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SEMIGROUPS CONTAINING SOVERED TWO-SIDED IDEALS

IMRICH FABRICI

In [5], semigroups containing one-sided covered ideals have been investigated. It turns out that a semigroup need not have one-sided covered ideals at all. As for two-sided ideals, the situation is different. The purpose of the paper is to investigate the structure of semigroups containing two-sided covered ideals.

Definition 1. A proper two-sided ideal $M$ of a semigroup $S$ is covered (briefly $C$-ideal) if $M \subset S(S - M)S$.

$$I^a = \{ x \in S/(a)_T = a \cup Sa \cup aS = x \cup Sx \cup xS \cup SxS = (x)_T \}$$

is the $J$-class of $S$ containing $a$.

The $J$-class $I^a$ is maximal, if $(a)_T$ is not a proper subset of any principal two-sided ideal of $S$.

It is known ([1]) that the $J$-class $I^a$ is maximal iff its complement is a maximal ideal of $S$.

When dealing with maximal ideals $M_a$, we shall denote the corresponding maximal $J$-classes by $M^a$.

Theorem 1. If $S$ contains two different proper ideals $M_1, M_2$ such that $M_1 \cup M_2 = S$, then none of them is a $C$-ideal.

Proof. If $M_1 \cup M_2 = S$, then $S - M_2 \subset M_1$, $S - M_1 \subset M_2$. If one of them were a $C$-ideal, e.g. $M_1$, then $M_1 \subset S(S - M_1)S \subset SM_2S \subset M_2$. Since $M_1 \cup M_2 = S$, it implies $M_2 = S$. Hence, we get a contradiction with our assumption that $M_1, M_2$ are proper two-sided ideals.

Corollary. If $S$ contains more than one maximal two-sided ideal, then none of them is a $C$-ideal of $S$.

Theorem 2. If $M_1$ and $M_2$ are two $C$-ideals of $S$, then $M_1 \cup M_2$ is a $C$-ideal of $S$.

Proof. We show that if $M_1 \subset S(S - M_1)S$, $M_2 \subset S(S - M_2)S$, then $M_1 \cup M_2 \subset S[S - (M_1 \cup M_2)]S$.

Let $x \in M_1$, then $M_1 \subset S(S - M_1)S$ implies that there is $a \in S - M_1$ such that $x \in SaS$. There are two possibilities:

1. $a \in S - (M_1 \cup M_2)$, then $x \in S[S - (M_1 \cup M_2)]S$. 

(2) \( a \in (S - M_1) \cap M_2 \), then \( a \in M_2 \subseteq S(S - M_2)S \). So there is \( b \in S - M_2 \) such that \( a \in SbS \). The element \( b \) does not belong to \( M_1 \), since otherwise we would have \( a \in SbS \subseteq SM_1S \subseteq M_1 \) and it is contradicting with the choice of \( a \). Therefore, \( b \in S - M_2 \), \( b \in S - M_1 \), so \( b \in (S - M_1) \cap (S - M_2) = S - (M_1 \cup M_2) \). We have \( x \in S \subseteq S(SbS) \subseteq SbS \subseteq S[ S - (M_1 \cup M_2)] \). Hence \( M_1 \subseteq S[ S - (M_1 \cup M_2)] \). And in the same way we can prove that \( M_2 \subseteq S[ S - (M_1 \cup M_2)] \).

**Theorem 3.** If \( M_1, M_2 \) are two \( C \)-ideals of \( S \), then \( M_1 \cap M_2 \) is a \( C \)-ideal of \( S \).

**Proof.** It is well known ([7]) that \( M_1 \cap M_2 \neq \emptyset \). It is enough to show that

\[
M_1 \cap M_2 \subseteq S[ S - (M_1 \cap M_2)]S.
\]

From the relation \( M_1 \subseteq S(S - M_1)S \) we have

\[
M_1 \cap M_2 \subseteq M_1 \subseteq S(S - M_1)S \subseteq S[ S - (M_1 \cap M_2)]S.
\]

If we consider both Theorem 2 and Theorem 3 we get:

**Corollary.** The set of all \( C \)-ideals of \( S \) is a sublattice of the lattice of all ideals of \( S \).

We have seen that if \( S \) contains more than one maximal ideal, then none of them can be a \( C \)-ideal of \( S \).

