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DOMINATION IN CUBES

BOHDAN ZELINKA

ABSTRACT. The graph of the $n$-dimensional cube is the graph whose vertex set is the set of all $n$-dimensional Boolean vectors and in which two vertices are adjacent if and only if they differ in exactly one coordinate. In the paper the $k$-domatic number and the edge-domatic number of these graphs are studied.

The graph $Q_n$ of the $n$-dimensional cube is the graph whose vertex set is the set of all $n$-dimensional vectors $(v_1, ..., v_n)$, where $v_i = 0$ or $v_i = 2$ for $i = 1, ..., n$, and in which two vertices are adjacent if and only if they differ exactly in one coordinate.

We shall study the edge-domatic number and the $k$-domatic number of these graphs.

The domatic number of an undirected graph $G$ was introduced by E. Cockayne and S. T. Hedetniemi in [1]. The edge-domatic number and the $k$-domatic number were introduced by the author of this paper in [2] and [3].

A subset $D$ of the vertex set $V(G)$ of a graph $G$ is called dominating if for each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ adjacent to $x$. If $k$ is a positive integer and if for each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ whose distance from $x$ in $G$ is at most $k$, then the set $D$ is called $k$-dominating. If $D$ is a subset of the edge set $E(G)$ of $G$ and for each edge $e \in E(G) - D$ there exists an edge $f \in D$ having a common end vertex with $e$, the set $D$ is called a dominating edge set of $G$.

A partition of $V(G)$, all of whose classes are dominating (or $k$-dominating) sets in $G$, is called a domatic (or $k$-domatic respectively) partition of $G$. A partition of $E(G)$, all of whose classes are dominating sets in $G$, is called an edge-domatic partition of $G$. The maximum number of classes of a domatic (or $k$-domatic, or edge-domatic respectively) partition of $G$ is called the domatic (or $k$-domatic, or edge-domatic respectively) number of $G$. The domatic number of $G$ is denoted by $d(G)$, the $k$-domatic number by $d_k(G)$, the edge-domatic number by $ed(G)$.

In the following the vector $(v_1, ..., v_n)$ will be denoted simply by $v_1 \ldots v_n$. The

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symbol \((v_1 \ldots v_n, v'_1 \ldots v'_n)\) will denote the edge in \(Q_n\) joining the vertices \(v_1 \ldots v_n, v'_1 \ldots v'_n\).

**Theorem 1.** Let \(k, n\) be integers, let \(1 \leq k \leq n\). Then

\[ d_k(Q_n) \geq 2^{k-1}d(Q_{n-k+1}). \]

**Proof.** Denote \(d(Q_{n-k+1}) = p\). Take the cube graph \(Q_{n-k+1}\) and choose a domatic partition \(D\) of it with \(p\) classes \(D_1, \ldots, D_p\). Now let \(M\) be the set of all ordered \(k\)-tuples \((i, h_1, \ldots, h_{k-1})\) of integers, where \(1 \leq i \leq p\) and each \(h_j\) is 0 or 1. The cardinality of \(M\) is \(2^{k-1}p\). Consider the cube graph \(Q_n\). We shall construct a partition \(D^*_1\) of the vertex set of \(Q_n\) whose classes will be \(D^*_m\) for all elements \(m \in M\). Every vertex \(v_1 \ldots v_n\) will be in \(D^*_m\) such that \(m = (v_{n-k+2}, \ldots, v_n)\), where \(i\) is the number such that \(v_1 \ldots v_{n-k+1} \in D_i\) in \(Q_{n-k+1}\). We shall prove that \(D^*_1\) is a \(k\)-domatic partition of \(Q_n\). Let \(m = (i, h_1, \ldots, h_{k-1}) \in M\) and let \(v = v_1 \ldots v_n\) be a vertex of \(Q_n\). Suppose that \(v \notin D_m\). Let \(w = v_1 \ldots v_{n-k+1}h_1 \ldots h_{k-1};\) this is a vertex of \(Q_n\). As the vectors \(v, w\) differ in at most \(k - 1\) coordinates, their distance in \(Q_n\) is at most \(k - 1\). In \(Q_{n-k+1}\) consider the vertex \(v' = v_1 \ldots v_{n-k+1}\). This vertex either belongs to \(D_i\) or is adjacent to a vertex \(z' \in D_i\) in \(Q_{n-k+1}\), because \(D_i\) is a dominating set in this graph. In the first case \(v\) has the distance at most \(k - 1\) from a vertex of \(D_m\) in \(Q_n\), namely the vertex \(v_1 \ldots v_{n-k+1}h_1 \ldots h_{k-1}\). In the second case let \(z\) be the vector obtained from \(z'\) by adding \(k - 1\) coordinates \(h_1, \ldots, h_{k-1}\) after the coordinates of \(z'\). The vertices \(w, z\) are adjacent in \(Q_n\) and therefore the distance between \(v\) and \(z\) is at most \(k\), while \(z \in D_m\). The set \(D_m\) is dominating in \(Q_n\). As \(m\) was chosen arbitrarily, \(D^*_1\) is a \(k\)-domatic partition of \(Q_n\) with \(2^{k-1}p = 2^{k-1}d(Q_{n-k+1})\) classes, which implies the assertion. \(\square\)

