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A COMPLETE METRIC ON THE SPACE
OF INTEGRABLE MULTIFUNCTIONS

DUŠAN HOLÝ

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ABSTRACT. The notion of a multivalued integral was introduced by Aumann and the notion of an integrable multifunction (which we use) by Hai. We find a complete metric on the space of integrable multifunctions with values in a Banach separable space.

1. Introduction

The notion of an integral for a multivalued function was introduced by Aumann. The convergence theorems for multivalued integrals were discussed by Aumann [A], Schmeidler [S], and Arzstein [Ar]. These authors aimed Fatou's lemma and Lebesgue's convergence theorem with the Kuratowski convergence for measurable multivalued functions having values in the closed subsets of \( \mathbb{R}^n \). Fatou's lemma is of some use in mathematical economics [S].

Hai [Hi] studies integrable multivalued functions with values in a Banach separable space. He proved Fatou's lemmas and Lebesgue's convergence theorems for multivalued integrals mainly with the Mosco convergence but in the exive spaces.

We find a complete metric on the space of integrable multifunctions with values in a Banach separable space, which can be a useful tool in integration theory.

2. Definitions and some elementary properties

Throughout the paper, \( \Omega \) will denote a measurable space with \( \sigma \)-algebra \( \mathcal{A} \). If there is a \( \sigma \)-finite measure defined on \( \mathcal{A} \), we say that \( \Omega \) is \( \sigma \)-finite. If there is a complete \( \sigma \)-finite measure defined on \( \mathcal{A} \), we call \( \Omega \) complete.

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$Y$ will be a topological space, $2^Y$, the space of all subsets of $Y$. Following Bourbaki, we will call $Y$: Polish, if $Y$ is separable and metrizable by a complete metric, Souslin, if $Y$ is metrizable and a continuous image of a Polish space.

A relation $F: \Omega \rightarrow Y$ is a subset of $\Omega \times Y$. Alternatively, $F$ may be regarded as a function from $\Omega$ to $2^Y$. A function $F: \Omega \rightarrow 2^Y - \{\emptyset\}$ is called a multifunction.

Let $F: \Omega \rightarrow Y$ be a relation and $B \subset Y$. Denote

$$F^{-1}(B) = \{ \omega \in \Omega : F(\omega) \cap B \neq \emptyset \} .$$

A relation $F: \Omega \rightarrow Y$ is measurable (weakly measurable) if and only if $F^{-1}(B)$ is measurable for each closed (open) subset $B$ of $Y$. We say that $F$ is graph measurable if

$$\text{Gr} F = \{(\omega, y) \in \Omega \times Y : y \in F(\omega)\} \in \mathcal{A} \times \mathcal{B} ,$$

where $\mathcal{B}$ is the $\sigma$-algebra of Borel subsets of $Y$, and $\mathcal{A} \times \mathcal{B}$ is understood in the usual sense.

Further we mention some properties from the papers [H], [W]:

We say that $\{f_n\}_{n \in \mathbb{Z}^+}$ is a Castaing representation of $F$ if, for all $n \in \mathbb{Z}^+$, $f_n$ is a measurable selector of $F$, and for all $\omega \in \Omega$

$$F(\omega) \subset \text{cl}\left\{ \bigcup_{n \geq 1} \{f_n(\omega)\} \right\} .$$

From [W; Theorem 5.10], we know that if $(\Omega, \mathcal{A})$ is a measurable space with $\mathcal{A}$ a Souslin family, $Y$ is a Souslin space and $F$ is a graph measurable multifunction, then $F$ admits a Castaing representation. Notice that $\mathcal{A}$ is a Souslin family ([KN]) if $\mathcal{A} = S(\mathcal{A})$, where $S(\mathcal{A})$ denotes the family of all sets obtained from $\mathcal{A}$ by the Souslin operation. In case that there is a $\sigma$-finite complete measure defined on the $\sigma$-algebra $\mathcal{A}$, $\mathcal{A}$ is a Souslin family ([KN]).

Further we will need the following proposition:

**Proposition A.** ([H]) Let $J$ be an at most countable set, and let $F_n: \Omega \rightarrow Y$ be a relation for each $n \in J$. Then if each $F_n$ is measurable (weakly measurable), so is the relation $\bigcup F_n: \Omega \rightarrow Y$ defined by $\left(\bigcup F_n\right)(\omega) = \bigcup F_n(\omega)$.

**Proposition B.** ([H]) A relation $F: \Omega \rightarrow Y$ is weakly measurable if and only if the relation $\text{cl} F: \Omega \rightarrow Y$, defined by $\text{cl} F(\omega) = \text{cl}\{F(\omega)\}$, is weakly measurable.

