Michal Fečkan
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THE RELATION BETWEEN A FLOW AND ITS DISCRETIZATION

MICHAL FEČKAN

ABSTRACT. It is proved that the $h$-time map of a hyperbolic flow and its $h$-discretization are uniformly topologically conjugate for each small positive $h$.

Introduction. Let $\Phi(t, x)$ be the flow generated by the equation

$$x' = Ax + g(x),$$

(1)

where $A \in \mathcal{L}(\mathbb{R}^m)$, $A$ has no eigenvalues on the imaginary axis, $g \in C^1(\mathbb{R}^m, \mathbb{R}^m)$, $g(0) = 0$, $\sup |g| < \infty$, $|Dg(x)| \leq b$ for each $x \in \mathbb{R}^m$ and $b$ sufficiently small. The equation (1) has the discretization

$$x_{n+1} = x_n + h \cdot Ax_n + h \cdot g(x_n), \quad h \neq 0$$

which gives us the mapping

$$G(h, x) = x + h \cdot Ax + h \cdot g(x).$$

(2)

It is not difficult to see that $I + h \cdot A$ has no eigenvalues on the unit circle for each small $h \neq 0$. Hence, if moreover $g(x) = o(|x|)$ as $x \to 0$, then the mapping (2) has local stable and unstable manifolds $W_h^s$, $W_h^u$ for the fixed point 0, respectively. Recently the author of this paper [1] has shown that the manifolds $W_h^s$, $W_h^u$ tend to $W^s$, $W^u$ as $h \to 0$, $h > 0$, where $W^s$, $W^u$ are local stable, unstable manifolds of (1) for the fixed point 0, respectively.

The purpose of this paper is to show that the mapping $\Phi(h, \cdot)$ and $G(h, \cdot)$ are uniformly topologically conjugate for each small positive $h$, i.e. the following theorem holds:


Key words: Discretization, Dynamical systems, Hartman-Grobman theorem.
Theorem 1. For sufficiently small $b$ and a compact set $K \subset \mathbb{R}^m$ there is a number $\delta > 0$ and a $C^0$-mapping
\[ H : (0, \delta) \to C^0(\mathbb{R}^m, \mathbb{R}^m) = \{ f : \mathbb{R}^m \to \mathbb{R}^m, \text{ f is continuous} \} \]
such that
\[ \Phi(h, \cdot) \cdot H(h, \cdot) = H(h, \cdot) \cdot G(h, \cdot) \quad \text{on K} \]
and
\begin{itemize}
  \item[i)] $H(h, \cdot)$ is a homeomorphism,
  \item[ii)] $\sup_{(0, \delta) \times K} |H(\cdot, \cdot)| < \infty$, $\sup_{(0, \delta) \times K} |H^{-1}(\cdot, \cdot)| < \infty$.
\end{itemize}
If $K = B_q = \{ x, |x| \leq q \}$ for $q$ large, then $B_{q/2} \subset \bigcap_{(0, \delta)} H(\cdot, K)$.

Proof. We divide the proof into several steps.

Step 1.
By the Hartman-Grobman theorem [2, p. 115] there is an $H_1 \in C^0_B(\mathbb{R}^m, \mathbb{R}^m) = \{ f \in C(\mathbb{R}^m, \mathbb{R}^m), f \text{ is bounded, i.e. } \sup |f| < \infty \}$ such that
\[ \Phi(h, \cdot) \cdot (I + H_1) = (I + H_1) \cdot e^{h \cdot A} \]
and $(I + H_1)^{-1} = I + H_1$ for some $\bar{H}_1 \in C^0_B(\mathbb{R}^m, \mathbb{R}^m)$.
Let $E^s$, $E^u$ be stable and unstable subspaces of $A$, respectively.

