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# COMMENT ON C. R. RAO'S MINQUE FOR REPLICATED OBSERVATIONS

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## Introduction

A replicated regression experiment [1] is a realization of a random vector $\boldsymbol{Y}=(\boldsymbol{y}_1', \boldsymbol{y}_2', \ldots, \boldsymbol{y}_n')' = (\boldsymbol{i}\otimes\boldsymbol{X})\boldsymbol{\beta}+(\boldsymbol{\varepsilon}_1', \boldsymbol{\varepsilon}_2', \ldots, \boldsymbol{\varepsilon}_n')'$, where $\boldsymbol{y}_j$ is an $N$-dimensional random vector, $j=1, \ldots, n$, $\boldsymbol{i}=(1, \ldots, 1)'$ is $n$-dimensional, $\boldsymbol{X}$ is a known $N\times k$ matrix (design matrix), $\otimes$ designates the tensor product of matrices, $\boldsymbol{\beta}$ is a $k$-dimensional unknown parameter, $\boldsymbol{\beta}\in\mathcal{R}^k$ ($k$-dimensional Euclidean space) and $\boldsymbol{\varepsilon}_i$, $i=1, \ldots, n$, is a vector of random errors. It is supposed that

$$E(\boldsymbol{\varepsilon}_i)=\mathbf{0},\quad E(\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_j')=\begin{cases}\mathbf{0} & \text{if } i\neq j\\ \sum_{r=1}^{p}\Theta_r\mathbf{V}_r & \text{if } i=j\end{cases}\quad i, j=1, \ldots, p.$$

The $N\times N$ matrices $\mathbf{V}_i$, $i=1, \ldots, p$, $(p>1)$ are known and $\boldsymbol{\Theta}=(\Theta_1, \ldots, \Theta_p)'\in\boldsymbol{\Theta}\subset\mathcal{R}^p$. The set $\boldsymbol{\Theta}$ is supposed to fulfil the condition

$$(*)\qquad \boldsymbol{\alpha}\in\boldsymbol{\Theta}\Rightarrow\mathbf{V}_{\alpha}=\sum_{i=1}^{p}\alpha_i\mathbf{V}_i \text{ is positive definite.}$$

The quantities $\Theta_i$, $i=1, \ldots, p$, are variance components. In an $n$-times replicated experiment an estimator of the variance components can be determined $n$-times from the different single component vectors $\boldsymbol{y}_i$, $i=1, \ldots, n$ (see [5]). Further the estimator can be based on the vector $\bar{\boldsymbol{y}}=(1/n)\sum_{i=1}^{n}\boldsymbol{y}_i$ (see [1], [5], [6]), on the matrix $\mathbf{S}=[1/(n-1)]\sum_{i=1}^{n}(\boldsymbol{y}_i-\bar{\boldsymbol{y}})(\boldsymbol{y}_i-\bar{\boldsymbol{y}})'$ see ([1], [2]) and mainly on the vector $\boldsymbol{Y}$ (see [1]).

The aim of this note is to compare dispersions of those estimators in the case when all variance components are unbiasedly estimable and errors are normally distributed.

## 1. NOTATIONS AND AUXILIARY STATEMENTS

According to [3] the class of estimators of a function $f(.)\colon \boldsymbol{\Theta} \to \mathscr{R}^1$, $f(\boldsymbol{\Theta}) = \boldsymbol{f}'\boldsymbol{\Theta}$, $\boldsymbol{f} \in \mathscr{R}^p$, is restricted to the following kinds of estimators

(1) $\hat{\gamma} = \boldsymbol{Y}'\mathbf{A}_1\boldsymbol{Y}$;

(2) $\hat{\gamma}_2 = \mathrm{Tr}(\mathbf{A}_2\mathbf{S})$ (Tr(.) means the trace);

(3) $\hat{\gamma}_3 = \bar{\boldsymbol{y}}'\mathbf{A}_3\bar{\boldsymbol{y}}$;

(4) $\hat{\gamma}_4 = (1/n)\sum_{i=1}^{n} \boldsymbol{y}_i'\mathbf{A}_4\boldsymbol{y}_i$.