Now we shall consider the case that \( S \) contains only one maximal two-sided ideal.

**Definition 2.** A two-sided ideal \( M \) of a semigroup \( S \) is said to be the greatest ideal of \( S \), if any proper two-sided ideal of \( S \) is contained in \( M \).

If such an ideal in \( S \) exists, then we shall denote it by \( M^* \).

**Theorem 4.** Let a semigroup \( S \) contain only one maximal two-sided ideal \( M \). If \( M \) is a \( C \)-ideal, then \( M = M^* \).

**Proof.** It is sufficient to show that any proper ideal of \( S \) is contained in \( M \). If \( T \) is any proper two-sided ideal of \( S \), then with regard to Theorem 1 we get that \( T \subseteq M \). It means that \( M = M^* \).

For one-sided ideals the converse statement holds too. The next example illustrates that for two-sided ideals it need not hold.

**Example 1.** Let \( S = \{a, b, c, d\} \) be the semigroup with the multiplication table:

<table>
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<th>a</th>
<th>b</th>
<th>c</th>
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<td>d</td>
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</table>
$M = \{a, b, c\}$ is the only maximal two-sided ideal of $S$. Any proper ideal of $S$ is contained in $M$, so $M = M^*$. $S - M^* = \{d\}$. $SdS = \{a, b\}$, so $M^* \triangleleft S(S - M^*)S$, hence $M^*$ is not a $C$-ideal of $S$.

**Theorem 5.** The ideal $M^*$ of a semigroup $S$ is a $C$-ideal iff $S^2 = S^3$.

**Proof.** (a) Let $M^*$ be a $C$-ideal, so $M^* \subseteq S(S - M^*)S$. Since $M^*$ is at the same time a maximal ideal, then $S - M^* = I^*$ is the unique maximal $\mathcal{J}$-class in $S$. Then either $S^2 \not\subseteq S$, or $S^2 = S$. If $S^2 = S$, then $S^3 = S^2$.

If $S^2 \not\subseteq S$, then either $S^3 = S^2$, or $S^3 \not\subseteq S^2$.

If $S^2 \not\subseteq S^2$, then $M^* \subseteq S(S - M^*)S \subseteq S^3$, hence $S - M^*$ would contain at least two different $\mathcal{J}$-classes, one in $S^2 - S^3$ and another in $S - S^2$. But this is a contradiction, since $S - M^*$ contains just one maximal $\mathcal{J}$-class. So, we have $S^2 = S^3$.

(b) Suppose that $S$ contains $M^*$ and $S^2 = S^3$. We show that $M^*$ is a $C$-ideal.

Let $x \in M^*$ be any element. Then for any element $a \in I^* = S - M^*$, $(a)_T = S$, therefore $x \in (a)_T$. However, $x \in M^*$ and $a \in I^* = S - M^*$, hence $x \neq a$. Then $x \in (aS \cup Sa \cup SaS)$.

If $x \in aS$ or $x \in Sa$, then $x \in S^2$. If $x \in SaS$, then $x \in S^3$. But according to the assumption $S^2 = S^3$, therefore $x \in S^3$.

Then there is $c \in S$ such that $x \in ScS$. Since $(a)_T = S$, we have $c \in (a)_T$. If $c = a$, then $x \in SaS$. If $c \neq a$, then $c \in (aS \cup Sa \cup SaS)$. If $c \in aS$, then $ScS \subseteq SaS$. The same relation can be obtained if $c \in Sa$, or $c \in SaS$. Hence, $x \in ScS \subseteq SaS$, $a \in I^* = S - M^*$ in all three cases. This implies: for any $x \in M^*$, $x \in SaS$ and $a \in I^* = S - M^*$, therefore $M^* \subseteq S(S - M^*)S$ i.e. $M^*$ is a $C$-ideal of $S$.

