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*Dedicated to Professor Sylvia Pulmannová
on the occasion of her 65th birthday*

ON THE STRONG LAW OF LARGE NUMBERS ON SOME ORDERED STRUCTURES

BELOSLAV RIEČAN

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ABSTRACT. Some limit theorems have been proved in the paper [RIEČAN, B.: *Probability theory on some ordered structures*, Atti Sem. Mat. Fis. Univ. Modena **47** (1999), 255–265] in a general ordered space. In the framework of the structure the strong law of large numbers is proved in this article.

1. Introduction

In [7] various mathematical models of quantum mechanical systems have been unified from the point of view of probability theory. More precisely a sequence of independent observables has been considered in [7].

Let us recall some basic notions. There is given a partially ordered set M with the least element 0 and the greatest element 1 and with a partial commutative binary operation $+$.

One of typical examples is the following. Let M be the set of all functions $f\colon \Omega \to \langle 0,1\rangle$ measurable with respect to a given σ-algebra of subsets of Ω. If the ordering is the usual one, then M evidently contains the least element 0_Ω and the greatest element 1_Ω. If we define the operation $+$ as the sum of functions, then evidently $+$ is only a partial binary operation.

The basic notions of the generalized probability theory are state and observable. The state corresponds to the probability measure, the observable corresponds to the notion of a random variable.

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DEFINITION 1. A *state* is a mapping $m\colon M \to \langle 0,1\rangle$ satisfying the following properties:

(i) $m(1)=1$, $m(0)=0$.

(ii) If $a,b,c \in M$, $b+c$ is defined and $a=b+c$, then

$$m(a)=m(b)+m(c).$$

(iii) If $(a_n)_{n=1}^{\infty} \subset M$, $a \in M$, $a_n \nearrow a$, then $m(a_n) \nearrow m(a)$.

DEFINITION 2. A *weak observable* is a mapping $x\colon \mathcal{B}(\mathbb{R}) \to M$ ($\mathcal{B}(\mathbb{R})$ is the σ-algebra of Borel subsets of the set $\mathbb{R}$ of real numbers) satisfying the following conditions:

(i) $m\big(x(\mathbb{R})\big)=1$.

(ii) If $A,B \in \mathcal{B}(\mathbb{R})$, $A \cap B = \emptyset$, then $x(A)+x(B)$ exists and $x(A\cup B) = x(A)+x(B)$.

(iii) If $(A_n)_{n=1}^{\infty} \subset \mathcal{B}(\mathbb{R})$ and $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

It is easy to see that for any state $m\colon M \to \langle 0,1\rangle$ and any weak observable $x\colon \mathcal{B}(\mathbb{R}) \to M$ the mapping $m_x\colon \mathcal{B}(\mathbb{R}) \to \langle 0,1\rangle$, defined by the formula $m_x = m\circ x$, is a probability measure.

If $(\Omega,\mathcal{S},P)$ is a probability space, then one can consider:

$$M=\{\chi_A:\ A\in\mathcal{S}\}, \qquad m\colon M\to\langle 0,1\rangle, \quad m(\chi_A)=P(A).$$

Moreover, if $\xi\colon \Omega \to \mathbb{R}$ is a random variable, then one can define an observable $x\colon \mathcal{B}(\mathbb{R}) \to M$ by the formula $x(B)=\chi_{\xi^{-1}(B)}$. Evidently $P_\xi = m_x$.

If ξ, η are two random variables, then they are independent if

$$P\big(\xi^{-1}(C)\cap\eta^{-1}(D)\big) = P\big(\xi^{-1}(C)\big)\cdot P\big(\eta^{-1}(D)\big) = P_\xi(C)\cdot P_\eta(D)$$

for any $C,D\in\mathcal{B}(\mathbb{R})$, what can be rewritten by the formula

$$P_T(C\times D) = P_\xi \times P_\eta(C\times D), \tag{$*$}$$

where $T=(\xi,\eta)$ is the corresponding random vector, $P_T(B) = P\big(T^{-1}(B)\big)$ ($B\in\mathcal{B}(\mathbb{R}^2)$) and $P_\xi\times P_\eta$ is the product of measures P_ξ, P_η. As a consequence of the equality $(*)$ we obtain the formula

$$P\Big((\xi+\eta)^{-1}\big((-\infty,t)\big)\Big) = P_\xi\times P_\eta\big(\{(u,v):\ u+v<t\}\big),$$

which can be rewritten by the formula

$$P\circ(\xi+\eta)^{-1}(B) = (P_\xi\times P_\eta)\circ g^{-1}(B), \qquad B\in\mathcal{B}(\mathbb{R}), \tag{$**$}$$

where $g\colon \mathbb{R}^2\to\mathbb{R}$ is given by $g(u,v)=u+v$.

