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NIKODÝM CONVERGENCE THEOREM
FOR UNIFORM SPACE VALUED
FUNCTIONS DEFINED ON D–POSETS

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ABSTRACT. Nikodým convergence type theorem with necessary and sufficient conditions for a sequence of functions defined on a D-poset and with values in a uniform space is proved.

1. Introduction

The classical Nikodým convergence theorem says that the limit of a sequence of countable additive measures is again a countable additive measure. This important theorem of the measure theory has many generalizations in different directions, even for non-additive set functions. E. Pap [24], [25] has investigated set functions with values in an arbitrary uniform space $Y$, without considering any algebraic operation on $Y$.

On the other hand, by the need of mathematical foundations of propositional calculus of quantum mechanics there were developed many structures as quantum logic (= orthomodular poset) [5], [6], [7], [8], [9], [12], [16], [26], [29], orthoalgebra [15] and very recently D-poset [18], [19].

In this paper, we obtain necessary and sufficient conditions for Nikodým convergence theorem to be true for a sequence of functions defined on a D-poset and with values in a uniform space.


Key words: D-poset, $\sigma(\oplus)$-D-poset, uniform space, orthomodular poset, MV-algebra, orthoalgebra.

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2. σ(⊕)-D-poset

We have by [18], [19], [14]

DEFINITION 2.1. A D-poset (difference poset) is a partially ordered set \( L \) with a partial ordering \( \leq \), maximal element \( 1 \), and with a partial binary operation \( \ominus: L \times L \to L \), called difference, such that, for \( a, b \in L \), \( b \ominus a \) is defined if and only if \( a \leq b \), for that the following axioms hold for \( a, b, c \in L \):

- (DP₁) \( b \ominus a \leq b \);
- (DP₂) \( b \ominus (b \ominus a) = a \);
- (DP₃) \( a \leq b \leq c \implies c \ominus b \leq c \ominus a \) and \( (c \ominus a) \ominus (c \ominus b) = b \ominus a \).

Then there exists also a minimal element \( 0 \) (\( = 1 \ominus 1 \)).

The following properties of the operation \( \ominus \) have been proved in [19]:

- (a) \( a \ominus 0 = a \).
- (b) \( a \ominus a = 0 \).
- (c) \( a \leq b \implies b \ominus a = 0 \iff b = a \).
- (d) \( a \leq b \implies b \ominus a = b \iff a = 0 \).
- (e) \( a \leq b \leq c \implies b \ominus a \leq c \ominus a \) and \( (c \ominus a) \ominus (b \ominus a) = c \ominus b \).
- (f) \( b \leq c, a \leq c \ominus b \implies b \leq c \ominus a \) and \( (c \ominus b) \ominus a = (c \ominus a) \ominus b \).
- (g) \( a \leq b \leq c \implies a \leq c \ominus (b \ominus a) \) and \( (c \ominus (b \ominus a)) \ominus a = c \ominus b \).

For an arbitrary but fixed element \( a \in L \) we define

\[ a^\perp := 1 \ominus a . \]

We have:

- (i) \( a^{\perp \perp} = a \);
- (ii) \( a \leq b \implies b^{\perp} \leq a^{\perp} \).

The elements \( a \) and \( b \) from \( L \) are orthogonal, denoted by \( a \perp b \), if and only if \( a \leq b^{\perp} \) (or \( b \leq a^{\perp} \)).

We define a partial binary operation \( \oplus: L \times L \to L \) for orthogonal elements \( a \) and \( b \) such that

\[ b \leq a \oplus b \quad \text{and} \quad a = (a \oplus b) \ominus b . \]

This operation \( \oplus \) is commutative and associative ([14]).

The notion of D-poset covers many important examples.

Example 2.2. ([6], [7], [8], [10], [11], [26]) An orthomodular poset is a partially ordered set \( O \) with an ordering \( \leq \), the least and greatest elements \( 0 \) and \( 1 \), respectively, and an orthocomplementation \( \perp: O \to O \) such that:

- (OM₁) \( a^{\perp \perp} = a \) (\( a \in O \));
- (OM₂) \( a \lor a^\perp = 1 \) (\( a \in O \));
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(OM₃) if \( a \leq b \), then \( b^\perp \leq a^\perp \);
(OM₄) if \( a \leq b^\perp \), then \( a \lor b \in O \);
(OM₅) if \( a \leq b \), then \( b = a \lor (a \lor b^\perp)^\perp \).

Taking for \( a \leq b \)

\[ b \ominus a := (a \lor b^\perp)^\perp, \]

we obtain that the orthomodular poset \( O \) is a D-poset.