It was proved in [4] that if \(n = 2^s\), where \(s\) is a positive integer, then \(d(Q_{n-1}) = d(Q_n) = n\). We have a corollary.

**Corollary.** Let \(s, k\) be positive integers, let \(n = 2^s + k\). Then \(d_j(Q_{n-2}) \geq 2^{s+k-1}\).

**Theorem 2.** Let \(n\) be a positive integer divisible by 3. Then

\[ ed(Q_n) \geq 4n/3. \]

**Proof.** First consider \(n = 3\). There exists an edge-domatic partition of \(Q_3\) consisting of the set \{\((000, 100), (010, 011), (101, 111)\}\} and the sets obtained from it by the iterations of the permutation given by \(000 \rightarrow 100 \rightarrow 110 \rightarrow 010 \rightarrow 000, 001 \rightarrow 101 \rightarrow 111 \rightarrow 011 \rightarrow 001\). (In geometry this permutation is the 90° rotation of the cube around its vertical axis.) This set has \(4n/3 = 4\) elements. Now consider the cube graph \(Q_n\), where \(n\) is divisible by 3 and \(n \geq 6\). For \(i = 1, \ldots, n/3\) let \(F_i\) be the set of edges which join vertices differing in the \((3i - 2)\)-th, the \((3i - 1)\)-th or the \(3i\)-th coordinate. The sets \(F_1, \ldots, F_{n/3}\) form a
partition of \( E(Q_n) \). The spanning subgraph of \( Q_n \) having the edge set \( F \) is a graph having \( 2^n-3 \) connected components which are all isomorphic to \( K_3 \); denote this graph by \( H_f \). The vertex set of each connected component of \( H_f \) consists of vertices for which the coordinates \( v_4, \ldots, v_n \) are the same. We shall call such a component even (or odd) if among the coordinates \( v_4, \ldots, v_n \) there is an even (odd, respectively) number of those which are equal to 1. In each even component of \( H_f \) we take the set of edges \((000v_4 \ldots v_n, 100v_4 \ldots v_n), (010v_4 \ldots v_n, 011v_4 \ldots v_n), (101v_4 \ldots v_n, 111v_4 \ldots v_n)\), in each odd component of \( H_f \) we take the set of edges \{(100v_4 \ldots v_n, 110v_4 \ldots v_n), (000v_4 \ldots v_n, 001v_4 \ldots v_n), (011v_4 \ldots v_n, 111v_4, \ldots v_n)\}. Let \( D \) be the union of all these sets for all connected components of \( H_f \).

Consider the set \( M \) of vertices of \( Q_n \) which are incident with no edge of \( D \). It consists of all vertices \( 001v_4, \ldots, v_n, 110v_4 \ldots v_n, \) where the number of coordinates equal to 1 among \( v_4, \ldots, v_n \) is even, and \( 010v_4 \ldots v_n, 101v_4 \ldots v_n, \) where this number is odd. It is easy to see that \( M \) is an independent set in \( Q_n \). Hence each edge of \( Q_n \) is incident with at most one vertex of \( M \) and with at least one vertex of \( V(Q_n) - M \). This implies that each edge of \( Q_n \) either belongs to \( D \) or has a common end vertex with an edge of \( D \) and thus \( D \) is a dominating set in \( Q_n \). We use the permutation given by \( 000v_4 \ldots v_n \mapsto 100v_4 \ldots v_n \mapsto 110v_4 \ldots v_n \mapsto 010v_4 \ldots v_n \mapsto 000v_4 \ldots v_n, 001v_4 \ldots v_n \mapsto 101v_4 \ldots v_n \mapsto 111v_4 \ldots v_n \mapsto 011v_4 \ldots v_n \mapsto 001v_4 \ldots v_n \) for any values of \( v_4, \ldots, v_n \). By this permutation and its iterations from \( D \) we obtain four pairwise disjoint dominating edge sets in \( Q_n \) (including \( D \) itself). Instead of \( H_f \) we may take other \( H_f \) and proceed analogously. In this way we obtain an edge-domatic partition of \( Q_n \) with \( 4n/3 \) classes, which implies the assertion. \( \Box \)

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