Let $F: \Omega \rightarrow Y$ be a relation and $B \subset Y$. Besides the notion $F^{-1}(B)$, we need also the notion of $F^+(B) = \{ \omega \in \Omega ; F(\omega) \subset B \}$.
3. Main results

**Definition 3.1.** ([HU]) Let \((\Omega, \mathcal{A})\) be complete. Let \(Y\) be a Banach separable space. Let \(F: \Omega \to Y\) be a multifunction with a measurable graph, such that there is an integrable function \(f: \Omega \to \mathbb{R}\) with the following property

\[ \forall \omega \in \Omega \quad \|F(\omega)\| \leq f(\omega), \]

(i.e. \(\|y\| \leq f(\omega)\) for all \(y \in F(\omega)\), where \(\|y\|\) is a norm of \(y\)).

Then we call \(F\) an integrable multifunction.

**Remark 3.2.** The assumptions of Definition 3.1 guarantee the existence of a Castaing representation of \(F\).

**Definition 3.3.** Let \(\Omega\) and \(Y\) be as in Definition 3.1. Denote by \(\mathcal{L}\) the space of all integrable multifunctions from \(\Omega\) to \(Y\). Define the function \(L: \mathcal{L} \times \mathcal{L} \to \mathbb{R}\) as follows:

\[
L(F, G) = \inf \left\{ \varepsilon : \text{for every measurable selector } f \text{ of } F \right. \\
\left. \quad \text{there exists a measurable selector } g \text{ of } G \text{ such that } \right. \\
\left. \quad \int_{\Omega} |f(\omega) - g(\omega)| \, d\mu \leq \varepsilon \quad \text{and } \right. \\
\left. \quad \text{for every measurable selector } g \text{ of } G \right. \\
\left. \quad \text{there exists a measurable selector } f \text{ of } F \text{ such that } \right. \\
\left. \quad \int_{\Omega} |g(\omega) - f(\omega)| \, d\mu \leq \varepsilon \right\}.
\]

This definition is a generalization of the definition introduced in [M].

What is a motivation for this definition? We show that a motivation for this definition is the Hausdorff metric. Since we will work with this notion further, we briefly mention some properties of this metric.

Let \((W, p)\) be a metric space. Denote \(B_\varepsilon[v] = \{z \in W : p(z, v) < \varepsilon\}\). If \(K\) is a subset of \(W\) and \(\varepsilon > 0\), let \(B_\varepsilon[K]\) denote the union of all open \(\varepsilon\)-balls whose centers run over \(K\). If \(K_1\) and \(K_2\) are nonempty subsets of \(W\) and, for some \(\varepsilon > 0\), both \(B_\varepsilon[K_1] \supseteq K_2\) and \(B_\varepsilon[K_2] \supseteq K_1\), we define the Hausdorff distance \(h_p\) between them to be

\[
h_p(K_1, K_2) = \inf \left\{ \varepsilon : B_\varepsilon[K_1] \supseteq K_2 \text{ and } B_\varepsilon[K_2] \supseteq K_1 \right\}.
\]

Otherwise, we write \(h_p(K_1, K_2) = \infty\). It is easy to check that \(h_p\) defines an infinite-valued pseudometric on the nonempty subset of \(W\), and that \(h_p(K_1, K_2) = 0\) if and only if \(K_1\) and \(K_2\) have the same closure. Thus, if we restrict \(h_p\) to closed subsets of \(W\), then \(h_p\) defines an infinite valued metric on such sets.
In the sequel, we shall denote the set of closed nonempty subsets of a metric space $W$ by $\text{CL}(W)$. If $(W,p)$ is complete, then so is $(\text{CL}(W), h_p)$.

If $(W,p)$ is a pseudometric space, we can also define the function $h_p$ on all nonempty subsets of $W$. Clearly $h_p$ is also a pseudometric.

In what follows, let $Y$ be a separable Banach space with norm $\|\cdot\|$. To simplify notation, we shall sometimes denote the norm on $Y$ by $|\cdot|$, rather than $\|\cdot\|$.

Put further $\varrho(x,y) = \|x-y\|$, $\varrho(x,A) = \inf\{\varrho(x,a) : a \in A\}$, and $\varrho(A,x) = \inf\{\varrho(a,x) : a \in A\}$ for a nonempty subset $A$ of $Y$. Further denote by $h_{||\cdot||}$ the Hausdorff metric on $\text{CL}(Y)$ induced by $\varrho$.

Let $\mathcal{B}$ denote the $\sigma$-algebra of Borel subsets of $Y$, and $(\Omega, \mathcal{A})$ be a measurable space. A function $f : \Omega \rightarrow Y$ is measurable if it is measurable with respect to $\mathcal{A}$ and $\mathcal{B}$.

It is easy to see that if $f$ is measurable with respect to $\mathcal{A}$ and $\mathcal{B}$, then $\omega \rightarrow |f(\omega)|$ is $\mathcal{A}$-measurable.

In our paper, we need the notion of an integrable function. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space, and let $Y$ be a Banach separable space. A function $f : \Omega \rightarrow Y$ is integrable if it is measurable and the function $\omega \rightarrow |f(\omega)|$ is integrable.

