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COMMENT ON C. R. RAO'S MINQUE FOR REPLICATED OBSERVATIONS

LUBOMÍR KUBÁČEK

Introduction

A replicated regression experiment [1] is a realization of a random vector \( Y = (y'_1, y'_2, ..., y'_n)' = (i \otimes X)\beta + (\varepsilon'_1, \varepsilon'_2, ..., \varepsilon'_n)' \), where \( y'_i \) is an \( N \)-dimensional random vector, \( j = 1, ..., n, i = (1, ..., 1)' \) is \( n \)-dimensional, \( X \) is a known \( N \times k \) matrix (design matrix), \( \otimes \) designates the tensor product of matrices, \( \beta \) is a \( k \)-dimensional unknown parameter, \( \beta \in \mathbb{R}^k \) (\( k \)-dimensional Euclidean space) and \( \varepsilon_i, i = 1, ..., n, \) is a vector of random errors. It is supposed that

\[
E(\varepsilon_i) = 0, \quad E(\varepsilon_i \varepsilon'_j) = \begin{cases} 0 & \text{if } i \neq j \\ \sum_{r=1}^p \Theta_r V_r & \text{if } i = j \end{cases} \quad i, \, j = 1, ..., p.
\]

The \( N \times N \) matrices \( V_i, i = 1, ..., p, (p > 1) \) are known and \( \Theta \neq (\Theta_1, ..., \Theta_p)' \in \Theta \subset \mathbb{R}^p \). The set \( \Theta \) is supposed to fulfil the condition

\begin{equation}
(\ast) \quad \alpha \in \Theta \Rightarrow V_\alpha = \sum_{i=1}^p \alpha_i V_i \text{ is positive definite.}
\end{equation}

The quantities \( \Theta_i, i = 1, ..., p, \) are variance components. In an \( n \)-times replicated experiment an estimator of the variance components can be determined \( n \)-times from the different single component vectors \( y_i, i = 1, ..., n \) (see [5]). Further the estimator can be based on the vector \( \hat{y} = (1/n)\sum_{i=1}^n y_i \) (see [1], [5], [6]), on the matrix \( S = [1/(n-1)]\sum_{i=1}^n (y_i - \hat{y})(y_i - \hat{y})' \) see ([1], [2]) and mainly on the vector \( Y \) (see [1]).

The aim of this note is to compare dispersions of those estimators in the case when all variance components are unbiasedly estimable and errors are normally distributed.
1. NOTATIONS AND AUXILIARY STATEMENTS

According to [3] the class of estimators of a function \( f(\cdot) : \Theta \rightarrow \mathbb{R}^k, f(\theta) = f'\theta, \)
\( f \in \mathbb{R}^p, \) is restricted to the following kinds of estimators

(1) \( \hat{\gamma} = Y'A_1Y; \)
(2) \( \hat{\gamma}_2 = \text{Tr}(A_2S) \) (\( \text{Tr}(. \) means the trace); \)
(3) \( \hat{\gamma}_3 = \hat{\gamma}'A_3\hat{\gamma}; \)
(4) \( \hat{\gamma}_4 = (1/n)\sum_{i=1}^{n} y'A_4y. \)

Statistical properties of those estimators are investigated in [1] (estimator of the type (1)), in [2] (estimator of the type (2)) and in [1], [5], [6] (estimators of the types (3) and (4), respectively).

According to [1] the following symbols are used: \( M = I - X(X'X)X'(X'X)^{-1} \) is a generalized inverse [4] of the matrix \( X'X; \)

\( \{S_{(MV,X)}\}_{i,j} \ (i, j) \)-th element of the matrix \( S_{(MV,X)} = \text{Tr}((MV,X)^{-1} (MV,X)^{-1}) \), \( i = 1, \ldots, p, \) \( (MV,X)^{-1} \) is the Moore—Penrose inverse of the matrix \( MV,X; \)

\( \{S_{V,e}\}_{i,j} = \text{Tr}(V_e^{-1}V_e^{-1}) \), \( i = 1, \ldots, p. \)

If all variance components are unbiasedly estimable by means of the estimator (1), (2), (3) and (4), then \( \mathcal{R}^p = \mathcal{M}(K_0) = \mathcal{M}(S_{V,e}) \), \( \{K_0\}_{i,j} = \text{Tr}(V_iV_i) = i, j = 1, \ldots, p \) (see Theorem 2.1 and Corollary in [2]). The symbol \( \mathcal{M}(K_0) \) denotes the column space of the matrix \( K_0. \) When MINQUE's (3) and (4) exist for all covariance components, then the matrix \( S_{(MV,X)} \) is regular.