Statistical properties of those estimators are investigated in [1] (estimator of the type (1)), in [2] (estimator of the type (2)) and in [1], [5], [6] (estimators of the types (3) and (4), respectively).

According to [1] the following symbols are used: $\mathbf{M} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})\ \mathbf{X}'((\mathbf{X}'\mathbf{X})$ is a generalized inverse [4] of the matrix $\mathbf{X}'\mathbf{X}$);

$\{\mathbf{S}_{(\mathbf{M}\mathbf{V}_\alpha\mathbf{M})^+}\}_{i,j}$ ($(i, j)$-th element of the matrix $\mathbf{S}_{(\mathbf{M}\mathbf{V}_\alpha\mathbf{M})^+}$) $=$ $\mathrm{Tr}[(\mathbf{M}\mathbf{V}_\alpha\mathbf{M})^+\mathbf{V}_i(\mathbf{M}\mathbf{V}_\alpha\mathbf{M})^+\mathbf{V}_j]$, $i, j = 1, \dots, p$, $(\mathbf{M}\mathbf{V}_\alpha\mathbf{M})^+$ is the Moore—Penrose inverse of the matrix $\mathbf{M}\mathbf{V}_\alpha\mathbf{M}$, $\{\mathbf{S}_{\mathbf{V}_\alpha^{\ 1}}\}_{i,j} = \mathrm{Tr}(\mathbf{V}_\alpha^{-1}\mathbf{V}_i\mathbf{V}_\alpha^{-1}\mathbf{V}_j)$, $i, j = 1, \dots, p$.

If all variance components are unbiasedly estimable by means of the estimator (1), (2), (3) and (4), then $\mathscr{R}^p = \mathscr{M}(\mathbf{K}_0) = \mathscr{M}(\mathbf{S}_{\mathbf{V}_\alpha^{\ 1}})$, where $\{\mathbf{K}_0\}_{i,j} = \mathrm{Tr}(\mathbf{V}_i\mathbf{M}\mathbf{V}_j) =$ $i, j = 1, \dots, p$ (see Theorem 2.1 and Corrollary in [2]). The symbol $\mathscr{M}(\mathbf{K}_0)$ denotes the column space of the matrix $\mathbf{K}_0$. When MINQUE's (3) and (4) exist for all covariance components, then the matrix $\mathbf{S}_{(\mathbf{M}\mathbf{V}_\alpha\mathbf{M})^+}$ is regular.

In the following the assumption of normality of the vector $\boldsymbol{Y}$ is used. The Rao—Cramér lower bound for dispersions is denoted as $R \cdot C \cdot [\boldsymbol{Y}, (\boldsymbol{0}', \boldsymbol{f}')(\boldsymbol{\beta}', \boldsymbol{\Theta}')']$ when the estimator of the function $f(.)$ is based on the vector $\boldsymbol{Y}$ (the parametric space is $\mathscr{R}^k \times \boldsymbol{\Theta}$; the notations $R.C.[\mathbf{S}, \boldsymbol{f}'\boldsymbol{\Theta}]$ in the case of $\mathbf{S}$ (the parametric space is $\boldsymbol{\Theta}$) and $R.C.[\bar{\boldsymbol{y}}, (\boldsymbol{0}', \boldsymbol{f}')(\boldsymbol{\beta}', \boldsymbol{\Theta}')']$ in the case of $\bar{\boldsymbol{y}}$ (the parametric space is $\mathscr{R}^k \times \boldsymbol{\Theta}$) is used.

