Ján Jakubík
Isometries of MV-algebras

Mathematica Slovaca, Vol. 54 (2004), No. 1, 43--48

Persistent URL: http://dml.cz/dmlcz/130187

Math. Slovaca, **54** (2004), No. 1, 43–48

Mathematica Slovaca

Dedicated to Professor Sylvia Pulmannová on the occasion of her 65th birthday

ISOMETRIES OF MV-ALGEBRAS

JÁN JAKUBÍK

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. In this note we prove that for each isometry f of an MV-algebra $\mathcal{A}$ and for each element x of the underlying set of $\mathcal{A}$ the relation $f(f(x)) = x$ is valid.

In [7], an explicit formula characterizing all 2-periodic isometries of MV-algebras has been deduced. In the present note we prove that the mentioned result remains valid without the assumption of 2-periodicity.

1. Preliminaries

For defining MV-algebras several equivalent systems of axioms have been applied. Let us apply, e.g., the definition from [2]; thus an MV-algebra is an algebraic structure

$$\mathcal{A} = (A; \oplus, \odot, \neg, 0, 1),$$

where A is a nonempty set, $\oplus$ and $\odot$ are binary operations, $\neg$ is a unary operation and 0, 1 are nulary operations on A such that the identities (M1)–(M8) from [2] (cf. also [7]) are satisfied. (In [2], the symbol $*$ instead of $\odot$ has been used.)

Let $\mathcal{A}$ be an MV-algebra. It is well known that if we put

$$x \vee y = (x \odot \neg y) \oplus y\,, \qquad x \wedge y = (x \oplus \neg y) \odot y$$

2000 Mathematics Subject Classification: Primary 06F35.
Keywords: MV-algebra, isometry, 2-periodicity, subdirect product decomposition.
Supported by VEGA 1/9056/22.

for each $x, y \in A$, then $(A; \vee, \wedge)$ turns out to be a distributive lattice with the least element 0 and the greatest element 1. The corresponding partial order on A will be denoted by $\leqq$.

Further, there exists an abelian lattice ordered group G with a strong unit u such that A is the interval $[0, u]$ of G, the above mentioned operations $\vee$ and $\wedge$ on A coincide with the lattice operations in G (reduced to the set $[0, u]$) and for $a, b \in A$ we have (cf. [9])

$$a \oplus b = (a + b) \wedge u, \qquad \neg a = u - a,$$
$$a \odot b = \neg(\neg a \oplus \neg b).$$

It is also clear that $u = 1$. If $x, y \in A$, $x \leqq y$ and if the symbol $-$ denotes the corresponding subtraction in G, then $y - x \in A$.

1.1. LEMMA. (cf. [5; Lemma 1.10]) *Let $x, y \in A$, $x \leqq y$. Then*

$$y - x = \neg(x \oplus \neg y).$$

For $x, y \in A$ we put

$$\rho(x, y) = (x \vee y) - (x \wedge y).$$

Hence $\rho(x, y)$ is an element of A. From 1.1 we conclude:

1.2. COROLLARY. *Let $x, y \in A$. Then*

$$\rho(x, y) = \neg\big((x \wedge y) \oplus \neg(x \vee y)\big).$$

Autometrized lattice ordered groups have been investigated in several papers (cf., e.g., [3], [4], [10], [12]). For other types of partially ordered algebraic structures, cf. [8], [11], and the references quoted there.

From the well-known properties of autometrized lattice ordered groups we infer:

1.3. LEMMA. *Let $x, y, z \in A$. Then we have*

(i) $\rho(x, y) \geqq 0$*; moreover,* $\rho(x, y) = 0$ *if and only if* $x = y$.

(ii) $\rho(x, y) = \rho(y, x)$.

(iii) $\rho(x, y) \leqq \rho(x, z) + \rho(z, y)$.

1.4. LEMMA. *Let $x, y, z \in A$. Then*

(iii') $\rho(x, y) \leqq \rho(x, z) \oplus \rho(z, y)$.

P r o o f. We have

$$\rho(x, z) \oplus \rho(z, y) = \big(\rho(x, z) + \rho(z, y)\big) \wedge u.$$

Since $\rho(x,y) \in A$, we get $\rho(x,y) \leqq u$. Then according to (iii), the relation (iii') must be valid. □

In view of (i), (ii) and (iii') we say that ρ is an *autometrization* of the MV-algebra $\mathcal{A}$; the pair $(\mathcal{A};\rho)$ is called an autometrized MV-algebra $\mathcal{A}$.