Following $(**)$ and $(*)$ in our general case [7] we have defined two kinds of independency.

DEFINITION 3. Weak observables $x_n\colon \mathcal{B}(\mathbb{R}) \to M$ $(n = 1,2,\dots)$ are called to be *weakly independent* if to any n there is a weak observable $y_n\colon \mathcal{B}(\mathbb{R}) \to M$ such that

$$m \circ y_n = (m_{x_1} \times \cdots \times m_{x_n}) \circ g_n^{-1},$$

where $g_n\colon \mathbb{R}^n \to \mathbb{R}$ is defined by the formula $g_n(u_1,\dots,u_n) = u_1 + \cdots + u_n$.

DEFINITION 4. Weak observables $x_n\colon \mathcal{B}(\mathbb{R}) \to M$ are called to be *strongly independent*, if to any n there exists a mapping $h_n\colon \mathcal{B}(\mathbb{R}^n) \to M$ satisfying the following properties:

(i) $m(h_n(\mathbb{R}^n)) = 1$.

(ii) $h_n(A \cup B) = h_n(A) + h_n(B)$, whenever $A, B \in \mathcal{B}(\mathbb{R}^n)$, $A \cap B = \emptyset$.

(iii) If $A_i \nearrow A$, $(A_i)_{i=1}^{\infty} \subset \mathcal{B}(\mathbb{R}^n)$, then $h_n(A_i) \nearrow h_n(A)$.

(iv) $m \circ h_n = m_{x_1} \times \cdots \times m_{x_n}$.

Using weak independency the weak law of large numbers and the central limit theorem have been proved in [7]. In the paper we prove the strong law of large numbers, of course, by the help of strong independency. A similar approach has been realized in [2], in the special case of D-posets ([4]).

2. Formulation

Recall that we work with an algebraic system $(M, \leq, +)$, where $(M, \leq)$ is a partial ordered set with the least element 0 and the greatest element 1 and $+$ is a commutative partial binary operation. State is defined with respect to Definition 1, observable with respect to Definition 2, strong independency of a sequence of observables with respect to Definition 4.

DEFINITION 5. We shall say that a weak observable $x\colon \mathcal{B}(\mathbb{R}) \to M$ *belongs to* L^1 if the following integral exists

$$E(x) = \int_{-\infty}^{\infty} t \,\mathrm{d}m_x(t).$$

It *belongs to* L^2 if the following integral exists

$$\int_{-\infty}^{\infty} t^2 \,\mathrm{d}m_x(t).$$

In this case we define the *dispersion* of x

$$\sigma^2(x) = \int_{-\infty}^{\infty} t^2 \, \mathrm{d}m_x(t) - E(x)^2$$
$$= \int_{-\infty}^{\infty} \big(t - E(x)\big)^2 \, \mathrm{d}m_x(t) .$$

DEFINITION 6. Let $x_1, \dots, x_n \colon \mathcal{B}(\mathbb{R}) \to M$ be strongly independent observables, $h_n \colon \mathcal{B}(\mathbb{R}^n) \to M$ the corresponding joint observable, $g_n \colon \mathbb{R}^n \to \mathbb{R}$ be a Borel function. Then we define an observable $g_n(x_1, \dots, x_n) \colon \mathcal{B}(\mathbb{R}) \to M$ by the formula

$$g_n(x_1, \dots, x_n)(B) = h_n \circ g_n^{-1}(B) .$$

DEFINITION 7. Let $(M, \leq, +)$ be a lattice. A sequence $(y_n)_{n=1}^{\infty}$ of observables converges m-a.e. to 0, if

$$\lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} y_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right) \right) = 1 .$$

THEOREM. *Let $(M, +, \leq)$ be the algebraic system stated above such that M is a lattice (with respect to $\leq$). Let $(x_n)_{n=1}^{\infty}$ be a strongly independent sequence of weak observables from L^2. Let $\sum_{n=1}^{\infty} \frac{\sigma^2(x_n)}{n^2} < \infty$. Then*

$$\left(\frac{1}{n} \sum_{i=1}^{n} \big(x_i - E(x_i)\big) \right)_{n=1}^{\infty}$$

converges m-a.e. to 0.