**Lemma 1.1.**

(1)
$$R.C.[\boldsymbol{Y}, (\boldsymbol{0}', \boldsymbol{f}')(\boldsymbol{\beta}', \boldsymbol{\Theta}')'] = (2/n)\boldsymbol{f}'\mathbf{S}^{-1}_{\mathbf{V}_\alpha^{-1}}\boldsymbol{f} \leqslant$$
$$\leqslant 2\boldsymbol{f}'[\mathbf{S}_{(\mathbf{M}\mathbf{V}_\Theta\mathbf{M})^+} + (n-1)\mathbf{S}_{\mathbf{V}_\Theta^{\ 1}}]^- \boldsymbol{f} = \mathscr{D}_{\boldsymbol{\Theta}}(\hat{\gamma}_1);$$

(2)
$$R.C.[\mathbf{S}, \boldsymbol{f}'\boldsymbol{\Theta}] = [2/(n-1)]\boldsymbol{f}'\mathbf{S}^{-1}_{\mathbf{V}_\Theta^{\ 1}}\boldsymbol{f} = \mathscr{D}_{\boldsymbol{\Theta}}(\hat{\gamma}_2);$$

(3)
$$R.C.[\bar{\boldsymbol{y}}, (\boldsymbol{0}', \boldsymbol{f}')(\boldsymbol{\beta}', \boldsymbol{\Theta}')'] = 2\boldsymbol{f}'\mathbf{S}^{-1}_{\mathbf{V}_\Theta^{-1}}\boldsymbol{f} \leqslant 2\boldsymbol{f}'\mathbf{S}^{-1}_{(\mathbf{M}\mathbf{V}_\Theta\mathbf{M})^+}\boldsymbol{f} = \mathscr{D}(\hat{\gamma}_3);$$

(4)
$$(2/n)\boldsymbol{f}'\mathbf{S}^{-1}_{(\mathbf{M}\mathbf{V}_\Theta\mathbf{M})^+}\boldsymbol{f} = \mathscr{D}(\hat{\gamma}_4).$$

Proof. It follows from the Remark 3.4 in [2] and from the definition of the Rao-Cramér lower bound for dispersions.

**Lemma 1.2.** *Let* $\mathbf{V}_\alpha$ *and* $\mathbf{S}_{(\mathbf{M}\mathbf{V}_\alpha\mathbf{M})^+}$ *be regular matrices; then for the matrices* $\mathbf{S}_{\mathbf{V}_\alpha^{-1}}$ *and* $\mathbf{S}_{(\mathbf{M}\mathbf{V}_\alpha\mathbf{M})^+}$ *there exists a regular* $p \times p$ *matrix* $\mathbf{G}$ *such that* $\mathbf{G}'\mathbf{S}_{\mathbf{V}_\alpha^{-1}}\mathbf{G} = \mathbf{I}$ *(identity* *matrix),* $\mathbf{G}'\mathbf{S}_{(\mathbf{MV}_\alpha\mathbf{M})^+}\mathbf{G}=\mathbf{D}$ *(diagonal matrix) and* $0<d_{i,i}=\{\mathbf{D}\}_{i,i}\leq 1$, $i=1, \ldots, p$.

Proof. The regularity of the matrix $\mathbf{V}_\alpha$ implies $(\mathbf{MV}_\alpha\mathbf{M})^+ = \mathbf{V}_\alpha^{-1} - \mathbf{V}_\alpha^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}_\alpha^{-1}\mathbf{X})^-\mathbf{X}'\mathbf{V}_\alpha^{-1}$; thus the matrix $(\mathbf{MV}_\alpha\mathbf{M})^+$ is positive semidefinite. That is why there exists a matrix $\mathbf{J}$ with the property $(\mathbf{MV}_\alpha\mathbf{M})^+=\mathbf{JJ}'$. Because of the relation $\{\mathbf{S}_{(\mathbf{MV}_\alpha\mathbf{M})^+}\}_{i,j} = \mathrm{Tr}(\mathbf{JJ}'\mathbf{V}_i\mathbf{JJ}'\mathbf{V}_j) = \mathrm{Tr}[(\mathbf{J}'\mathbf{V}_i\mathbf{J})(\mathbf{J}'\mathbf{V}_j\mathbf{J})]$ the matrix $\mathbf{S}_{(\mathbf{MV}_\alpha\mathbf{M})^+}$ is the Gramm matrix of the elements $\mathbf{J}'\mathbf{V}_i\mathbf{J}$, $i=1, \ldots, p$, and therefore it is positive semidefinite. Under assumption of regularity it is positive definite. Now the existence of the matrix $\mathbf{G}$ follows from the symmetry of the matrices $\mathbf{S}_{\mathbf{V}_\alpha^{-1}}$ and $\mathbf{S}_{(\mathbf{MV}_\alpha\mathbf{M})^+}$. The positive definiteness of the matrix $\mathbf{S}_{(\mathbf{MV}_\alpha\mathbf{M})^+}$ implies the relations $0<d_{i,i}$, $i=1, \ldots, p$ and the relations $d_{i,i}\leq 1$, $i=1, \ldots, p$, follow from (3) of Lemma 1.1.