In the following the assumption of normality of the vector \( Y \) is used. The Rao—Cramér lower bound for dispersions is denoted as \( R \cdot C \cdot [Y, (0', f') (\beta', \Theta')'] \) when the estimator of the function \( f(.) \) is based on the vector \( Y \) (the parametric space is \( \mathbb{R}^k \times \Theta; \) the notations \( R \cdot C \cdot [S, f'\Theta] \) in the case of \( S \) (the parametric space is \( \Theta \) ) and \( R \cdot C \cdot [\hat{\gamma}, (0', f') (\beta', \Theta')'] \) in the case of \( \hat{\gamma} \) (the parametric space is \( \mathbb{R}^k \times \Theta \) is used.

Lemma 1.1.

\( R \cdot C \cdot [Y, (0', f') (\beta', \Theta')'] = (2/n) f'S_{V,e}^{-1}f \leq 2f'[S_{(MV,X)} + (n - 1)S_{V,e}]^{-1}f = D_{\theta} (\hat{\gamma}_1); \)

(2) \( R \cdot C \cdot [S, f'\Theta] = [2/(n - 1)] f'S_{V,e}^{-1}f = D_{\theta} (\hat{\gamma}_2); \)

(3) \( R \cdot C \cdot [\hat{\gamma}, (0', f') (\beta', \Theta')'] = 2f'S_{V,e}^{-1}f \leq 2f'S_{(MV,X)}^{-1}f = D_{\theta} (\hat{\gamma}_3); \)

(4) \( (2/n) f'S_{(MV,X)}^{-1}f = D_{\theta} (\hat{\gamma}_4). \)

Proof. It follows from the Remark 3.4 in [2] and from the definition of the Rao-Cramér lower bound for dispersions.

Lemma 1.2. Let \( V_e \) and \( S_{(MV,X)} \) be regular matrices; then for the matrices \( S_{V,e}^{-1} \) and \( S_{(MV,X)}^{-1} \) there exists a regular \( p \times p \) matrix \( G \) such that \( G'S_{V,e}^{-1}G = I \) (identity
matrix), \( \mathbf{G} \mathbf{S}_{(M\mathbf{V},M)} \mathbf{G} = \mathbf{D} \) (diagonal matrix) and \( 0 < d_{i,i} = \{ \mathbf{D} \}_{i,i} \leq 1, \ i = 1, \ldots, p \).

Proof. The regularity of the matrix \( \mathbf{V}_a \) implies \( (M\mathbf{V}_a,M)^+ = \mathbf{V}^{-1}_a - \mathbf{V}^{-1}_a \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}_a \mathbf{X})^{-1} \mathbf{X}' \mathbf{V}^{-1}_a \); thus the matrix \( (M\mathbf{V}_a,M)^+ \) is positive semidefinite. That is why there exists a matrix \( \mathbf{J} \) with the property \( (M\mathbf{V}_a,M)^+ = \mathbf{J}\mathbf{J}' \). Because of the relation \( \{ \mathbf{S}_{(M\mathbf{V},M)^+} \}_{i,j} = \text{Tr}(\mathbf{J}\mathbf{V}_i \mathbf{J}' \mathbf{V}_j) = \text{Tr}(\mathbf{J}' \mathbf{V}_j \mathbf{J} \mathbf{V}_i) \) the matrix \( \mathbf{S}_{(M\mathbf{V},M)^+} \) is the Gramm matrix of the elements \( \mathbf{J}' \mathbf{V}_j \mathbf{J} \), \( i = 1, \ldots, p \), and therefore it is positive semidefinite. Under assumption of regularity it is positive definite. Now the existence of the matrix \( \mathbf{G} \) follows from the symmetry of the matrices \( \mathbf{S}_{\mathbf{V}_a^{-1}} \) and \( \mathbf{S}_{(M\mathbf{V},M)^+} \). The positive definiteness of the matrix \( \mathbf{S}_{(M\mathbf{V},M)^+} \) implies the relations \( 0 < d_{i,i}, \ i = 1, \ldots, p \) and the relations \( d_{i,i} \leq 1, \ i = 1, \ldots, p \), follow from (3) of Lemma 1.1.

