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On \(d\)-algebras

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ON $d$-ALGEBRAS

J. NEGGERS* — HEE SIK KIM**

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ABSTRACT. In this paper we introduce the notion of $d$-algebras which is another generalization of $BCK$-algebras, and investigate several relations between $d$-algebras and $BCK$-algebras. Furthermore, we show that the class of oriented digraphs corresponds in a simple way to the class of edge $d$-algebras and that arbitrary $d$-algebras also determine unique edge $d$-algebras in a natural manner.

1. Introduction

Y. Imaï and K. Iséki introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras ([1], [2]). It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. In [3], [4] Q. P. Hu and X. Li introduced a broad class of abstract algebras: $BCH$-algebras. They have shown that the class of $BCI$-algebras is a proper subclass of the class of $BCH$-algebras. $BCK$-algebras also have some connections with other areas: D. Mundici [6] proved that $MV$-algebras are categorically equivalent to bounded commutative $BCK$-algebras, and J. Meng [5] proved that implicative commutative semigroups are equivalent to a class of $BCK$-algebras. We introduce the notion of $d$-algebras, which is another useful generalization of $BCK$-algebras, and then we investigate several relations between $d$-algebras and $BCK$-algebras as well as some other interesting relations between $d$-algebras and oriented digraphs.

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Key words: $BCK$-algebra, $d$-algebra, edge, $d$-transitive, digraph.
2. d-algebras

A d-algebra is a non-empty set $X$ with a constant 0 and a binary operation $*$ satisfying the following axioms:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y$ in $X$.

A BCK-algebra is a d-algebra $(X; *, 0)$ satisfying the following additional axioms:

(IV) $((x * y) * (x * z)) * (z * y) = 0$,
(V) $(x * (x * y)) * y = 0$ for all $x, y, z$ in $X$.

Example 2.1.

(a) Every BCK-algebra is a d-algebra.

(b) Let $X := \{0, 1, 2\}$ be a set with the following Table 1.

\[
\begin{array}{ccc}
* & 0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
1 & 2 & 0 & 2 \\
2 & 1 & 1 & 0 \\
\end{array}
\]

Table 1.

Then $(X; *, 0)$ is a d-algebra, but not a BCK-algebra, since $(2 * (2 * 2)) * 2 = (2 * 0) * 2 = 1 * 2 = 2 \neq 0$.

(c) Let $\mathbb{R}$ be the set of all real numbers and define $x * y := x \cdot (x - y)$, $x, y \in \mathbb{R}$, where $\cdot$ and $-$ are the ordinary product and substraction of real numbers. Then $x * x = 0$, $0 * x = 0$, $x * 0 = x^2$. If $x * y = y * x = 0$, then $x(x - y) = 0$ and $x^2 = xy$, $y(y - x) = 0$, $y^2 = xy$. Thus if $x = 0$, $y^2 = 0$, $y = 0$; if $y = 0$, $x^2 = 0$, $x = 0$ and if $xy \neq 0$, then $x = y$. Hence $(\mathbb{R}; *, 0)$ is a d-algebra, but not a BCK-algebra, since $(2 * 0) * 2 \neq 0$.

Remark.

1. If a d-algebra $(X; *, 0)$ is associative, then $0 * x = 0 = (x * x) * x = x * (x * x) = x * 0$, and thus by (III) $x = 0$, i.e., d-algebras are the “most non-associative” algebras.

2. Let $(X; *, 0)$ be a d-algebra. If $S \subseteq X$ is closed under $*$, then $x \in S$ implies $x * x = 0 \in S$, so that $(S; *, 0)$ is a d-algebra.

Definition 2.2. Let $(X; *, 0)$ be a d-algebra and $x \in X$. Define $x * X := \{x * a \mid a \in X\}$. $X$ is said to be edge if for any $x$ in $X$, $x * X = \{x, 0\}$. 