Step 2.
Lemma 2. There is a $\delta_1 > 0$ and a $C^0$-mapping
\[ H_3 : (0, \delta_1) \to C^0_B(\mathbb{R}^m, \mathbb{R}^m) \]
such that
\[ (I + H_3(h, \cdot)) \cdot (I + h \cdot A) = G(h, \cdot) \cdot (I + H_3(h, \cdot)), \quad (3) \]
where $I + H_3(h, \cdot)$ is a homeomorphism for each $h \in (0, \delta_1)$ and $(I + H_3(h, \cdot))^{-1} = I + \bar{H}_3(h, \cdot)$, $\bar{H}_3(h, \cdot) \in C^0_B(\mathbb{R}^m, \mathbb{R}^m)$. Moreover
\[ \sup_{(0, \delta_1) \times \mathbb{R}^m} |H_3(\cdot, \cdot)| < \infty, \quad \sup_{(0, \delta_1) \times \mathbb{R}^m} |\bar{H}_3(\cdot, \cdot)| < \infty. \]

Proof of Lemma 2. We shall follow [2, Theorem 5.15.]. We can rewrite the equation (3) in the form
\[ H_3^s = (I + hA)^s \cdot H_3^s \cdot (I + hA)^{-1} + h \cdot g^s \cdot (I + H_3) \cdot (I + hA)^{-1} \]
\[ H_3^u = ((I + hA)^u)^{-1} \cdot H_3^u \cdot (I + hA) - h \cdot ((I + hA)^u)^{-1} \cdot g^u \cdot (I + H_3), \quad (4) \]
THE RELATION BETWEEN A FLOW AND ITS DISCRETIZATION

where for any mapping \( S: \mathbb{R}^m \to \mathbb{R}^m \) we write \( S^s = P_s S, S^u = P_u S \) and \( P_u, P_s \) are projections to \( E^u, E^s \), respectively. We solve (4) in the space \( C_B^0(\mathbb{R}^m, \mathbb{R}^m) \). It is clear that the mapping

\[
T_h: C_B^0(\mathbb{R}^m, \mathbb{R}^m) \to C_B^0(\mathbb{R}^m, \mathbb{R}^m)
\]

\[
T_h(H) = \left((I + hA)^s \cdot H^s \cdot (I + hA)^{-1}, ((I + hA)^u)^{-1} \cdot H^u \cdot (I + hA)\right)
\]

has the property

\[
|T_h(H) - T_h(F)| \leq (1 - c \cdot h) \cdot |H - F|
\]  

(5)

for some constant \( c > 0 \), small positive \( h \) and each \( H, F \in C_B^0(\mathbb{R}^m, \mathbb{R}^m) \). Indeed, we can choose norms \( \| \cdot \|_1, \| \cdot \|_2 \) on the space \( E^s, E^u \) respectively [3, p. 145] such that

\[
\|(I + h \cdot A)^s\|_1 \leq (1 - h \cdot c)
\]

\[
\|(I + h \cdot A)^u\|_2 \leq (1 - h \cdot c)
\]

for each small positive \( h \) and we put

\[
|f| = \sup_{\mathbb{R}^m}(\|f^s(\cdot)\|_1 + \|f^u(\cdot)\|_2)
\]

for each \( f \in C_B^0(\mathbb{R}^m, \mathbb{R}^m) \).

Hence (4) has the form

\[
H = T_h(H) + h \cdot F_h(H),
\]

where \( F_h(H) = \left(g^s \cdot (I + H) \cdot (I + hA)^{-1}, -(I + hA)^{-1} \cdot g^u \cdot (I + H)\right) \).

Thus

\[
H = h \cdot (I - T_h)^{-1} \cdot F_h(H).
\]  

(6)

Since by (5) and the Banach fixed point theorem

\[
|(I - T_h)^{-1}| \leq \frac{c}{h}
\]

we can apply uniformly the implicit function theorem to (6) for each small positive \( h \). (Note that \( b \) is sufficiently small). Hence (3) has a unique solution.

On the other hand, let us consider the equation

\[
(I + H) \cdot (I + h \cdot A + h \cdot g) = (I + h \cdot A) \cdot (I + H)
\]

125
which is equivalent to

\[ H^* - (I + hA)^* \cdot H^* \cdot (I + hA + hg)^{-1} = -hg^* \cdot (I + hA + hg)^{-1} \]
\[ H^u - ((I + hA)^u)^{-1} \cdot H^u \cdot (I + hA + hg) = h \cdot ((I + hA)^u)^{-1} \cdot g^u. \]

Since this equation is similar to \((4)\) we obtain by the above results that this equation has a unique solution \(H(h, \cdot) \in C_B^0(\mathbb{R}^m, \mathbb{R}^m)\) for each small positive \(h\). Using a standard procedure [2, Theorem 5.15] we have \(I + H = (I + H)^{-1}\), where \(H\) is a solution of \((4)\). This gives us the proof of Lemma 2.