Lemma 1.3. Let \( h(.,.): \mathbb{R}^2_+ \rightarrow \mathbb{R}_+ \), where \( \mathbb{R}^2_+ = \{(y,z): \ y \geq 0, \ z \geq 0\} \) \( - \{(0,0)\} \), \( \mathbb{R}^+_1 = \{x: x \geq 0\} \), be defined by the relation \( h(y,z) = yz/(y + z) \). For \( s = 1, 2, \ldots \) there holds:

\[
\begin{align*}
\text{(a)} & \quad \forall \{y, z \in \mathbb{R}^+: (y_i = \{y\}_i, z_i = \{z\}_i) \in \mathbb{R}^+_s, \ i = 1, \ldots, s\} \forall \{c \in \mathbb{R}^+: c_i = \{c\}_i \in \mathbb{R}^+_s, \ i = 1, \ldots, s\}, (c'y)^2 + (c'z)^2 \neq 0
\end{align*}
\]

\[
\sum_{i=1}^{l} c_i h(y_i, z_i) \leq h(c'y, c'z);
\]

(b) if \( y, z, c \in \mathbb{R}^+ \) \( (s \geq 2) \) fulfill the conditions from (a), then

\[
\left\{ \sum_{i=1}^{l} c_i h(y_i, z_i) = h(c'y, c'z) \right\} \Leftrightarrow \left\{ \exists \{k_i \geq 0\} \forall \{i = 1, \ldots, s\} y_i = k_i z_i \right\}
\]

or

\[
\left\{ \exists \{k_2 \geq 0\} \forall \{i = 1, \ldots, s\} z_i = k_2 y_i \right\}.
\]

Proof. (a) The tangential plane of the function \( h(y,z) = yz/(y+z) \) at the point \( (y_0, z_0) \in \mathbb{R}^+_2 \) is \( x = [z_0/(y_0 + z_0)]^2 y + [y_0/(y_0 + z_0)]^2 z \). The relation \( \min \{ p^2 y + q^2 z: p + q = 1, \ p \geq 0, \ q \geq 0 \} = yz/(y+z) \), where \( y \geq 0, \ z \geq 0 \) and \( y^2 + z^2 > 0 \) implies \( x \geq h(y,z) \), \( (y,z) \in \mathbb{R}^+_2 \). Therefore the function \( h(.,.) \) is concave. Suppose a vector \( c \) satisfies all conditions listed in (a) together with the additional condition \( \sum_{i=1}^{l} c_i = 1 \). Then assertion (a) is obviously true. Since \( h(ky, kz) = kh(y, z) \) for \( k \geq 0 \) and \( (y,z) \in \mathbb{R}^+_2 \) the proof of the statement (a) is obviously concluded.

(b) The equality \( \sum_{i=1}^{l} c_i h(y_i, z_i) = h(c'y, c'z) \) holds if and only if all triples \( (y_i, z_i, h(y_i, z_i)), i = 1, \ldots, s \), fulfill the equation of some tangential plane \( [z_0/(y_0 + z_0)]^2 y_i + [y_0/(y_0 + z_0)]^2 z_i = y_i z_i/(y_i + z_i) \Leftrightarrow p^2 y_i^2 + q^2 z_i^2 - y_i z_i (1 - p^2 - q^2) = 0 \). Because of \( 1 = (p+q)^2 = p^2 + q^2 + 2pq \), we get \( p^2 y_i^2 + q^2 z_i^2 - 2pq y_i z_i = 0 \Leftrightarrow (p y_i - q z_i)^2 = 0 \). Thus either \( y_i/z_i = y_0/z_0 \) or \( z_i/y_i = z_0/y_0 \) for \( i = 1, \ldots, s \), and (b) is proved.
2. Comparison of estimators