Let $\mathcal{I}(\Omega, \mathcal{A}, \mu, Y)$ be the set of all integrable functions from $\Omega$ to $Y$. Then $\mathcal{I}(\Omega, \mathcal{A}, \mu, Y)$ is a vector space. The formula

$$\|f\| = \int_{\Omega} |f(\omega)| \, d\mu$$

induces a seminorm on $\mathcal{I}(\Omega, \mathcal{A}, \mu, Y)$, and clearly

$$d(f,g) = \int_{\Omega} |f(\omega) - g(\omega)| \, d\mu$$

induced a pseudometric on $\mathcal{I}(\Omega, \mathcal{A}, \mu, Y)$.

Let $(\Omega, \mathcal{A}, \mu)$ be a complete space, and let $(Y, \mathcal{B})$ be a Banach separable space. Let $F : \Omega \rightarrow Y$ be an integrable multifunction. Put

$$S_F = \{f \in \mathcal{I}(\Omega, \mathcal{A}, \mu, Y) : f(\omega) \in F(\omega) \text{ almost everywhere}\}.$$  

Then $S_F \neq \emptyset$, and $S_F$ is a closed set in $(\mathcal{I}(\Omega, \mathcal{A}, \mu, Y), d)$ for every multifunction $F$ with closed values.

We can identify $F$ with $S_F$. Let $F$, $G$ be two integrable multifunction. It is easy to verify that

$$L(F,G) = h_d(S_F, G_F).$$

If $F : \Omega \rightarrow Y$ is an integrable multifunction, then the integral or mean $E[F]$ of $F$ is defined by

$$E[F] = \int_{\Omega} F(\omega) \, d\mu = \left\{ E(f) = \int_{\Omega} f(\omega) \, d\mu : f \in S_F \right\},$$

where $E(f)$ denotes the mean of $f$.  

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where \( E[f] = \int_{\Omega} f(\omega) \, d\mu \) is the usual Bochner integral. This multivalued integral was introduced by Áumann [A].

It is easy to verify that if \( F, G \) are two integrable multifunctions, then

\[
h_{|\cdot|}(E[F], E[G]) \leq L(F, G).
\]

**Theorem 3.4.** The function \( L: \mathcal{L} \times \mathcal{L} \to \mathbb{R} \) defined in the Definition 3.3 is a pseudo-metric.

**Proof.** The proof is similar as in [M].

**Theorem 3.5.** Let \((\Omega, \mathcal{A})\) be complete and let \( Y \) be a Banach separable space. Let \( F, G \) be integrable multifunctions from \( \Omega \) to \( Y \). Then \( L(F, G) = 0 \) if and only if \( \text{cl}\{F(\omega)\} = \text{cl}\{G(\omega)\} \) almost everywhere.

**Proof.**

\( \Rightarrow \) : Denote by \( \text{CL}(Y) \) the space of all nonempty closed subsets of \( Y \) and \( h_{|\cdot|} \) the Hausdorff metric on \( \text{CL}(Y) \). Let \( \mu \) be a complete \( \sigma \)-finite measure on \( \mathcal{A} \). We prove that

\[
\left\{ \omega \in \Omega : h_{|\cdot|}(\text{cl}\{F(\omega)\}, \text{cl}\{G(\omega)\}) > 0 \right\}
\]

is a measurable set with measure zero.

Let \( \varepsilon > 0 \). It is easy to verify that

\[
\left\{ \omega \in \Omega : h_{|\cdot|}(\text{cl}\{F(\omega)\}, \text{cl}\{G(\omega)\}) > \varepsilon \right\}
\]

\[
= \left( \bigcup_{n} \bigcup_{k} \left( \text{cl}^{-1}\left(B_{1/k}[y_n]\right) \cap \text{cl}G^+(Y \setminus B_{\varepsilon+1/k}[y_n]) \right) \right)
\]

\[
\cup \left( \bigcup_{n} \bigcup_{k} \left( \text{cl}^{-1}\left(B_{1/k}[y_n]\right) \cap \text{cl}F^+(Y \setminus B_{\varepsilon+1/k}[y_n]) \right) \right),
\]

where \( \{y_n : n \in \mathbb{Z}^+\} \) is a countable dense set in \( Y \). Thus

\[
\left\{ \omega \in \Omega : h_{|\cdot|}(\text{cl}\{F(\omega)\}, \text{cl}\{G(\omega)\}) > 0 \right\}
\]

is measurable.

Now we show that \( \mu\left\{ \omega \in \Omega : h_{|\cdot|}(\text{cl}\{F(\omega)\}, \text{cl}\{G(\omega)\}) > \varepsilon \right\} = 0 \) for every \( \varepsilon > 0 \). Let \( \varepsilon > 0 \). Put

\[
A_\varepsilon = \bigcup_{n} \bigcup_{k} \left( \text{cl}^{-1}\left(B_{1/k}[y_n]\right) \cap \text{cl}G^+(Y \setminus B_{\varepsilon+1/k}[y_n]) \right),
\]

and

\[
B_\varepsilon = \bigcup_{n} \bigcup_{k} \left( \text{cl}^{-1}\left(B_{1/k}[y_n]\right) \cap \text{cl}F^+(Y \setminus B_{\varepsilon+1/k}[y_n]) \right).
\]
Suppose \( \mu(A_\varepsilon \cup B_\varepsilon) > 0 \). Then either \( \mu(A_\varepsilon) > 0 \) or \( \mu(A_\varepsilon) > 0 \). Without loss of generality we can suppose that \( \mu(A_\varepsilon) > \delta \).