A bijection $f\colon A \to A$ is defined to be an isometry of the MV-algebra $\mathcal{A}$, if

$$\rho(x,y) = \rho\big(f(x), f(y)\big)$$

is valid for each $x, y \in A$.

Further, f is called 2*-periodic* if $f\big(f(x)\big) = x$ for each $x \in A$. (Sometimes we write $f^2(x)$ instead of $f\big(f(x)\big)$.)

Since the lattice $(A; \vee, \wedge)$ is distributive, for each $a \in A$ there exists at most one complement (i.e., an element $b \in A$ with $a \wedge b = 0$, $a \vee b = u$); if such element b does exist, we denote it by a'.

The following result has been proved in [7].

(α) *Let f be an isometry of an MV-algebra $\mathcal{A}$. Suppose that f is 2-periodic. Put $f(0) = a$. Then there exists the element a' in A and for each $x \in A$ the formula*

$$f(x) = \big(a - (x \wedge a)\big) \vee (a' \wedge x)$$

is valid.

In the present paper we prove:

(β) *Each isometry of an MV-algebra is 2-periodic.*

Hence the assumption of 2-periodicity in (α) can be omitted. We remark that a result analogous to (β) fails to be valid for isometries in autometrized lattice ordered groups.

2. Proof of (β)

Let $\mathcal{A}$ be as above. Our considerations would be trivial in the case $A = \{0\}$; thus we assume that A fails to be a one-element set.

For proving (β) we need some auxiliary results.

It is well known that $\mathcal{A}$ can be represented as a subdirect product of linearly ordered MV-algebras (cf., e.g., [1], [6]). Hence we can suppose that there exists a system $\{A_i\}_{i\in I}$ of non-zero linearly ordered MV-algebras and a monomorphism

$$\varphi\colon \mathcal{A} \to \prod_{i\in I} \mathcal{A}_i \tag{1}$$

such that φ is a homomorphism, and for each $i \in I$ and $x^i \in A_i$ (where A_i is the underlying set of $\mathcal{A}_i$) there exists $a \in A$ with $\varphi(a)_i = x^i$. We denote

$\varphi(a)_i = a_i$; thus

$$\varphi(a) = (a_i)_{i \in I} .$$

The corresponding autometrization of the MV-algebra $\mathcal{A}_i$ will be denoted by ρ_i. Further, we denote by 0^i and 1^i the least and the greatest element of $\mathcal{A}_i$, respectively. Clearly, $0^i = 0_i$ and $1^i = 1_i$.

In view of 1.2 we have:

2.1. LEMMA. *Let $x, y \in A$ and $i \in I$. Then*

$$\rho(x,y)_i = \rho_i(x_i, y_i) .$$

2.2. LEMMA. *Let f be an isometry of $\mathcal{A}$. Suppose that $x, y \in A$, $i \in I$, $f(x)_i = f(y)_i$. Then $x_i = y_i$.*

Proof. We have

$$\rho(x,y) = \rho\big(f(x), f(y)\big) ,$$

whence

$$\rho(x,y)_i = \rho\big(f(x), f(y)\big)_i .$$

Thus according to 2.1,

$$\rho_i(x_i, y_i) = \rho_i\big(f(x)_i, f(y)_i\big) .$$

The assumption yields $\rho_i\big(f(x)_i, f(y)_i\big) = 0$ and thus $x_i = y_i$. □

Let f be an isometry of $\mathcal{A}$ and $i \in I$. We define a mapping $f_i \colon A_i \to A_i$ as follows. Let $x^i \in A_i$. There exists $x \in A$ with $x_i = x^i$. We put

$$f_i(x^i) = f(x)_i . \tag{2}$$

Then in view of 2.2, the mapping f_i is correctly defined; moreover, it is a bijection.