**Theorem.** (a) Let \( f(.), i = 1, ..., p, \) be functions such that \( f_i(\Theta) = f'_i\Theta \) and that \( G^Tf_i = e_i = (0, ..., 0_{i-1}, 1, 0_{i+1}, ..., 0_p)' \), where \( G \) is the matrix from Lemma 1.2. Then there exist real numbers \( c_{i,2}, c_{i,3}, c_{i,2} \geq 0, c_{i,3} \geq 0, c_{i,2} + c_{i,3} = 1 \) so that

\[
D_a(\hat{\gamma}_1^{(Y)}) = D_a[c_{i,2} \hat{\gamma}_2^{(Y)} + c_{i,3} \hat{\gamma}_3^{(Y)}].
\]

(b) Let a function \( f(\Theta) = f'\Theta, \Theta \in \Theta, \) not be a constant multiple of some function from (a). Then a necessary and sufficient condition for the existence of numbers \( c_{i,2} \geq 0, c_{i,3} \geq 0, c_{i,2} + c_{i,3} = 1 \) with the property 

\[
D_a(\hat{\gamma}_2^{(Y)}) = D_a(c_{i,2} \hat{\gamma}_2^{(Y)} + c_{i,3} \hat{\gamma}_3^{(Y)}) = D_a(\hat{\gamma}_1^{(Y)}) = \text{the existence of a number } d_0, 0 < d_0 \leq 1 \text{ such that } \forall j: \{G^Tf_j\} \neq 0 \}
\]

**Proof.** (a) Owing to Lemma 1.1 we have

\[
D_a[\hat{\gamma}_1^{(Y)}] = 2f'_iGG^{-1}[S_{(MV,M)}^T + (n - 1)S_{V_a}]^{-1}G^{-1}G^Tf =
\]

\[
= 2e'(D + (n - 1)I)^{-1}e = 2/(d_{i,i} + n - 1),
\]

\[
D_a[\hat{\gamma}_2^{(Y)}](S) = 2/(n - 1), D_a[\hat{\gamma}_3^{(Y)}](\hat{Y}) = 2/d_{i,i}.
\]

By taking into account the definitions \( c_{i,2} = (n - 1)/(n - 1 + d_{i,i}), \ c_{i,3} = d_{i,i}/(n - 1 + d_{i,i}) \) and the stochastic independence of the estimators \( \hat{\gamma}_2(S) \) and \( \hat{\gamma}_3(\hat{Y}) \) we easily get (a).

(b) If the function \( f(.) \) is not a constant multiple of some function from (a), then in the expression \( D_a[\hat{\gamma}_1^{(Y)}] = \sum_{i=1}^p \{G^Tf_i\}^2/(n - 1 + d_{i,i}) \) at least two members differ from zero. From Lemma 1.1, Lemma 1.2 and from the stochastic independence of the estimators \( \hat{\gamma}_2(S) \) and \( \hat{\gamma}_3(\hat{Y}) \) it follows that

\[
D_a[\hat{\gamma}_1^{(Y)}] \leq \min \{ D_a[k_{i,2} \hat{\gamma}_2^{(Y)}(S) + k_{i,3} \hat{\gamma}_3^{(Y)}(\hat{Y})]: k_{i,2} \geq 0,
\]

\[
k_{i,3} \geq 0, k_{i,2} + k_{i,3} = 1 \}
\]

\[
= D_a[\hat{\gamma}_2^{(Y)}(S)]D_a[\hat{\gamma}_3^{(Y)}(\hat{Y})]/\{ D_a[\hat{\gamma}_2^{(Y)}(S)] + D_a[\hat{\gamma}_3^{(Y)}(\hat{Y})]\}).
\]

That is why the numbers \( c_{i,2}, c_{i,3}, c_{i,2} \) can exist iff

\[
D_a[\hat{\gamma}_1^{(Y)}] = D_a[\hat{\gamma}_2^{(Y)}(S)]D_a[\hat{\gamma}_3^{(Y)}(\hat{Y})]/\{ D_a[\hat{\gamma}_2^{(Y)}(S)] + D_a[\hat{\gamma}_3^{(Y)}(\hat{Y})]\}).
\]