3. Proof

Let $\big(\mathbb{R}^{\mathbb{N}}, \sigma(\mathcal{C}), P\big)$ be the probability space, where $\mathcal{C}$ is the family of all cylinders in $\mathbb{R}^{\mathbb{N}}$ and P is the infinite product of probability measures $m_{x_1}, m_{x_2}, \dots$, i.e.

$$P\big(\{(t_i)_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \ t_1 \in A_1, \dots, \ t_n \in A_n\}\big) = m_{x_1}(A_1) \cdot m_{x_2}(A_2) \cdot \dots \cdot m_{x_n}(A_n) .$$

Define $\xi_n \colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ by the formula

$$\xi_n\big((t_i)_{i=1}^{\infty}\big) = t_n .$$

Then ξ_n is a random variable and

$$P_{\xi_n}(A) = P\big(\xi_n^{-1}(A)\big) = P\big(\{(t_i)_{i=1}^{\infty} :\ t_n \in A\}\big) = m_{x_n}(A)\,,$$

hence $P_{\xi_n} = m_{x_n}$. Therefore

$$\int\limits_{\mathbb{R}^{\mathbb{N}}} \xi_n^2 \,\mathrm{d}P = \int\limits_{\mathbb{R}} t^2 \,\mathrm{d}P_{\xi_n}(t) = \int\limits_{\mathbb{R}} t^2 \,\mathrm{d}m_{x_n}(t) < \infty\,,$$

hence $\xi_n \in L^2$ and $E(\xi_n) = E(x_n)$, $\sigma^2(\xi_n) = \sigma^2(x_n)$. Moreover, $\xi_1, \xi_2, \xi_3, \dots$ are independent. Indeed,

$$\begin{aligned} P\left(\bigcap_{i=1}^{n} \xi_i^{-1}(A_i)\right) &= P\big(\{(t_i)_{i=1}^{\infty} :\ t_1 \in A_1, \dots,\ t_n \in A_n\}\big) \\ &= m_{x_1}(A_1) \cdots m_{x_n}(A_n) = P_{\xi_1}(A_1) \cdots P_{\xi_n}(A_n) \\ &= P\big(\xi_1^{-1}(A_1)\big) \cdots P\big(\xi_n^{-1}(A_n)\big)\,. \end{aligned}$$

Therefore $(\xi_n)_{n=1}^{\infty}$ satisfies the assumptions of the strong law of large numbers, hence

$$\frac{1}{n} \sum_{i=1}^{n} \big(\xi_i - E(\xi_i)\big) \to 0 \quad P\text{-a.e.}.$$

Define $g_n \colon R^n \to R$ by the equality

$$g_n(u_1, \dots, u_n) = \frac{1}{n} \sum_{i=1}^{n} \big(u_i - E(\xi_i)\big) = \frac{1}{n} \sum_{i=1}^{n} \big(u_i - E(x_i)\big)$$

and put

$$\begin{aligned} \eta_n &= g_n(\xi_1, \dots, \xi_n) = g_n \circ T_n\,, \\ y_n &= g_n(x_1, \dots, x_n) = h_n \circ g_n^{-1}\,. \end{aligned}$$

We have proved that $\eta_n \to 0$ P-a.e.. It is equivalent to the equality

$$\lim_{p \to \infty} \lim_{k \to \infty} \lim_{i \to \infty} P\left(\bigcap_{n=k}^{k+i} \eta_n^{-1}\left(\left(-\tfrac{1}{p}, \tfrac{1}{p}\right)\right)\right) = 1\,.$$

But

$$
\begin{aligned}
&P\left(\bigcap_{n=k}^{k+i} \eta_n^{-1}\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) \\
&= P\left(\bigcap_{n=k}^{k+i} T_n^{-1} \circ g_n^{-1}\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) \\
&= m_{x_1} \times \cdots \times m_{x_{k+i}}\left(\bigcap_{n=k}^{k+i}\left\{(u_1, \ldots, u_{k+i}) :\ (u_1, \ldots, u_n) \in g_n^{-1}\left(\left(-\tfrac{1}{p}, \tfrac{1}{p}\right)\right)\right\}\right) \\
&= m\left(h_{k+i}\left(\bigcap_{n=k}^{k+i}\left\{(u_1, \ldots, u_{k+i}) :\ (u_1, \ldots, u_n) \in g_n^{-1}\left(\left(-\tfrac{1}{p}, \tfrac{1}{p}\right)\right)\right\}\right)\right) \\
&\le m\left(\bigwedge_{n=k}^{k+i} h_{k+i}\left(\left\{(u_1, \ldots, u_{k+i}) :\ (u_1, \ldots, u_n) \in g_n^{-1}\left(\left(-\tfrac{1}{p}, \tfrac{1}{p}\right)\right)\right\}\right)\right) \\
&= m\left(\bigwedge_{n=k}^{k+i} h_n \circ g_n^{-1}\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = m\left(\bigwedge_{n=k}^{k+i} y_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
1 &= \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} P\left(\bigcap_{n=k}^{k+i} \eta_n^{-1}\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) \\
&\le \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} y_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right).
\end{aligned}
$$

We have proved that $y_n \to 0$ m-a.e.. But

$$
y_n = g_n(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n}(x_i - E(x_i)).
$$

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