2.3. LEMMA. *Let i and f_i be as above. Then f_i is an isometry of $\mathcal{A}_i$.*

Proof. This is a consequence of 2.1. □

2.4. LEMMA. *Let $i \in I$ and let g be an isometry of $\mathcal{A}_i$. Then we have either $g(0_i) = 0_i$, or $g(0_i) = 1_i$.*

Proof. From the fact that $\mathcal{A}_i$ is linearly ordered we easily conclude that whenever $y, z \in A_i$ and $z \neq 0_i$, then

$$\rho(y,z) < \rho(0_i, 1_i) = 1_i .$$

By way of contradiction, assume that $0_i \neq g(0_i) \neq 1_i$. Then for each $y \in A_i$ we have $\rho\big(y, g(0_i)\big) < 1_i$. In particular,

$$1_i > \rho\big(g(1_i), g(0_i)\big) = \rho(1_i, 0_i) = 1_i ,$$

which is a contradiction. □

2.5. LEMMA. *Let i and g be as in 2.4 and $x^i \in A_i$. If $g(0_i) = 0_i$, then $g(x^i) = x^i$. If $g(0_i) = 1_i$, then $g(x^i) = 1_i - x^i$.*

Proof. At first assume that $g(0_i) = 0_i$. Then

$$\begin{aligned} x^i = x^i - 0_i = \rho(x^i, 0_i) &= \rho(g(x^i), g(0_i)) \\ &= \rho(g(x^i), 0_i) = g(x^i). \end{aligned}$$

Further, suppose that $g(0_i) = 1_i$. If $g(1_i) \neq 0_i$, then

$$1_i = \rho(1_i, 0_i) = \rho(g(1_i), g(0_i)) = \rho(g(1_i), 1_i) < 1_i,$$

which is impossible. Hence $g(1_i) = 0_i$. Clearly

$$\rho(x^i, 1_i) = 1_i - x^i,$$

therefore

$$\begin{aligned} g(x^i) = \rho(g(x^i), 0_i) &= \rho(g(x^i), g(1_i)) \\ &= \rho(x^i, 1_i) = 1_i - x^i. \end{aligned}$$

□

2.6. COROLLARY. *Let i, g and x^i be as in 2.5. Then $g^2(x^i) = x^i$.*

Proof of (β). Assume that f is an isometry of $\mathcal{A}$. Let $i \in I$ and let f_i be as above. In view of 2.3, f_i is an isometry of $\mathcal{A}_i$. Hence according to 2.6,

$$f_i^2(x^i) = x^i$$

for each $x^i \in A_i$.

Let $x \in A$. In view of (2) we get

$$(f^2(x))_i = f_i^2(x_i) = x_i$$

for each $i \in I$. Therefore $f^2(x) = x$. □

According to (β), the assumption of 2-periodicity of f can be omitted in (α).

REFERENCES

[1] CIGNOLI, R.—D'OTTAVIANO, I. M. I.—MUNDICI, D.: *Algebraic Foundations of Many-valued Reasoning*, Kluwer Academic Publishers, Dordrecht, 2000.

[2] GLUSHANKOF, D.: *Cyclic ordered groups and MV-algebras*, Czechoslovak Math. J. **44** (1994), 725–739.

[3] HOLLAND, CH.: *Intrinsic metrics for lattice ordered groups*, Algebra Universalis **19** (1984), 142–150.

[4] JAKUBÍK, J.: *Isometries of lattice ordered groups*, Czechoslovak Math. J. **30** (1980), 142–152.

[5] JAKUBÍK, J.: *Sequential convergences on MV-algebras*, Czechoslovak Math. J. **45** (1995), 709–726.

[6] JAKUBÍK, J.: *Subdirect product decompositions of MV-algebras*, Czechoslovak Math. J. **49** (1999), 163–173.

[7] JAKUBÍK, J.: *On intervals and isometries of MV-algebras*, Czechoslovak Math. J. **52** (2002), 651–663.

[8] JASEM, M.: *Weak isometries and direct decompositions of dually residuated lattice ordered semigroups*, Math. Slovaca **43** (1993), 119–136.

[9] MUNDICI, D.: *Interpretation of AFC*-algebras in Łukasiewicz sentential calculus*, J. Funct. Anal. **65** (1986), 15–63.

[10] POWELL, W. B.: *On isometries in abelian lattice ordered groups*, J. Indian Math. Soc. **46** (1982), 189–194.

[11] RACHŮNEK, J.: *Izometries in ordered groups*, Czechoslovak Math. J. **34** (1984), 334–341.

[12] SWAMY, K. L.: *Izometries in autometrized lattice ordered groups*, Algebra Universalis **8** (1977), 58–64.

Received November 18, 2002

Matematický ústav SAV
Grešákova 6
SK-040 01 Košice
SLOVAKIA
E-mail: kstefan@saske.sk