Define a function \( f: \Omega \times Y \to \mathbb{R} \) by \( f(\omega, y) = \varrho(\text{cl}\{G(\omega)\}, y) \). The function \( f \) is measurable in \( \omega \) for each \( y \in Y \) ([H]) and continuous in \( y \) for every \( \omega \in \Omega \). Thus \( f \) is measurable ([H]), i.e. the set \( C = \{ (\omega, y) : \varrho(\text{cl}\{G(\omega)\}, y) \geq \varepsilon \} \) is measurable. Put further \( D = C \cap \text{Gr} F \). Then the set \( P_{\Omega}(D) \) contains \( A_\varepsilon \), where \( P_{\Omega}(\omega, y) = \omega \) for every \( (\omega, y) \).

Now define the following set \( E \subset \Omega \times Y \):

\[
E = \{ (\omega, y) : (\omega, y) \in D \text{ and } \omega \in A_\varepsilon \} \cup \{ (\omega, y) : (\omega, y) \in \text{Gr} F \text{ and } \omega \notin A_\varepsilon \}.
\]

Further define a multifunction \( K: \Omega \to Y \) by

\[
K(\omega) = E_\omega = \{ y \in Y : (\omega, y) \in E \}.
\]

Clearly the multifunction \( K \) has a measurable graph and \( \text{Gr} K \subset \text{Gr} F \). The assumptions of the theorem guarantee the existence of a Castaing representation \( \{ k_n \}_{n \in \mathbb{Z}^+} \) of \( K \).

Let \( k_n \) be a measurable selector of \( K \) from the Castaing representation of \( K \), and let \( g \) be a measurable selector of a multifunction \( G \). Then we have

\[
\int_{\Omega} |k_n(\omega) - g(\omega)| \, d\mu = \int_{\Omega \setminus A_\varepsilon} |k_n(\omega) - g(\omega)| \, d\mu + \int_{A_\varepsilon} |k_n(\omega) - g(\omega)| \, d\mu > \delta \cdot \varepsilon,
\]

and that is a contradiction.

\( \iff \): Let \( f \) be a selector of \( F \). We show that for every \( \varepsilon > 0 \) there is a selector \( g \) of \( G \) such that \( \int_{\Omega} |f(\omega) - g(\omega)| \, d\mu < \varepsilon \). The multifunctions \( F \) and \( G \) are integrable, and \( \text{cl} F = \text{cl} G \) almost everywhere. Thus there is an integrable function \( h: \Omega \to \mathbb{R} \) such that \( \| \text{cl}\{F(\omega)\} \| \leq h(\omega) \) and \( \| \text{cl}\{G(\omega)\} \| \leq h(\omega) \).

There is a measurable set \( A \) such that \( \mu(A) < \infty \) and \( \int_{\Omega \setminus A} h(\omega) \, d\mu < \frac{\varepsilon}{6} \).

Put

\[
M = \left\{ (\omega, y) : \varrho(f(\omega), y) = \frac{\varepsilon}{6\mu(A)} \right\}.
\]

Then \( M \) is a measurable set. Put \( N = M \cap \text{Gr} G \) and define a multifunction \( K: \Omega \to Y \) by

\[
K(\omega) = N_\omega = \{ y \in Y : (\omega, y) \in N \}.
\]
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There is a Castaing representation of $K$. Let $g^*$ be a function from the Castaing representation of $K$. Then we have:

$$\int_{\Omega} |f(\omega) - g^*(\omega)| \, d\mu = \int_{\Omega \setminus A} |f(\omega) - g^*(\omega)| \, d\mu + \int_{A} |f(\omega) - g^*(\omega)| \, d\mu$$

$$< \int_{\Omega \setminus A} |2h(\omega)| \, d\mu + \int_{A} |f(\omega) - g^*(\omega)| \, d\mu$$

$$\leq \frac{2\varepsilon}{6} + \frac{2\varepsilon\mu(A)}{6\mu(A)} < \varepsilon.$$

On the space $\mathcal{L}$, define a relation $\approx$ by $F \approx G$ if and only if $\text{cl}\{F(\omega)\} = \text{cl}\{G(\omega)\}$ almost everywhere. Let $\mathcal{L}_1$ be a space of all integrable multifunctions with closed values; put $\mathcal{L}^\sim = \mathcal{L}_1/\approx$ and define

$$L^\sim : \mathcal{L}^\sim \times \mathcal{L}^\sim \to \mathbb{R} \quad \text{by} \quad L^\sim(F^\sim, G^\sim) = L(F_1, G_1),$$

where $F_1, G_1 \in \mathcal{L}_1$ and $F^\sim, G^\sim \in \mathcal{L}^\sim$. The standard proof of [K] shows that $L^\sim$ is well defined and $L^\sim$ is a metric on $\mathcal{L}^\sim$.