It was already mentioned that $S$ need not have one-sided covered ideals at all. For two-sided ideals we have:

**Theorem 6.** If $S$ is not a simple semigroup, then $S$ contains at least one $C$-ideal of $S$.

**Proof.** Let $T$ be any proper two-sided ideal of $S$. Consider $S(S - T)S$. $S(S - T)S$ is an ideal of $S$, and it is known [7] that the intersection of two two-sided ideals is non-empty. Therefore, $T \cap S(S - T)S \neq \emptyset$. If we denote $M = T \cap S(S - T)S$, then $M$ is an ideal of $S$ and for $M$ we get $M \subseteq S(S - T)S$. Since $S - T \subseteq S - M$, then the relation $M \subseteq S(S - T)S$ implies

$$M \subseteq S(S - T)S \subseteq S(S - M)S,$$

hence $M$ is a $C$-ideal of $S$.

We now investigate the case that $S$ contains more than one maximal ideal.

**Definition 3.** A two-sided ideal $M$ of $S$ will be called the greatest covered ideal of $S$ if any covered ideal of $S$ is contained in $M$.

If $S$ contains the greatest covered ideal of $S$, this ideal will be denoted by $M^\circ$.

**Remark.** Let a semigroup $S$ contain maximal two-sided ideals. If $M_\alpha$, $\alpha \in \lambda$ are
all maximal two-sided ideals of \( S \), then \( \bigcap_{\alpha \in \Lambda} \mathcal{M}_\alpha \neq \emptyset \). Denote \( \hat{\mathcal{M}} = \bigcap_{\alpha \in \Lambda} \mathcal{M}_\alpha \). If \( S \) contains \( \mathcal{M}^o \), then necessarily \( \mathcal{M}^o \subseteq \hat{\mathcal{M}} \) holds. For if there is at least one \( \mathcal{M}_\alpha \) such that \( \mathcal{M}^o \nsubseteq \mathcal{M}_\alpha \), then by Theorem 1 \( \mathcal{M}^o \) is not a \( C \)-ideal and it is a contradiction.

However, we can show that if \( S \) contains maximal two-sided ideals, it does not mean that \( S \) must contain \( \mathcal{M}^o \).

**Example 2.** Let \( N = \{0, 1, 2, \ldots, n, \ldots\} \). Let \( S = N \cup \{z_1\} \cup \{z_2\} \). Define the binary operation \( \circ \) in \( S \) as follows:

\[
\begin{align*}
a \circ b &= \min \{a, b\} \text{ if } a, b \in N \\
&= 0 \text{ if } (1) \ a \in N, \ b \in \{z_1, z_2\}, \ (2) \ a = z_1, \ b = z_2 \\
&= z_i \text{ if } a = b = z_1, \ i = 1, 2.
\end{align*}
\]

It can be easily verified that \( S \) with the operation \( \circ \) is a semigroup.

\( (z_1)_T = \{0, z_1\}, \ (z_2)_T = \{0, z_2\}. \ M_1 = S - \{z_1\}, \ M_2 = S - \{z_2\} \)

are all maximal ideals of \( S \). \( \hat{\mathcal{M}} = M_1 \cap M_2 = N \). The subset \( T = \{0, 1, \ldots, k\} \) is an ideal of \( S \) and moreover \( S(S - T)S = \{0, 1, 2, \ldots, k, k + 1, \ldots\} \supseteq \{0, 1, 2, \ldots, k\} \), so \( T \) is a \( C \)-ideal of \( S \). But \( S \) does not contain \( \mathcal{M}^o \).

For one-sided covered ideals the following statement holds: If \( \hat{\mathcal{L}} = \bigcap_{\alpha \in \Lambda} \mathcal{L}_\alpha \neq \emptyset \),

where \( \mathcal{L}_\alpha, \ \alpha \in \Lambda \) are all maximal left ideals, then \( \hat{\mathcal{L}} \) is the greatest covered left ideal of \( S \).