**Lemma 1.3.** *Let* $h(., ..,): \mathscr{R}_+^2 \to \mathscr{R}_+^1$, *where* $\mathscr{R}_+^2=\{(y, z)\colon\ y\geq 0,\ z\geq 0\} - \{(0, 0)\}$, $\mathscr{R}_+^1=\{x\colon x\geq 0\}$, *be defined by the relation* $h(y, z) = yz/(y+z)$. *For* $s=1, 2, \ldots$ *there holds:*

$$\text{(a)}\ \forall\{\boldsymbol{y}, \boldsymbol{z}\in\mathscr{R}^s\colon (y_i=\{\boldsymbol{y}\}_i,\ z_i=\{\boldsymbol{z}\}_i)\in\mathscr{R}_+^2,\ i=1, \ldots, s\}$$
$$\forall\{\boldsymbol{c}\in\mathscr{R}^s\colon c_i=\{\boldsymbol{c}\}_i\in\mathscr{R}_+^1,\ i=1, \ldots, s,\ (\boldsymbol{c}'\boldsymbol{y})^2+(\boldsymbol{c}'\boldsymbol{z})^2\neq 0\}$$
$$\sum_{i=1}^{s} c_i h(y_i, z_i)\leq h(\boldsymbol{c}'\boldsymbol{y}, \boldsymbol{c}'\boldsymbol{z});$$

(b) *if* $\boldsymbol{y}$, $\boldsymbol{z}$, $\boldsymbol{c}\in\mathscr{R}^s$ $(s\geq 2)$ *fulfil the conditions from (a), then*

$$\left\{\sum_{i=1}^{s} c_i h(y_i, z_i)=h(\boldsymbol{c}'\boldsymbol{y}, \boldsymbol{c}'\boldsymbol{z})\right\}\Leftrightarrow\{\exists\{k_1\geq 0\}\ \forall\{i=1, \ldots, s\}\,y_i=k_1z_i\}$$

*or*

$$\{\exists\{k_2\geq 0\}\ \forall\{i=1, \ldots, s\}\,z_i=k_2y_i\}.$$

Proof. (a) The tangential plane of the function $h(y, z)=yz/(y+z)$ at the point $(y_0, z_0)\in\mathscr{R}_+^2$ is $x=[z_0/(y_0+z_0)]^2y + [y_0/(y_0+z_0)]^2z$. The relation $\min\{p^2y+q^2z\colon p+q=1,\ p\geq 0, q\geq 0\} = yz/(y+z)$, where $y\geq 0$, $z\geq 0$ and $y^2+z^2>0$ implies $x\geq h(y, z)$, $(y, z)\in\mathscr{R}_+^2$. Therefore the function $h(., ..)$ is concave. Suppose a vector $\boldsymbol{c}$ satisfies all conditions listed in (a) together with the additional condition $\sum_{i=1}^{p} c_i=1$. Then assertion (a) is obviously true. Since $h(ky, kz)=kh(y, z)$ for $k\geq 0$ and $(y, z)\in R_+^2$ the proof of the statement (a) is obviously concluded.