**Step 3.**

In the last step we try to find a homeomorphism \(I + H_4(h, \cdot) : \mathbb{R}^m \to \mathbb{R}^m\) such that

\[ e^{h \cdot A} (I + H_4(h, \cdot)) = (I + H_4(h, \cdot)) \cdot (I + h \cdot A) \quad \text{on} \quad K \subset \mathbb{R}^m \quad (7) \]

for \(h > 0\) small and \(H_4(h, \cdot) \in C_B^0(\mathbb{R}^m, \mathbb{R}^m)\), \(K\) is a compact set. Since

\[ e^{h \cdot A} = I + h \cdot A + f(h \cdot A), \]

where \(f(x) = e^x - 1 - x\), we have \(f(h \cdot A) = O(h^2)\) as \(h \to 0\). Without loss of generality we can suppose \(K = B_q\) for \(q\) large. Since \(f(h \cdot A) \notin C_B^0(\mathbb{R}^m, \mathbb{R}^m)\), we modify \(f(h \cdot A)\) in the following way

\[ \tilde{f}(h, x) = s(x) \cdot f(h \cdot A)x, \]

where \(s\) is a function having the property

i) \(s \in C^\infty\)

ii) \(s = 1\) on \(B_L\)

iii) \(s = 0\) on \(B_{2L}\)

for \(L \gg q\) sufficiently large. Thus a modified equation of \((7)\) has the form

\[ (I + h \cdot A + \tilde{f}(h, \cdot))(I + H_4(h, \cdot)) = (I + H_4(h, \cdot))(I + h \cdot A). \quad (8) \]

To solve \((8)\) we follow the above step 2. Hence \((8)\) has the form

\[ H_4 = T_h(H_4) + O(h^2) \]

and

\[ H_4 = (I - T_h)^{-1} \cdot O(h^2). \]
THE RELATION BETWEEN A FLOW AND ITS DISCRETIZATION

We see that $H_4$ exists for each small positive $h$ and $H_4(h,\cdot) \to 0$ as $h \to 0$. It follows also by the step 2 that

$$(I + H_4(h,\cdot))^{-1} = I + \tilde{H}_4(h,\cdot), \quad \tilde{H}_4(h,\cdot) \in C^0_B(\mathbb{R}^m, \mathbb{R}^m)$$

and $\tilde{H}_4(h,\cdot) \to 0$ as $h \to 0$.

Summing up we see that

$$(I + H_1(\cdot)) \cdot (I + H_4(h,\cdot)) \cdot (I + \tilde{H}_3(h,\cdot))$$

is the desired mapping $H(h,\cdot)$ satisfying

$$\Phi(h,\cdot) \cdot H(h,\cdot) = H(h,\cdot) \cdot G(h,\cdot) \quad \text{on} \quad K. \quad \quad (9)$$

Indeed, since $L \gg q$ is large, $H_4(h,\cdot) = O(h)$, $\tilde{H}_3(h,\cdot)$ is bounded and $h$ is small we have

$$(I + H_4(h,\cdot)) \cdot (I + \tilde{H}_3(h,\cdot)) K \subset B_L,$$

and thus $\tilde{f}(h,\cdot) = f(h\cdot A)$ on $(I + H_4(h,\cdot)) \cdot (I + \tilde{H}_3(h,\cdot)) K$. Moreover, $H_1(\cdot)$ is also bounded on $\mathbb{R}^m$. These facts imply both

$$B_{\delta/2} \subset \bigcap_{(0,\delta)} H(\cdot, K)$$

for $\delta$ small and (9).

REFERENCES


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Matematický ústav SAV
Štefánikova 49
814 73 Bratislava