The following example illustrates that for two-sided ideals this need not be true.

**Example 3.** Let \( S = \{a, b, c, d\} \) with the multiplication table:

\[
\begin{array}{c|cccc}
   & a & b & c & d \\
\hline
a & a & a & a & a \\
b & a & a & a & a \\
c & a & a & a & a \\
d & a & a & b & b \\
\end{array}
\]

\( S \) contains two maximal ideals: \( M_1 = \{a, b, c\}, \ M_2 = \{a, b, d\} \). \( \hat{\mathcal{M}} = M_1 \cap M_2 = \{a, b\} \). Then \( S - \hat{\mathcal{M}} = \{c, d\} \). But \( S(S - \hat{\mathcal{M}})S = S\{c, d\}S = \{a\} \nsubseteq \{a, b\} \). Therefore, \( \hat{\mathcal{M}} \) is not a \( C \)-ideal of \( S \).

If we want to describe conditions for existence of \( \mathcal{M}^o \) we need the notion of a two-sided base of a semigroup.

**Definition 4.** A non-empty subset \( A \) of \( S \) is a two-sided base of \( S \) if

1. \( (A)_T = A \cup SA \cup AS \cup SAS = S \)
2. There is no proper subset \( B \subseteq A \) such that \( (B)_T = S \).
The condition (2) implies that a two-sided base $A$ of $S$ does not contain two different elements of $\mathcal{J}$-class $I^a$. Hence, for $a, b \in A$, $a \neq b$, $I^a \cap I^b = \emptyset$. Further, we shall denote maximal $\mathcal{J}$-class as a complement of a maximal two-sided ideal $M_a$ by $M^a$.

**Theorem 7.** If a semigroup $S$, which is not simple, contains a two-sided base $A$ of $S$, then $S$ contains the ideal $M^0$. Moreover, $M^0 = S^3 \cap \hat{M}$, where $\hat{M} = \bigcap_{a \in \lambda} M_a$.

**Proof.** The existence of a two-sided base $A$ implies the existence of maximal ideals ([3]) and $M_a = S - M^a$ where $M^a$ is a maximal $\mathcal{J}$-class. Since $\emptyset \neq \hat{M} = \bigcap_{a \in \lambda} (S - M^a) = S - \bigcup_{a \in \lambda} M^a$, then $\hat{M}$ is an ideal of $S$ ([7]). And $S^3$ is an ideal of $S$ too. Denote by $M = \hat{M} \cap S^3 \neq \emptyset$. We shall show that $M$ is a $C$-ideal of $S$.

Let $x \in M$ be any element. Hence, $x \in S^3$ and therefore there is $c \in S$ such that $x \in ScS$. If $c \in A$, then there is $b \in A$ such that $c \in (b)_T$, hence $c \in (Sb \cup bS \cup SbS)$ and $c$ is at least in one of the subsets: $Sb, bS, SbS$. Then $ScS \subseteq SbS$ and $x \in ScS \subseteq SbS$ implies $x \in SbS$ for $b \in A$. We have got that for any $x \in M$, there is $b \in A$ such that $x \in SbS \subseteq SAS \subseteq S(S - \hat{M})S \subseteq S(S - M)S$, therefore $M \subseteq S(S - M)S$. It remains to show that $M$ is the greatest $C$-ideal of $S$.

We shall show that any $C$-ideal of $S$ is contained in $M = S^3 \cap \hat{M}$, i.e. that $M$ is the greatest covered ideal of $S$.

Let $T$ be any $C$-ideal of $S$. Then $T \subseteq S(S - T)S \subseteq S^3$, therefore $T \subseteq S^3$. Since $T$ is a $C$-ideal of $S$, then $T$ cannot contain any maximal $\mathcal{J}$-class. It means, $T \subseteq S - M^a$ for every $a \in \lambda$. Hence we have $T \subseteq \bigcap_{a \in \lambda} (S - M^a) = \bigcap_{a \in \lambda} M_a = \hat{M}$. The relations $T \subseteq S^3$ and $T \subseteq \hat{M}$ imply $T \subseteq S^3 \cap \hat{M} = M$, therefore, any $C$-ideal $T$ is contained in $M$, i.e. $M = M^0$.