20
Remark. If \((X, \leq)\) is an ordered set (poset), then the operation \(*\) on \(X\) given by \(x * y = 0\) if and only if \(x \leq y\) and \(x * y = x\) otherwise defines a BCK-algebra. On the other hand, from our viewpoint it has the "edge" property. Although edge \(d\)-algebras are not in general BCK-algebras, they come close to being so, as we note below.

**Lemma 2.3.** Let \((X; *, 0)\) be an edge \(d\)-algebra. Then \(x * 0 = x\) for any \(x \in X\).

**Proof.** Since \((X; *, 0)\) is an edge \(d\)-algebra, either \(x * 0 = x\) or \(x * 0 = 0\) for any \(x \in X\). If \(x \neq 0\) and \(x * 0 = 0\), then by (III) \(x = 0\), a contradiction. \(\Box\)

**Proposition 2.4.** If \((X; *, 0)\) is an edge \(d\)-algebra, then the condition (V) holds.

**Proof.** If \(x = 0\), then \((x * (x * y)) * y = 0\) by (II). Let \(x \neq 0\). Assume \((x * (x * y)) * y \neq 0\) for some \(y \in X\). Let \(\alpha := x * (x * y)\). Then \(\alpha * y \neq 0\) and \(\alpha \neq 0\). This means that \(x \neq x * y \in x * X = \{x, 0\}\) and hence \(x * y = 0\). It follows that, by Lemma 2.3, \((x * (x * y)) * y = (x * 0) * y = x * y = 0\), a contradiction. \(\Box\)

**Definition 2.5.** A \(d\)-algebra \((X; *, 0)\) is said to be \(d\)-transitive if \(x * z = 0\) and \(z * y = 0\) imply \(x * y = 0\).

**Theorem 2.6.** Let \((X; *, 0)\) be a \(d\)-transitive edge \(d\)-algebra. Then \((X; *, 0)\) is a BCK-algebra.

**Proof.** By Proposition 2.4, it is enough to show that condition (IV) holds. Assume that \(((x * y) * (x * z)) * (z * y) \neq 0\) for some \(x, y, z \in X\). Since \((x * y) * (x * z) \in (x * y) * X = \{x * y, 0\}\),

\[
(x * y) * (x * z) = x * y.
\]  
(a)

If \(x * y = 0\), then \(0 \neq ((x * y) * (x * z)) * (z * y) = (0 * (x * z)) * (z * y) = 0 * (z * y) = 0\), a contradiction. It follows that

\[
x * y = x.
\]  
(b)

Hence

\[
x = x * y \quad \text{[by (b)]}
= (x * y) * (x * z) \quad \text{[by (a)]}
= x * (x * z) \quad \text{[by (b)]}
\]

that is,

\[
x = x * (x * z).
\]  
(c)
If \( x \ast z \neq 0 \), then \( x \ast z = x \), since \( X \) is an edge \( d \)-algebra. By applying (III),

\[
x = x \ast (x \ast z) = x \ast x = 0.
\]

This means that

\[
0 \neq ((x \ast y) \ast (x \ast z)) \ast (z \ast y)
= (x \ast x) \ast (z \ast y) \quad \text{[by (b) and } x \ast z = x]\n= 0 \ast (z \ast y)
= 0,
\]
a contradiction. Thus we conclude

\[
x \ast z = 0. \tag{d}
\]

We claim that \( z \ast y = 0 \). If \( z \ast y = z \), then

\[
0 \neq ((x \ast y) \ast (x \ast z)) \ast (z \ast y)
= ((x \ast y) \ast 0) \ast z \quad \text{[by (d) and } z \ast y = z]\n= (x \ast y) \ast z \quad \text{[by Lemma 2.3]}\n= x \ast z \quad \text{[by (b)]}\n= 0, \quad \text{[by (d)]}
\]
a contradiction. Thus we obtain that \( x \ast z = 0 \) and \( z \ast y = 0 \). Since \( X \) is \( d \)-transitive, \( x \ast y = 0 \), and hence \( 0 \neq ((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0 \), a contradiction. This proves the theorem. \( \square \)

**Remark.** Both conditions, i.e., to have the \( d \)-transitive and edge properties, are necessary for a \( d \)-algebra of this type to be a \( BCK \)-algebra. Thus, arbitrary \( BCK \)-algebras do not always have the edge property even if the standard examples derived from posets do indeed possess it.