Thus, with notation \( \{G^Tf_i\}_i = g_i, i = 1, ..., p \) (\( G \) and \( D \) are matrices from Lemma 1.2), we obtain

\[
(**) 2 \sum_{i=1}^p g_i^2/(n - 1 + d_{i,i}) =
\]

\[
= [2/(n - 1)] \sum_{i=1}^p g_i^2 \sum_{i=1}^p (g_i^2/d_{i,i})/[2/(n - 1)] \sum_{i=1}^p g_i^2 + 2 \sum_{i=1}^p (g_i^2/d_{i,i})].
\]

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In Lemma 1.3 we substitute for \( c, y_i, z_i, i = 1, \ldots, p \):
\[
c = (1, \ldots, 1)' \in \mathbb{R}^p, \quad y_i = 2g^2/(n-1), \quad z_i = 2g^2/d_{i,i}, \quad i = 1, \ldots, p.
\]
Because of
\[
2g^2/(n-1 + d_{i,i}) = [2g^2/(n-1)](2g^2/d_{i,i})/[2g^2/(n-1) + 2g^2/d_{i,i}],
\]
for \( i = 1, \ldots, p \), the equality \((**)\) is valid iff there exists the real number \( k \geq 0 \) from Lemma 1.3 with the property
\[
\forall \{j \in \{i: g_i \neq 0\}\} 2g^2/(n-1) = k2g^2/d_{i,i} \Leftrightarrow
\]
\[
\Rightarrow \forall \{j \in \{i: g_i \neq 0\}\} d_{i,i} = k(n-1) \quad (= d_0).
\]

**Corollary 1.** Let \( \mathbf{D} \) be the matrix from Lemma 1.2. Then
\[
\{\mathbf{D}\}_{i,i} < 1 \text{ implies } \exists \{n_i\} \forall \{n \geq n_i\} \mathbb{D}_a[\hat{\gamma}^{(0)}(y_1, \ldots, y_s)] > \mathbb{D}_a[\hat{\gamma}^{(0)}(\mathbf{S})].
\]
The proof follows from (4) of Lemma 1.1, from the Theorem and from Lemma 1.2.

**Corollary 2.** The implication \( n \uparrow \infty \Rightarrow c_{i,2} \uparrow 1 \& c_{i,3} \downarrow 0 \) (see [1] p. 194) shows the growing importance of the estimator \( \hat{\gamma}^{(0)}(\mathbf{S}) \) with the growing number of replication of the experiment and the relation \( \mathbb{D}_a[\hat{\gamma}^{(0)}(\mathbf{S})]/\mathbb{D}_a[\hat{\gamma}^{(0)}(\mathbf{Y})] \downarrow 1 \). Following the Theorem these facts are valid for an arbitrary function \( f(\Theta) = f' \Theta, \Theta \in \Theta \).

**Corollary 3.** The relations
\[
c_{i,2} = (n-1)/(n-1 + d_{i,i}) =
\]
\[
= \{1/\mathbb{D}_a[\hat{\gamma}_2(\mathbf{S})]\}/\{1/\mathbb{D}_a[\hat{\gamma}_2(\mathbf{S})]\} + \{1/\mathbb{D}_a[\hat{\gamma}_3(\mathbf{Y})]\}
\]
and
\[
c_{i,3} = d_{i,i}/(n-1 + d_{i,i}) =
\]
\[
= \{1/\mathbb{D}_a[\hat{\gamma}_3(\mathbf{Y})]\}/\{1/\mathbb{D}_a[\hat{\gamma}_3(\mathbf{Y})]\} + \{1/\mathbb{D}_a[\hat{\gamma}_3(\mathbf{S})]\}
\]
imply
\[
\mathbb{D}_a[\hat{\gamma}_2(\mathbf{S})]/\mathbb{D}_a[\hat{\gamma}_3(\mathbf{Y})] = c_{i,3}/c_{i,2} = d_{i,i}/(n-1) \downarrow 0.
\]

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ЗАМЕТКА К MINQUE Ц. Р. РАО ДЛЯ ПОВТОРАЮЩИХСЯ НАБЛЮДЕНИЙ

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Резюме

В работе приводятся необходимые и достаточные условия для равенства между двумя оценками. Первая оценка — MINQUE Rao в повторяющемся регрессионном эксперименте, созданная Клefeфе; вторая оценка является оптимальной комбинацией оценок, основанных на матрице Уишарта и на векторе арифметических средних наблюдений.