**Lemma 1.** Let $S$ contain $M^0$ and $M^0 \subseteq S^3$. Then every $\mathcal{J}$-class in $S^3 - M^0$ is a maximal one and for any $a \in S^3 - M^a$ $(a)_T = SaS$.

**Proof.** Let $S^3 - M^0 \neq \emptyset$. Since both $M^0$ and $S^3$ are ideals in $S$, then $S^3 - M^0$ consists of some $\mathcal{J}$-classes of $S$. Let $M'$ be any one of them. Since $M' \subseteq S^3$ it implies that any $a \in M'$ is of the form: $a = xby$ for $x, y, b \in S$. Then $a \in SbS$. Next we show that $b \in M^a$. If $b \in M^a$, $\delta \neq \gamma$, then $a \in SbS$ would imply $(a)_T \subseteq (b)_T$. The element $b$ does not belong to $(a)_T$ (otherwise $b \in (a)_T$ would imply $(b)_T \subset (a)_T$ and we get $(a)_T = (b)_T$ which is contradicting with $\delta \neq \gamma$). Therefore $(a)_T$ is a $C$-ideal and $M^a \cup (a)_T$ is a $C$-ideal too, properly containing $M^a$, what is impossible.

So we have got: for any $a \in M'$ there is $b \in M'$ such that $a \in SbS$. This implies $(a)_T \subset SbS \subset (b)_T = (a)_T$, hence $(a)_T = SaS$. We show that $(a)_T = SaS$. Since $(a)_T = SbS = (b)_T$, from there we get: if $a = b$, $(a)_T = SaS$. If $a \neq b$, then $b \in (Sa \cup aS \cup SaS)$. If $b \in Sa$, then $SbS \subset SaS$. If $b \in aS$ or $b \in SaS$ we get again
SbS \subseteq SaS. The relation \((a) = (b) = SbS \subseteq SaS \subseteq (a)\) implies \((a) = SaS\). This is true for any \(\mathcal{J}\)-class \(M^r\) from \(S^3 - M^o\).

If we suppose that for \(a \in M^r \subseteq S^3 - M^o\), \((a) = SaS \subseteq (c)\) for \(c \in S\), then \(a \in (Sc \cup cS \cup ScS)\) and \(a\) is contained in at least one of the sets: \(Sc, cS, ScS\). Then \(SaS \subseteq ScS\) and \((a) \subset ScS\). But \(c \in (a)\), i.e. \((a)\) is a \(C\)-ideal and \(M^o \cup (a)\) is a \(C\)-ideal too, properly containing \(M^o\) and we have a contradiction.

We have proved that any \(\mathcal{J}\)-class in \(S^3 - M^o\) is maximal.

**Theorem 8.** If a semigroup \(S\) contains the ideal \(M^o\), then \(S\) contains a two-sided base.

**Proof.**

\[M^o \subseteq S(S - M^o)S \subseteq S^3 \subseteq S\] 

Denote by \(M^o\) \(\mathcal{J}\)-classes from \(S - S^2\), by \(M^p\) \(\mathcal{J}\)-classes from \(S^2 - S^3\) and by \(M^y\) \(\mathcal{J}\)-classes from \(S^3 - M^o\). Construct a subset \(A\) in the following way: from each \(\mathcal{J}\)-class \(M^a\) and \(M^y\) choose just one element into \(A\). Denote by

\[(A) = A \cup SA \cup AS \cup SAS.

\(\mathcal{J}\)-classes \(M^a \subseteq S - S^2\) are of the form: \(M^a = \{a\}\), where \(a\) is undecomposable element, so any \(M^a \subseteq S - S^2\) is a maximal \(\mathcal{J}\)-class.