**Theorem 3.6.** Let $(\Omega, A)$ be complete, and let $Y$ be a Banach separable space. Then the space $(\mathcal{L}^\sim, L^\sim)$, defined as above, is complete.

**Proof.** Let $\{F_n\}_{n \in \mathbb{Z}^+}$ be a Cauchy sequence from $\mathcal{L}^\sim$. Without loss of generality we can suppose that for every $n \in \mathbb{Z}^+$ is

$$L^\sim(F_n^\sim, F_{n+1}^\sim) < \frac{1}{2n+1}.$$

For every $n \in \mathbb{Z}^+$ choose $F_n \in F_n^\sim$. Clearly

$$L(F_n, F_{n+1}) < \frac{1}{2n+1}$$

for every $n \in \mathbb{Z}^+$.

Let $n \in \mathbb{Z}^+$ and let $\{f_{n,l}\}_{l \in \mathbb{Z}^+}$ be a Castaing representation of $F_n$. For every selector $f_{n,l}$ of $F_n$, we choose a $d$-Cauchy sequence $\{f_{n,l,p}\}_{p \geq n}$ ($d(f, g) = \int |f - g| \, d\mu$) in the following way:

Let $f_{n,l,p}$ be a selector of $F_p$ such that

$$\int_{\Omega} |f_{n,l,p}(\omega) - f_{n,l,p+1}(\omega)| \, d\mu < \frac{1}{2^p}.$$

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For every sequence \( \{f_{n,i,p}\}_{p>n} \) there is a measurable function \( \bar{f}_{n,l} \) such that \( \{f_{n,i,p}\}_{p>n} \) d-converges to \( \bar{f}_{n,l} \). Now define the multifunction \( F \) by

\[
F(\omega) = \text{cl}\left\{ \bigcup \{ \bar{f}_{n,l} : n \in \mathbb{Z}^+, l \in \mathbb{Z}^+ \} \right\}.
\]

The multifunction \( F \) has a measurable graph ([H]), and \( \{\bar{f}_{n,l}\}_{n,l \in \mathbb{Z}^+} \) is a Castaing representation of \( F \). Now we show that \( F \) is an integrable multifunction. It is sufficient to prove that there is an integrable function \( h, h : \Omega \to \mathbb{R} \), such that \( |F(\omega)| \leq h(\omega) \) for every \( \omega \in \Omega \).

Denote \( P_K(\mathbb{R}) \) the family of all compact subsets of \( \mathbb{R} \). Define the family \( c \) of multifunctions \( \{G_n : n \in \mathbb{Z}^+\}, G_n : \Omega \to P_K(\mathbb{R}) \) by

\[
G_n(\omega) = \text{cl}\left\{ \bigcup \{|f_{n,l}(\omega)| : l \in \mathbb{Z}^+\} \right\}
\]

for every \( n \in \mathbb{Z}^+ \). The multifunctions are measurable ([H]).

On the family of all multifunctions with real values and bounded by an integrable function, we have, by Definition 3.3, defined a metric, which is in this real case denoted by \( L_\mathbb{R} \).

Since

\[
\int_{\Omega} | |f(\omega)| - |g(\omega)| | \, d\mu \leq \int_{\Omega} |f(\omega) - g(\omega)| \, d\mu,
\]

we also have

\[
L_\mathbb{R}(G_n, G_m) \leq L(F_n, F_m).
\]

Thus the sequence \( \{G_n\} \) is \( L_\mathbb{R} \)-Cauchy, and from the proof of Theorem 6.15 [M], the assumptions of which are satisfied, it is possible to see that there is an integrable function \( h : \Omega \to \mathbb{R} \) such that \( ||G_n(\omega)|| \leq h(\omega) \) for each \( n \in \mathbb{Z}^+ \) and \( \omega \in \Omega \).

Now we prove that \( \{F_n\} \) \( L \)-converges to \( F \). We show that for every \( \varepsilon > 0 \) there is \( N(\varepsilon) \) such that, for every \( n > N(\varepsilon) \), \( L(F_n, F) < \varepsilon \).

Let \( h \) be an integrable function from \( \Omega \) to \( \mathbb{R} \) such that, for every \( n \in \mathbb{Z}^+ \), \( ||F_n(\omega)|| \leq h(\omega) \) and \( ||F(\omega)|| \leq h(\omega) \) \( \forall \omega \in \Omega \).

There is a measurable set \( A \) of finite measure such that

\[
\int_{\Omega \setminus A} h(\omega) \, d\mu < \frac{\varepsilon}{6}.
\]

Let \( g \) be an arbitrary selector of \( F \). Put

\[
P(\omega) = \left\{ y \in Y : g(y, g(\omega)) \leq \frac{\varepsilon}{3\mu(A)} \right\}.
\]
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There is a selector \( f_{n_1,l_1} \) of a multifunction \( F \) from the above Castaing representation \( \{ f_{n,l} \}_{n,l \in \mathbb{Z}^+} \) of \( F \) such that

\[
\{ f_{n_1,l_1}(\omega) \} \cap P(\omega) \neq \emptyset
\]
on a subset \( A_1 \subseteq A \) of nonzero measure. (This is very easy to see from the fact that \( \{ f_{n,l} \}_{n,l \in \mathbb{Z}^+} \) is a Castaing representation of \( F \) and thus \( F(\omega) \subseteq \text{cl}\{ \bigcup \{ f_{n,l}(\omega) : n,l \in \mathbb{Z}^+ \} \} \).

