\ll r_2+1 \gg$ into two factors such that the factor F_2 with radius r_2 contains the edges:

(1) $v_i v_{i+1}$ for $i=0, 1, 2, \ldots, r_2-1$,

(2) $v_{r_2} v_0$.

The distance between arbitrary vertices of F_2 is less than or equal to the radius r_2 (because all the vertices of F_2 are on a cycte(1) of the length r_2+1).

The factor F_1 with the radius r_1 is complementary to F_2 in $\ll r_2+1 \gg$ and it is easy to see that $r_1=1$.

The second relation is evident, the third follows from Lemma 3.

To prove the statement $D(1, \infty)=2$, it is sufficient to decompose the graph $\ll 2 \gg$ into two factors in the following way: F_1 contains the edge v_0v_1 and v_1v_0 and F_2 contains no edge.

To prove the statement $D(2, \infty)=4$ (it is easy to see that $D(2, \infty)\neq 3$ by a systematic examination of all possibilities), it is sufficient to decompose the graph $\ll 4 \gg$ into factors in the following way:

F_1 contains the edges: v_0v_1, v_1v_0, v_1v_2, v_2v_1, v_2v_3, v_3v_2, v_0v_3, v_3v_0 and F_2 contains the edges: v_0v_2, v_2v_0, v_1v_3, v_3v_1.

To prove the statement $D(\infty, \infty)=2$, it is sufficient to decompose the graph $\ll 2 \gg$ into factors in the following way: F_1 contains the edge v_0v_1 and F_2 contains the edge v_1v_0.

From Theorem 1 the following corollary follows:

Corollary 1. *The graph* $\ll n \gg$ *is decomposable into two factors with radii* r_1 *and* r_2 *if and only if* $n\geqslant D(r_1, r_2)$, *where* $D(r_1, r_2)$ *is the same as in Theorem 2.*

(1) = directed circuit [2]

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О РАЗЛОЖЕНИИ ПОЛНЫХ ОРИЕНТИРОВАННЫХ ГРАФОВ НА ФАКТОРЫ С ДАННЫМИ РАДИУСАМИ

Eliška Tomová

Резюме

Рассматривается проблема разложения полных ориентированных графов на факторы с данными радиусами. Проблема решена полностью для случая двух факторов.