Equally, any \(\mathcal{J}\)-class \(M^r \subseteq S^3 - M^o\) is a maximal \(\mathcal{J}\)-class (by Lemma 1).

We wish to show that for any \(x \in M^o \subseteq S^2 - S^3\) there is some \(u \in M^a \subseteq S - S^2\) such that \(x \in (u)\). Note that \(S^2 - S^3 \neq \emptyset\) implies \(S - S^2 \neq \emptyset\). Since \(x \in M^o \subseteq S^2 - S^3\), then \(x = uv\), where \(u, v \in S - S^2\), it means that both \(u\) and \(v\) are undecomposable. Now \(x = uv\) implies \(x \in (u)\), \(u \in S - S^2\), i.e. \(u \in M^a\).

Till now we have got: \(M^o \subseteq S(S - M^o)S\), it means for any \(y \in M^o\) there is \(z \in S - M^o\) such that \(y \in (z)\). Since \(z \in S - M^o\), then \(z\) is from \(M^o\), or \(M^p\), or \(M^y\). If \(z \in M^o\) or \(z \in M^y\), then in both cases we can choose \(z \in A\). If \(z \in M^p\), then there is \(u \in M^a\) such that \(z \in (u)\), hence, \(y \in (z) \subseteq (u)\), and \(u \in A\).

We have shown that for any \(y \in M^o\) there is \(a \in A\) such that \(y \in (a)\), hence \(M^o \subseteq (A)\). And with regard to the construction of \(A\) we have: \(S^3 - M^o \subseteq (A)\), \(S^2 - S^3 \subseteq (A)\) and \(S - S^2 \subseteq (A)\) so together we have \(S \subseteq (A)\), therefore

\[(A) = S,

hence, the subset \(A\) generates \(S\).

To prove that \(A\) is a two-sided base of \(S\), it remains to show that there is no proper subset \(B \neq A\) with the property

\[(B) = B \cup SB \cup BS \cup SBS = S.

But this is evident, because \(A\) has been constructed by means of elements of maximal \(\mathcal{J}\)-classes of \(S\), and from each maximal \(\mathcal{J}\)-class just one element was chosen into \(A\). Therefore, \(A\) is a two-sided base of \(S\).

Theorem 7 and Theorem 8 imply:
Corollary. If $S$ is not simple, then $S$ contains the ideal $M^o$ iff $S$ has a two-sided base.

The question arises, whether the relation $M^o \subseteq S^3$ does not mean $M^o = S^3$ always. Next example illustrates that it need not be so.

Example 4. $T$ is the multiplicative semigroup of real numbers: $T = \{x - 1/x : 0 \leq x \leq 1\}$, $G$ an arbitrary commutative group. Define in $S = T \cup G$ an associative binary operation $\circ$ as follows: $x \circ y = 0$ if $x \in T$, $y \in G$ and the products in $T$ and $G$ remain old ones. Then $S$ is a semigroup. $T$ is a maximal ideal in $S$. However, there is an infinite number of further maximal ideals of the form: $M_a = S - \{\alpha\}$, where $\frac{1}{4} < \alpha \leq \frac{1}{2}$. The intersection of all maximal ideals in $S$ is:

$$\hat{M} = \bigcap_a M_a \cap T = \langle 0, \frac{1}{4} \rangle.$$

For $S$, $S \neq S^2$, $S^3 = \langle 0, \frac{1}{8} \rangle \cup G$. Then $M^o = S^3 \cap \hat{M} = \langle 0, \frac{1}{8} \rangle$. Hence, $M^o \neq S^3$.

Theorem 9. Let any proper two-sided ideal of a semigroup $S$ be covered. Then just one of the following conditions holds:

1. $S$ contains the ideal $M^*$.
2. $S = S^2$ and for any proper two-sided ideal $M$ and for any principal two-sided ideal $(a)_T \subseteq M$, there is a principal proper two-sided ideal $(b)_T$ such that $(a)_T \not\subseteq (b)_T$ and $b \in S - M$.