(b) The equality $\sum_{i=1}^{s} c_ih(y_i, z_i) = h(\boldsymbol{c}'\boldsymbol{y}, \boldsymbol{c}'\boldsymbol{z})$ holds if and only if all triples $(y_i, z_i, h(y_i, z_i))$, $i=1, \ldots, s$, fulfil the equation of some tangential plane $[z_0/(y_0+z_0)]^2y_i + [y_0/(y_0+z_0)]^2z_i = y_iz_i/(y_i+z_i) \Leftrightarrow p^2y_i^2 + q^2z_i^2 - y_iz_i(1-p^2-q^2)=0$ $(p=z_0/(y_0+z_0),\ q=y_0/(y_0+z_0))$. Because of $1=(p+q) = (p+q)^2 = p^2+q^2+2pq$, we get $p^2y_i^2 + q^2z_i^2 - 2pqy_iz_i=0 \Leftrightarrow (py_i - qz_i)^2=0$. Thus either $y_i/z_i = y_0/z_0$ or $z_i/y_i=z_0/y_0$ for $i=1, \ldots, s$, and (b) is proved.

## 2. Comparison of estimators

**Theorem.** *(a) Let $f_i(.)$, $i=1, \dots, p$, be functions such that $f_i(\boldsymbol{\Theta}) = \mathbf{f}_i'\boldsymbol{\Theta}$ and that $\mathbf{G}'\mathbf{f}_i = \mathbf{e}_i = (0_1, \dots, 0_{i-1}, 1_i, 0_{i+1}, \dots, 0_p)'$, where $\mathbf{G}$ is the matrix from Lemma 1.2. Then there exist real numbers $c_{i,2}$, $c_{i,3}$, $c_{i,2} \geq 0$, $c_{i,3} \geq 0$, $c_{i,2} + c_{i,3} = 1$ so that*

$$\mathscr{D}_\alpha(\hat{\gamma}_1^{(i)}) = \mathscr{D}_\alpha[c_{i,2}\hat{\gamma}_2^{(i)} + c_{i,3}\hat{\gamma}_3^{(i)}].$$

*(b) Let a function $f(\boldsymbol{\Theta}) = \mathbf{f}'\boldsymbol{\Theta}$, $\boldsymbol{\Theta} \in \boldsymbol{\Theta}$, not be a constant multiple of some function from (a). Then a necessary and sufficient condition for the existence of numbers $c_{2,f} \geq 0$, $c_{3,f} \geq 0$, $c_{2,f} + c_{3,f} = 1$ with the property $\mathscr{D}_\alpha[c_{2,f}\hat{\gamma}_2^{(f)} + c_{3,f}\hat{\gamma}_3^{(f)}] = \mathscr{D}_\alpha(\hat{\gamma}_3^{(f)})$ is the existence of a number $d_0$, $0 < d_0 \leq 1$ such that $\forall\{j\colon \{\mathbf{G}'\mathbf{f}\}_j \neq 0\}\, d_{j,j} = \{\mathbf{G}'\mathbf{S}_{(\mathbf{MV}_\alpha\mathbf{M})^+}\mathbf{G}\}_{j,j} = d_0$.*

Proof. (a) Owing to Lemma 1.1 we have

$$\mathscr{D}_\alpha[\hat{\gamma}^{(i)}(\mathbf{Y})] = 2\mathbf{f}_i'\mathbf{G}\mathbf{G}^{-1}[\mathbf{S}_{(\mathbf{MV}_\alpha\mathbf{M})^+} + (n-1)\mathbf{S}_{\mathbf{V}_\alpha^{-1}}]^{-1}\mathbf{G}'^{-1}\mathbf{G}'\mathbf{f}_i =$$

$$= 2\mathbf{e}_i'[\mathbf{D} + (n-1)\mathbf{I}]^{-1}\mathbf{e}_i = 2/(d_{i,i} + n - 1),$$

$$\mathscr{D}_\alpha[\hat{\gamma}_2^{(i)}(\mathbf{S})] = 2/(n-1),\ \mathscr{D}_\alpha[\hat{\gamma}_3^{(i)}(\bar{\mathbf{y}})] = 2/d_{i,i}.$$