Suppose \( \mu(A \setminus A_1) > 0 \). Then by the same argument as above, there is a selector \( f_{n_2,l_2} \) from \( \{ f_{n,l} \}_{n,l \in \mathbb{Z}^+} \setminus \{ f_{n_1,l_1} \} \) such that

\[
\{ f_{n_2,l_2}(\omega) \} \cap P(\omega) \neq \emptyset
\]
on a subset \( A_2 \subseteq A \setminus A_1 \) of nonzero measure.

In this way, we obtain a sequence of disjoint subsets \( \{ A_n : n \in \mathbb{Z}^+ \} \) of \( A \) such that

\[
A = \bigcup \{ A_n : n \in \mathbb{Z}^+ \},
\]
and a sequence \( \{ f_{n_k,l_k} \}_{k \in \mathbb{Z}^+} \) of measurable selectors of \( F \).

Since \( h \) is an integrable function, then from the absolute continuity of integral it follows, that for \( \varepsilon / 6 \) there is \( \delta > 0 \) such that for arbitrary measurable set \( B \) with \( \mu(B) < \delta \) it holds

\[
\int_B 2h(\omega) \, d\mu < \frac{\varepsilon}{6}.
\]

Since \( \mu(A) = \sum_{k=1}^{\infty} \mu(A_k) < \infty \), then there is \( k_0 \) such that

\[
\mu\left( \bigcup_{k=k_0}^{\infty} A_k \right) = \sum_{k=k_0}^{\infty} \mu(A_k) < \delta.
\]

So

\[
\int_{\bigcup_{k=k_0}^{\infty} A_k} 2h(\omega) \, d\mu < \frac{\varepsilon}{6}.
\]

For \( k = 1, \ldots, k_0 \), choose \( p_k \) such that

\[
\int_\Omega \left| f_{n_k,l_k}(\omega) - f_{n_k,l_k,p}(\omega) \right| \, d\mu < \frac{\varepsilon}{k_06} \quad \text{for all} \quad p > p_k.
\]

Let \( M > \max\{ p_1, \ldots, p_{k_0} \} \). For \( p > M \), produce a selector of the multifunction \( F_p \) as follows:
Let $f_p$ be a measurable selector of $F_p$. Put

$$g_p(\omega) = \begin{cases} f_{n_k,l_k,p}(\omega) & \text{for } \omega \in A_k, \ k = 1, 2, \ldots, k_0, \\ f_p(\omega) & \text{otherwise}. \end{cases}$$

Now we show that $g_p$ is the needed selector of $F_p$.

$$\int_{\Omega} |g(\omega) - g_p(\omega)| \ d\mu$$

$$= \int_{A} |g(\omega) - g_p(\omega)| \ d\mu + \int_{\Omega \setminus A} |g(\omega) - g(\omega)| \ d\mu$$

$$= \sum_{k=1}^{\infty} \int_{A_k} |g(\omega) - g_p(\omega)| \ d\mu + \int_{\Omega \setminus A} |g(\omega) - g_p(\omega)| \ d\mu$$

$$\leq \sum_{k=1}^{\infty} \frac{\varepsilon \mu(A_k)}{3} + \sum_{k=1}^{k_0} \int_{A_k} |f_{n_k,l_k}(\omega) - g_p(\omega)| \ d\mu$$

$$+ \sum_{k=k_0}^{\infty} \int_{A_k} |f_{n_k,l_k}(\omega) - g_p(\omega)| \ d\mu + \int_{\Omega \setminus A} |g(\omega) - g_p(\omega)| \ d\mu$$

$$\leq \frac{\varepsilon}{3} + \sum_{k=1}^{k_0} \int_{A_k} |f_{n_k,l_k}(\omega) - f_{n_k,l_k,p}(\omega)| \ d\mu$$

$$+ \sum_{k=k_0}^{\infty} \int_{A_k} |f_{n_k,l_k}(\omega) - f_p(\omega)| \ d\mu + \int_{\Omega \setminus A} |g(\omega) - g_p(\omega)| \ d\mu$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \int_{\bigcup_{k=k_0}^{\infty} A_k} |f_{n_k,l_k}(\omega) - f_p(\omega)| \ d\mu + \frac{\varepsilon}{3}$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \int_{\bigcup_{k=k_0}^{\infty} A_k} 2h(\omega) \ d\mu + \frac{\varepsilon}{3} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{3} = \varepsilon$$
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The proof of the opposite inclusion is similar.