**Example 2.7.** Consider the following \( d \)-algebra \( X \) with the Table 2.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2.

We can easily see that \( 1 \ast 2 = 0 \), \( 2 \ast 3 = 0 \), but \( 1 \ast 3 = 1 \), and hence \((X; \ast, 0)\) is non-\( d \)-transitive edge \( d \)-algebra. Since \((1 \ast 3) \ast (1 \ast 2) \ast (2 \ast 3) = 1 \neq 0\), \((X; \ast, 0)\) is not a \( BCK \)-algebra.
ON $d$-ALGEBRAS

EXAMPLE 2.8. Let $X := \{0, 1, 2, \ldots\}$ and let the binary operation $*$ be defined as follows:

$$x * y := \begin{cases} 0 & \text{if } x \leq y, \\ 1 & \text{otherwise.} \end{cases}$$

Then $x * z = 0$, $z * y = 0$ implies $x \leq z$, $z \leq y$ and in particular $x \leq y$, i.e., $x * y = 0$ also. Furthermore, $x * x = 0$, $0 * x = 0$ and $x * y = y * x = 0$ if $x \leq y$, $y \leq x$, whence $x = y$. Thus, the algebra $(X; *, 0)$ is a $d$-transitive non-edge $d$-algebra. Also, $(2 * (2 * 0)) * 0 = (2 * 1) * 0 = 1 * 0 = 1$, so that $(X; *, 0)$ is not a $BCK$-algebra.

3. Construction of edge $d$-algebras

Suppose that $(X; *, 0)$ is an arbitrary $d$-algebra. Assume that $(X; *, 0)$ is not an edge $d$-algebra. Define a binary operation $\oplus: X \times X \to X$ by

$$x \oplus y := \begin{cases} x & \text{if } x * y \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can see easily that $(X; \oplus, 0)$ is a $d$-algebra. Suppose now that $x \oplus X = \{0\}$. Then $x * y = 0$ for all $y \in X$. In particular, $x * 0 = 0 = 0 * x$, so that also $x = 0$. Hence, if $x \neq 0$, then $x \oplus X = \{x, 0\}$. We summarize:

**Theorem 3.1.** Given a $d$-algebra $(X; *, 0)$ we can construct an edge $d$-algebra $(X; \oplus, 0)$, called the extended edge $d$-algebra.

**Proposition 3.2.** A $d$-algebra $(X; *, 0)$ is $d$-transitive if and only if its extended edge $d$-algebra $(X; \oplus, 0)$ is $d$-transitive.

**Proof.** If $(X; *, 0)$ is $d$-transitive then $x \oplus z = 0$ and $z \oplus y = 0$ imply $x * z = 0 = z * y$, so that $x * y = 0$ and $x \oplus y = 0$ as well. Conversely, if $(X; \oplus, 0)$ is $d$-transitive, then $x \oplus z = 0$ and $z * y = 0$ imply $x \oplus z = 0 = z \oplus x$, so that $x \oplus y = 0$ and $x * y = 0$ as well. \qed

**Example 3.3.** There are 27 $d$-algebras as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>u</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>v</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>0</td>
<td>w</td>
<td>0</td>
</tr>
</tbody>
</table>
where $u, v, w \in \{a, b, c\}$. All of these algebras have the same unique edge $d$-algebra as follows:

\[
\begin{array}{ccc}
\oplus & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & a \\
b & b & b & 0 & 0 \\
c & c & c & 0 & 0 \\
\end{array}
\]

This $d$-algebra is not $d$-transitive since $a \oplus b = b \oplus c = 0$, while $a \oplus c = a \neq 0$. It also has the following $d$-chain property: $x \oplus y \neq 0$ implies $y \oplus x = 0$.