By taking into account the definitions $c_{i,2} = (n-1)/(n-1+d_{i,i})$, $c_{i,3} = d_{i,i}/(n-1+d_{i,i})$ and the stochastic independence of estimators $\hat{\gamma}_2(\mathbf{S})$ and $\hat{\gamma}_3(\bar{\mathbf{y}})$ we easily get (a).

(b) If the function $f(.)$ is not a constant multiple of some function from (a), then in the expression $\mathscr{D}_\alpha[\hat{\gamma}_1^{(f)}(\mathbf{Y})] = \sum_{i=1}^{p} \{\mathbf{G}'\mathbf{f}\}_i^2/(n-1+d_{i,i})$ at least two members differ from zero. From Lemma 1.1, Lemma 1.2 and from the stochastical independence of the estimators $\hat{\gamma}_2^{(f)}(\mathbf{S})$ and $\hat{\gamma}_3^{(f)}(\bar{\mathbf{y}})$ it follows that

$$\mathscr{D}_\alpha[\hat{\gamma}_1^{(f)}(\mathbf{Y})] \leq \min\{\mathscr{D}_\alpha[k_{2,f}\hat{\gamma}_2^{(f)}(\mathbf{S}) + k_{3,f}\hat{\gamma}_3^{(f)}(\bar{\mathbf{y}})]\colon k_{2,f} \geq 0,$$

$$k_{3,f} \geq 0,\ k_{2,f} + k_{3,f} = 1\}$$

$$(= \mathscr{D}_\alpha[\hat{\gamma}_2^{(f)}(\mathbf{S})]\mathscr{D}_\alpha[\hat{\gamma}_3^{(f)}(\bar{\mathbf{y}})]/\{\mathscr{D}_\alpha[\hat{\gamma}_2^{(f)}(\mathbf{S})] + \mathscr{D}_\alpha[\hat{\gamma}_3^{(f)}(\bar{\mathbf{y}})]\}).$$

That is why the numbers $c_{2,f}$, $c_{3,f}$ can exist iff

$$\mathscr{D}_\alpha[\hat{\gamma}_1^{(f)}(\mathbf{Y})] = \mathscr{D}_\alpha[\hat{\gamma}_2^{(f)}(\mathbf{S})]\mathscr{D}_\alpha[\hat{\gamma}_3^{(f)}(\bar{\mathbf{y}})]/\{\mathscr{D}_\alpha[\hat{\gamma}_2^{(f)}(\mathbf{S})] + \mathscr{D}_\alpha[\hat{\gamma}_3^{(f)}(\bar{\mathbf{y}})]\}.$$

Thus, with notation $\{\mathbf{G}'\mathbf{f}\}_i = g_i$, $i = 1, \dots, p$ ($\mathbf{G}$ and $\mathbf{D}$ are matrices from Lemma 1.2), we obtain

$$(**) \qquad 2\sum_{i=1}^{p} g_i^2/(n-1+d_{i,i}) =$$

$$= [2/(n-1)]\sum_{i=1}^{p} g_i^2 2\sum_{j=1}^{p}(g_j^2/d_{j,j})\Big/\left\{[2/(n-1)]\sum_{i=1}^{p} g_i^2 + 2\sum_{j=1}^{p}(g_j^2/d_{j,j})\right\}.$$

In Lemma 1.3 we substitute for $\boldsymbol{c}$, $y_i$, $z_i$, $i=1, \ldots, p$:

$$\boldsymbol{c}=(1, \ldots, 1)' \in \mathscr{R}^p, \quad y_i=2g_i^2/(n-1), \quad z_i=2g_i^2/d_{i,i}, \quad i=1, \ldots, p.$$