Proof. First we show that if any proper two-sided ideal of $S$ is covered, then $S$ cannot contain even two different maximal $\mathcal{J}$-classes.

If $M^a$, $M^b$ were maximal $\mathcal{J}$-classes, $M^a \neq M^b$, then as we know $M_a = S - M^a$, $M_b = S - M^b$ would be maximal proper two-sided ideals of $S$ and none of them is a $C$-ideal of $S$.

Let $S$ contain just one maximal $\mathcal{J}$-class $M^a$. Then $M_a = S - M^a$ is again a maximal proper two-sided ideal of $S$ and moreover it is a $C$-ideal. By Theorem 4 we have that $M_a = M^a$.

Suppose that $S$ does not contain maximal $\mathcal{J}$-classes. First we show that $S^2 = S$. If $S^2 \not\subseteq S$, then for $y \in S - S^2$ $(y)_T \neq S$, because if $(y)_T = S$, then $S$ would contain a maximal $\mathcal{J}$-class. So $(y)_T \not\subseteq S$, and since any proper two-sided ideal is covered, then $(y)_T \subseteq S[S - (y)_T]$, hence $y \in S \cap S$ for $z \in S - (y)_T$, therefore $y \in S^3$. But $S^3 \subset S^2$, it means $y \in S^2$, which is a contradiction with the fact that $y \in S - S^2$. Therefore, $S = S^2$.

Let $M$ be any proper two-sided ideal of $S$. Then $M \subseteq S(S - M)S$. Let $a \in M$ be any element. Then there is $b \in S - M$ such that $a \in SbS$. This implies $(a)_T \subseteq SbS \subseteq (b)_T$. The ideal $(b)_T \neq S$, since $S$ does not contain maximal $\mathcal{J}$-classes. Moreover $(a)_T \neq (b)_T$, because $a \in M$, $b \in S - M$. Therefore, $(b)_T \not\subseteq M$. 

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Theorem 10. Let a semigroup $S$ satisfy just one of the following conditions:
(1) $S$ contains $M^*$ and it is a $C$-ideal.
(2) $S = S^2$ and for any proper two-sided ideal $M$ and for every principal two-sided ideal $(a)_T \subseteq M$, there is a principal two-sided proper ideal $(b)_T$, whose generator $b \in S - M$ and $(a)_T \subseteq (b)_T$.

Then any two-sided proper ideal is covered.

Proof. Let $M$ be any proper two-sided ideal of $S$. If (1) holds, then $M \subseteq M^*$, and $S - M^* \subseteq S - M$. Then $M \subseteq M^* \subseteq S(S - M^*)S \subseteq S(S - M)S$. Hence, $M$ is a $C$-ideal.

Let (2) be satisfied. If $x \in M$, then $(x)_T \subseteq M$. Then there is $b \in S - M$ such that $(x)_T \subseteq (b)_T$. It is evident that $(x)_T \neq (b)_T$. Since $S = S^3$, then $S = S^3$, and $b \in SdS$ for some $d \in S$. We show that $d \in M$. If $d \in M$, then $SdS \subseteq M$, and $b \in SdS \subseteq M$, hence, $b \in M$, which is a contradiction with the fact $b \in S - M$. Therefore, for arbitrary $x \in M$, there is $d \in S - M$ such that $x \in SdS$. This means that

$$M \subseteq S(S - M)S,$$

hence, $M$ is a $C$-ideal.

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ПОЛУГРУППЫ, СОДЕРЖАЩИЕ ЗАКРЫТЫЕ ДВУСТОРОННИЕ ИДЕАЛЫ

Imrich Fabrici

Резюме

Двусторонний идеал $M$ полугруппы $S$ называется закрытым, если $M \subseteq S(S - M)S$.
В работе доказано, что множество всех двусторонних закрытых идеалов полугруппы $S$
является подструктуру структуры всех идеалов в $S$.
Приведено необходимое и достаточное условие для того, чтобы:
(1) полугруппа $S$ содержала наибольший закрытый идеал.
(2) каждый идеал полугруппы $S$ был закрытым.