Properties of edge $d$-algebras are properties of $d$-algebras having the edge property. This may be a useful observation in the reduction of certain questions from $d$-algebras to the simpler situation where one has to deal with edge $d$-algebras.

Let $\Gamma'$ be a digraph such that if $(x, y)$ is an edge, then $(y, x)$ is not an edge. Such a digraph $\Gamma'$ is said to be oriented. Let $0$ be adjoined to $\Gamma'$ and denote by $\Gamma$ the oriented digraph with edges $(0, x), x \in \Gamma'$ added to those of $\Gamma'$. On $\Gamma$ define the operation $*$ by $x * x = 0; x * 0 = x; 0 * x = 0$ and $x * y = x$ if $(x, y)$ is not an edge, $x * y = 0$ if $(x, y)$ is an edge. Then $(\Gamma; *, 0)$ is an edge $d$-algebra. We summarize:

**Theorem 3.4.** Every edge $d$-algebra $(X; *, 0)$ produces an oriented digraph $\Gamma = \Gamma' \cup \{0\}$ and conversely.

### 4. Direct sum(product) of $d$-algebras

Let $\{(X_i; *, 0_i) \mid i \in I\}$ be a non-empty family of $d$-algebras and let $\prod_{i \in I} X_i$ consist of all vectors $(x_i)_{i \in I}, x_i \in X_i$. Then $(x_i = 0_i)_{i \in I} = 0$ serves as $0$ if we define $(x_i)_{i \in I} * (y_i)_{i \in I} := (x_i * y_i)_{i \in I}$ and $(\prod_{i \in I} X_i; *, 0)$ is a $d$-algebra, called the direct product of the $d$-algebras $\{(X_i; *, 0) \mid i \in I\}$. Similarly, $\bigoplus_{i \in I} X_i$ consisting of all vectors $(x_i)_{i \in I}, x_i \in X_i$, such that $x_i = 0_i$ except for a finite number of $i$, is a subset of $\prod_{i \in I} X_i$ which is closed under $*$, whence $\left( \bigoplus_{i \in I} X_i; *, 0 \right)$ is a $d$-algebra, called the direct sum of the $d$-algebras $\{(X_i; *, 0) \mid i \in I\}$. Let $(X; *, 0)$ and $(Y; *, 0)$ be $d$-algebras. A mapping $f: X \to Y$ is called a
**ON d-ALGEBRAS**

d-morphism if \( f(x \ast y) = f(x) \ast f(y) \) for any \( x, y \in X \). Note that \( f(0_X) = 0_Y \). Using this concept we study some edge properties.

**PROPOSITION 4.1.** Let \( f : (X; *, 0) \rightarrow (Y; *, 0) \) be an onto d-morphism and let \((X; *, 0)\) be an edge d-algebra. Then \((Y; *, 0)\) is also an edge d-algebra.

**Proof.** Consider \( y = f(x) \), \( b = f(a) \). Then \( y \ast b = f(x) \ast f(a) = f(x \ast a) \in \{f(x), f(a)\} = \{y, 0\} \), whence the conclusion follows. \( \Box \)

Even though \((X; *, 0)\) and \((Y; *, 0)\) are edge d-algebras, their direct sum \( X \oplus Y \) need not have the edge property.