Let us remark (see the end of this paper) that the space \( \mathcal{L}^\sim \) of integrable multifunctions from \( \Omega \to Y \) was studied also by H i a i and Umegaki in [HU]. They consider other metric \( \Delta \) on \( \mathcal{L}^\sim \).

If \( A \) and \( B \) are two nonempty closed subsets of \( Y \), put
\[
\delta(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\},
\]
the Hausdorff distance between \( A \) and \( B \) ([Be]), where \( d \) is the metric induced by the norm of \( Y \).

Let \( F_1, F_2 \in \mathcal{L}^\sim \). Taking two sequences \( \{f_{1i}\} \) and \( \{f_{2j}\} \) of measurable functions such that
\[
F_1(\omega) = \text{cl}(\{f_{1i}(\omega); i \in \mathbb{Z}^+\}) \quad \text{and} \quad F_2(\omega) = \text{cl}(\{f_{2j}(\omega); j \in \mathbb{Z}^+\}) \quad \text{for all} \ \omega \in \Omega,
\]
we have
\[
\delta(F_1(\omega), F_2(\omega)) = \max \left\{ \sup_{i} \inf_{j} \|f_{1i}(\omega) - f_{2j}(\omega)\|, \sup_{j} \inf_{i} \|f_{1i}(\omega) - f_{2j}(\omega)\| \right\},
\]
so that the function \( \omega \to \delta(F_1(\omega), F_2(\omega)) \) is measurable. Since
\[
\delta(F_1(\omega), F_2(\omega)) \leq \|F_1(\omega)\| + \|F_2(\omega)\|,
\]
the function \( \omega \to \delta(F_1(\omega), F_2(\omega)) \) is also integrable. H i a i and Umegaki define in [HU] the metric \( \Delta \) on \( \mathcal{L}^\sim \) as follows
\[
\Delta(F_1, F_2) = \int_{\Omega} \delta(F_1(\omega), F_2(\omega)) \, d\mu.
\]

A natural question is to find relations between metrics \( L \) and \( \Delta \). First we introduce some auxiliary relations.

Let \( f \) be a measurable function from \( \Omega \) to \( Y \), and let \( \sigma \) be a measurable function from \( \Omega \) to \( [0, \infty] \). Then, by literature, there is a sequence of simple measurable functions \( \{f_n\}_{n \in \mathbb{Z}^+} \) such that
\[
f(\omega) = \lim_{n} f_n(\omega) \quad \text{and} \quad \|f_n(\omega)\| \leq \|f(\omega)\|, \quad n = 1, 2, \ldots, \quad \text{for each} \ \omega \in \Omega.
\]
Here, by a simple function, we mean a function with finitely many values.

Also there is a sequence of simple measurable functions
\[
\{\sigma_n\}, \quad \sigma_n: \Omega \to [0, \infty),
\]

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for every $n \in \mathbb{Z}^+$, such that
\[
\sigma(\omega) = \lim_n \sigma_n(\omega) \quad \text{for each } \omega \in \Omega.
\]

The function $f\sigma$ is also measurable, since
\[
f(\omega)\sigma(\omega) = \lim_n f_n(\omega)\sigma_n(\omega)
\]
and $f_n\sigma_n$ is a simple measurable function.

Further, let $B$ be a unit ball in $Y$ (i.e. $B = \{y \in Y : \|y\| \leq 1\}$), and let $\{a_i\}$ be a countable dense set in $B$.

Put
\[
g_i(\omega) = f(\omega) + a_i \quad \text{for every } \omega \in \Omega, \quad i = 1, 2, ...
\]

Clearly
\[
\|g_i(\omega) - f(\omega)\| \leq 1 \quad \text{for every } \omega \in \Omega, \quad i = 1, 2, ...
\]
and
\[
\text{cl}(\{g_i(\omega) : i \in \mathbb{Z}^+\}) = \{y : \|y - f(\omega)\| \leq 1\} \quad \text{for every } \omega \in \Omega.
\]

Define the multifunction $H : \Omega \to Y$ by
\[
H(\omega) = \{y : \|y - f(\omega)\| \leq \sigma(\omega)\} \quad \text{for every } \omega \in \Omega.
\]

We show that $H$ is a weakly measurable multifunction.

For every $i \in \mathbb{Z}^+$, let $h_i : \Omega \to Y$ be the following function:
\[
h_i(\omega) = (g_i(\omega) - f(\omega))\sigma(\omega) + f(\omega) \quad \text{for every } \omega \in \Omega.
\]

Clearly, the function $h_i$ is measurable for every $i \in \mathbb{Z}^+$. It is very easy to verify that $\|h_i(\omega) - f(\omega)\| \leq \sigma(\omega)$ for every $\omega \in \Omega$ and every $i \in \mathbb{Z}^+$.