Because of

$$2g_i^2/(n-1+d_{i,i})=[2g_i^2/(n-1)](2g_i^2/d_{i,i})/\{[2g_i^2/(n-1)]+2g_i^2/d_{i,i}\},$$

for $i=1, \ldots, p$, the equality $(**)$ is valid iff there exists the real number $k \geq 0$ from Lemma 1.3 with the property

$$\forall\{j \in \{i\colon g_i \neq 0\}\} 2g_j^2/(n-1)=k2g_j^2/d_{j,j} \Leftrightarrow$$
$$\Leftrightarrow \forall\{j \in \{i\colon g_i \neq 0\}\} d_{j,j}=k(n-1)\ (=d_0).$$

**Corollary 1.** *Let* $\mathbf{D}$ *be the matrix from Lemma 1.2. Then*

$$\{\mathbf{D}\}_{i,i}<1 \textit{ implies } \exists\{n_i\}\ \forall\{n \geq n_i\} \mathscr{D}_\alpha[\hat{\gamma}_4^{(i)}(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n)]>\mathscr{D}_\alpha[\hat{\gamma}_2^{(i)}(\mathbf{S})].$$

The proof follows from (4) of Lemma 1.1, from the Theorem and from Lemma 1.2.

**Corollary 2.** *The implication* $n \uparrow \infty \Rightarrow c_{i,2} \uparrow 1$ & $c_{i,3} \downarrow 0$ *(see* [1] p. 194) *shows the growing importance of the estimator* $\hat{\gamma}_3^{(i)}(\mathbf{S})$ *with the growing number of replication of the experiment and the relation* $\mathscr{D}_\alpha[\hat{\gamma}_2^{(i)}(\mathbf{S})]/\mathscr{D}_\alpha[\gamma_1^{(i)}(\mathbf{Y})] \downarrow 1$. *Following the Theorem these facts are valid for an arbitrary function* $f(\boldsymbol{\Theta})=\boldsymbol{f}'\boldsymbol{\Theta}$, $\boldsymbol{\Theta} \in \boldsymbol{\Theta}$.

**Corollary 3.** *The relations*

$$c_{i,2}=(n-1)/(n-1+d_{i,i})=$$
$$=\{1/\mathscr{D}_\alpha[\hat{\gamma}_2(\mathbf{S})]\}/\langle\{1/\mathscr{D}_\alpha[\hat{\gamma}_2(\mathbf{S})]\}+\{1/\mathscr{D}_\alpha[\hat{\gamma}_3(\bar{y})]\}\rangle$$

*and*

$$c_{i,3}=d_{i,i}/(n-1+d_{i,i})=$$
$$=\{1/\mathscr{D}_\alpha[\hat{\gamma}_3(\bar{\boldsymbol{y}})]\}/\langle\{1/\mathscr{D}_\alpha[\hat{\gamma}_2(\mathbf{S})]\}+\{1/\mathscr{D}_\alpha[\hat{\gamma}_3(\bar{\boldsymbol{y}})]\}\rangle$$

*imply*

$$\mathscr{D}_\alpha[\hat{\gamma}_2(\mathbf{S})]/\mathscr{D}_\alpha[\hat{\gamma}_3(\bar{\boldsymbol{y}})]=c_{i,3}/c_{i,2}=d_{i,i}/(n-1) \downarrow 0.$$

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## ЗАМЕТКА К MINQUE Ц. Р. РАО ДЛЯ ПОВТОРЯЮЩИХСЯ НАБЛЮДЕНИЙ

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Резюме

В работе приводятся необходимые и достаточные условия для равенства между двумя оценками. Первая оценка – MINQUE Рао в повторяющемся регрессионном эксперименте, созданная Клеффе; вторая оценка является оптимальной комбинацией оценок, основанных на матрице Уишарта и на векторе арифметических средних наблюдений.