Let \( x \in X \) and \( y \in Y \), and let \( x \ast a = x, y \ast b = 0 \) for some \( a \in X \) and \( b \in Y \). Then \( (x, y) \ast (a, b) = (x \ast a, y \ast b) = (x, 0) \notin \{(x, y), (0, 0)\} \) if \( y \neq 0 \). In order for a Cartesian product of two d-algebras to have the edge property, we introduce a new binary operation \( \oplus \). Let \((X; *, 0)\) and \((Y; *, 0)\) be d-algebras. Define the binary operation \( \oplus \) on \( X \times Y \) as follows: \( (x, y) \oplus (a, b) := (x, y) \) unless \( x \ast a = 0 = y \ast b \), when \( (x, y) \oplus (a, b) := (0, 0) \). Then we can easily see that \((X \times Y; \oplus, 0_{X \times Y})\) is an edge d-algebra, denoted by \( X \oplus Y \), and called the edge product of d-algebras \((X; *, 0)\) and \((Y; *, 0)\). Given \( X \oplus Y \) and \( X \oplus Y \), there are inclusion mappings \( \iota_X \) and \( \iota_Y \), and projections \( \pi_X \) and \( \pi_Y \). Now, \( \iota_X(x \ast a) = (x \ast a, 0) = (x, 0) \ast (a, 0) = \iota_X(x) \ast \iota_X(a) \). Similarly, \( \iota_Y(y \ast b) = \iota_Y(y) \ast \iota_Y(b) \). Moreover, \( \pi_X(x \ast a, y \ast b) = x \ast a = \pi_X(x, y) \ast \pi_X(a, b) \). Similarly, \( \pi_Y(x \ast a, y \ast b) = \pi_Y(x, y) \ast \pi_Y(a, b) \). We summarize:

**PROPOSITION 4.2.** The inclusion mappings and projections relative to \( X \oplus Y \) are d-morphisms.

**THEOREM 4.3.** Let \((X; *, 0)\) and \((Y; *, 0)\) be d-algebras. Then \( X \) (or \( Y \), respectively) is an edge d-algebra if and only if the inclusion mapping \( \iota_X \) (or \( \iota_Y \), respectively) is a d-morphism relative to \( X \oplus Y \).

**Proof.** Suppose \( \iota_X \) is a d-morphism relative to \( X \oplus Y \). Then \( (x \ast a, 0) = \iota_X(x \ast a) = \iota_X(x) \oplus \iota_X(a) = (x, 0) \oplus (a, 0) \), and hence \( x \ast a \in \{x, 0\} \) for any \( a \in X \). This means \( x \ast x \in \{x, 0\} \) for all \( x \in X \). Thus \( X \) is an edge d-algebra. Similarly, if \( \iota_Y \) is a d-morphism relative to \( X \oplus Y \), then \( Y \) is an edge d-algebra. Conversely, assume \( X \) is an edge d-algebra. Consider the inclusion mapping \( \iota_X \) relative to \( X \oplus Y \). Then \( \iota_X(x \ast a) = (x \ast a, 0) = (x, 0) \) or \( (0, 0) \), and \( \iota_X(x) \oplus \iota_X(a) = (x, 0) \oplus (a, 0) = (x, 0) \) or \( (0, 0) \) both according as to \( x \ast a = x \) or \( x \ast a = 0 \). Thus \( \iota_X \) is a d-morphism. Similarly, if \( Y \) is an edge d-algebra, then \( \iota_Y \) is a d-morphism. \( \Box \)

Since \( X \oplus Y \) is an edge d-algebra, the following proposition is an immediate consequence of Proposition 4.1.
PROPOSITION 4.4. If the projection $\pi_X$ (or $\pi_Y$, respectively) is a $d$-morphism relative to $X \otimes Y$, then $X$ (or $Y$, respectively) is an edge $d$-algebra.

Remark. Even though $X$ and $Y$ are edge $d$-algebras, the projections $\pi_X$ and $\pi_Y$ relative to $X \otimes Y$ need not be $d$-morphisms.

Indeed, suppose that $y \ast 0 = y \neq 0$. Then $\pi_X((x, y) \otimes (a, 0)) = \pi_X(x, y) = x$. On the other hand, $\pi_X(x, y) \ast \pi_X(a, 0) = x \ast a$, so that if $x \ast a = 0$, then $x \neq 0$ implies $\pi_X((x, y) \otimes (a, 0)) \neq \pi_X(x, y) \ast \pi_X(a, 0)$, i.e., $\pi_X$ is not a $d$-morphism, nor is $\pi_Y$ a $d$-morphism.

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