Now we show that
\[
\text{cl}(\{h_i(\omega) : i \in \mathbb{Z}^+\}) = H(\omega) \quad \text{for every } \omega \in \Omega.
\]

If $\sigma(\omega) = 0$, then clearly $H(\omega) = \text{cl}(\{h_i(\omega) : i \in \mathbb{Z}^+\})$. Now let $\omega \in \Omega$ be such that $\sigma(\omega) \neq 0$. It is sufficient to prove that
\[
H(\omega) \subset \text{cl}(\{h_i(\omega) : i \in \mathbb{Z}^+\}).
\]

Let $y \in H(\omega)$ and $\varepsilon > 0$. We show that for the set
\[
O_y = \{z \in Y : \|y - z\| < \varepsilon\}
\]
the following relation holds:
\[
O_y \cap (\{h_i(\omega) : i \in \mathbb{Z}^+\}) \neq \emptyset.
\]
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Clearly, we can write \( y \) as \( f(\omega) + c \), where \( c \) is an element from \( Y \) with \( \|c\| \leq \sigma(\omega) \). Further, put

\[
y_1 = \frac{y}{\sigma(\omega)} + f(\omega) - \frac{f(\omega)}{\sigma(\omega)}.
\]

Then we have

\[
\|y_1 - f(\omega)\| = \left\| \frac{y}{\sigma(\omega)} + f(\omega) - \frac{f(\omega)}{\sigma(\omega)} - f(\omega) \right\| = \frac{1}{\sigma(\omega)} \| y - f(\omega) \| \leq 1.
\]

Put

\[
O_{y_1} = \left\{ z \in Y : \|z - y_1\| < \frac{\varepsilon}{\sigma(\omega)} \right\}.
\]

There is \( i \in \mathbb{Z}^+ \) such that \( g_i(\omega) \in O_{y_1} \). We show that \( \|h_i(\omega) - y\| < \varepsilon \).

\[
\begin{align*}
\|(g_i(\omega) - f(\omega))\sigma(\omega) + f(\omega) - ((y_1 - f(\omega))\sigma(\omega) + f(\omega))\| \\
= \|g_i(\omega)\sigma(\omega) - y_1\sigma(\omega)\| = \|g_i(\omega) - y_1\|\sigma(\omega) < \varepsilon.
\end{align*}
\]

The multifunction \( H : \Omega \to Y \) is weakly measurable because the multifunction \( P : \Omega \to Y \) defined by \( P(\omega) = \{h_i(\omega) : i \in \mathbb{Z}^+\} \) is weakly measurable ([H]).

The following example shows that there are two multifunctions \( F \) and \( G \), for which \( L^\sim(F, G) < \Delta(F, G) \).

**Example.** Let \( \Omega = Y = \mathbb{R} \) with the usual metric. Put

\[
\begin{align*}
F(\omega) &= 0 \quad \text{if } \omega \in (-\infty, -1) \cup (0, \infty), \\
F(\omega) &= \{1, 2\} \quad \text{if } \omega \in (-1, 0), \\
G(\omega) &= 0 \quad \text{if } \omega \in (-\infty, 0) \cup (1, \infty), \\
G(\omega) &= \{0, -2\} \quad \text{if } \omega \in (0, 1).
\end{align*}
\]

It is very easy to verify that \( \Delta(F, G) = 4 \) and \( L^\sim(F, G) = 3 \).

**Proposition 3.7.** \( L^\sim(F, G) \leq \Delta(F, G) \) for all multifunctions \( F \), \( G : \Omega \to Y \).

**Proof.** Suppose that there are multifunctions \( F \), \( G \) for which

\[
L^\sim(F, G) > \Delta(F, G), \quad \text{where } \Delta(F, G) = \int \sigma(\omega) \, d\mu = a,
\]

and \( \sigma(\omega) \) is the Hausdorff distance between \( F(\omega) \) and \( G(\omega) \).

Hence, one of the following possibilities is true:
1. There is $f$, a selector of the multifunction $F$ such that
\[
\int_{\Omega} \|g(\omega) - f(\omega)\| \, d\mu > a
\]
for every selector of the multifunction $G$.

2. There is $g$, a selector of the multifunction $G$ such that
\[
\int_{\Omega} \|g(\omega) - f(\omega)\| \, d\mu > a
\]
for every selector $f$ of the multifunction $F$.

Suppose condition 1 is true. The multifunction
\[
H(\omega) = \{y : \|f(\omega) - y\| \leq \sigma(\omega)\}
\]
is weakly measurable, as we proved above; so $H$ has a measurable graph. Hence
\[
H(\omega) \cap G(\omega) \neq \emptyset \quad \text{for every } \omega \in \Omega
\]
because $\sigma(\omega)$ is the Hausdorff distance between the sets $F(\omega)$ and $G(\omega)$ and $f(\omega) \in F(\omega)$. Put
\[
P(\omega) = H(\omega) \cap G(\omega) \quad \text{for every } \omega \in \Omega.
\]

Then $P$ is a graph measurable multifunction. There is a selector $p$ of the multifunction $P$ for which
\[
\int_{\Omega} \|f(\omega) - p(\omega)\| \, d\mu \leq \int_{\Omega} \sigma(\omega) \, d\mu = a
\]
because $p$ is a selector of the multifunction $H$. But that is a contradiction because $p$ is also a selector of the multifunction $G$. \hfill \Box

REFERENCES


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