Example 2.3. ([2], [21]) An \( MV\)-algebra is a set \( M \) endowed with two binary operations \( \oplus \) and \( \odot \), an unary operation \( \star \) and two elements 0 and 1 such that, for all \( a, b, c \in M \),

- (MV₁) \( a \oplus b = b \oplus a \);
- (MV₂) \( (a \oplus b) \oplus c = a \oplus (b \oplus c) \);
- (MV₃) \( a \oplus 0 = a \);
- (MV₄) \( a \oplus 1 = 1 \);
- (MV₅) \( (a^\star)^\star = a \);
- (MV₆) \( 0^\star = 1 \);
- (MV₇) \( a \oplus a^\star = 1 \);
- (MV₈) \( (a^\star \odot b)^\star \odot b = (a \odot b^\star)^\star \odot a \);
- (MV₉) \( a \odot b = (a^\star \odot b^\star)^\star \).

Taking

\[ a \leq b \iff (a \odot b^\star) \odot b = b \]

and for \( a \leq b \)

\[ b \ominus a := (a \oplus b^\star)^\star, \]

we obtain that the MV-algebra \( M \) is a D-poset.

Example 2.4. ([15], [14]) An orthoalgebra is a set \( A \) with two particular elements \( 0, 1 \), and with a partial binary operation \( \oplus : A \times A \to A \) such that for all \( a, b, c \in A \),

- (OA₁) if \( a \oplus b \in A \), then \( b \oplus a \in A \) and \( a \oplus b = b \oplus a \);
- (OA₂) if \( b \oplus c \in A \) and \( a \oplus (b \oplus c) \in A \), then \( a \oplus b \in A \) and \( (a \oplus b) \oplus c \in A \),
   and \( a \oplus (b \oplus c) = (a \oplus b) \oplus c \);
- (OA₃) for any \( a \in A \) there is a unique \( b \in A \) such that \( a \oplus b \) is defined, and
   \( a \oplus b = 1 \);
- (OA₄) if \( a \oplus a \) is defined, then \( a = 0 \).

We have \( a \leq b \) if and only if there exists an element \( c \in A \) such that \( a \oplus c \) is defined in \( A \) and \( a \oplus c = b \). An element \( b \) is the orthocomplement of \( a \) (denoted by \( a^\perp \)) if and only if \( b \) is a (unique) element of \( A \) such that \( b \oplus a \) is defined in \( A \) and \( a \oplus b = 1 \).

Taking for \( a \leq b \)

\[ b \ominus a := (a \oplus b^\perp)^\perp, \]

we obtain that the orthomodular poset \( O \) is a D-poset.
we obtain that the orthoalgebra $A$ is a D-poset. We remark that each orthomodular poset (Example 2.2) is an orthoalgebra, but the opposite is not true (see example of R. Wright in [15]).

**Example 2.5.** ([19], [14]) Let $\mathcal{E}(H)$ be the set of all Hermitian operators $T$ on a Hilbert space $H$ with $O \leq T \leq I$, where $O$ and $I$ are the zero and identity operators, respectively, on $H$. The set $\mathcal{E}(H)$ is a D-poset, which is not an orthoalgebra.

**Example 2.6.** ([18]) Let $\Omega$ be a nonempty set and $\mathcal{F}$ the family of all fuzzy sets on $\Omega$, i.e., $\mathcal{F} = [0,1]^{\Omega}$. We have for $f, g \in \mathcal{F}$

$$f \leq g \iff f(\omega) \leq g(\omega) \quad (\omega \in \Omega).$$

Let $\Phi: [0,1] \to [0,\infty)$ be an injective increasing continuous function such that $\Phi(0) = 0$. Taking for $f \leq g$

$$(g \ominus f)(\omega) = \Phi^{-1}\left(\Phi(g(\omega)) - \Phi(f(\omega))\right) \quad (\omega \in \Omega)$$

we obtain that $\mathcal{F}$ is a D-poset.

$L$ will always denote a D-poset. Let $\{a_1, \ldots, a_n\} \subseteq L$. We define

$$a_1 \oplus \cdots \oplus a_n = \begin{cases} 
0 & \text{for } n = 0, \\
 a_1 & \text{for } n = 1, \\
 (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n & \text{for } n \geq 3,
\end{cases}$$

supposing that $a_1 \oplus \cdots \oplus a_{n-1}$ and $a_1 \oplus \cdots \oplus a_n$ exist in $L$. We have by [14]

**Definition 2.7.** A finite subset $\{a_1, \ldots, a_n\}$ of $L$ is $\oplus$-orthogonal if $a_1 \oplus \cdots \oplus a_n$ exists in $L$.

We say that an $\oplus$-orthogonal subset $\{a_1, \ldots, a_n\}$ of $L$ has a $\oplus$-sum $\bigoplus_{i=1}^{n} a_i$, defined by

$$\bigoplus_{i=1}^{n} a_i := a_1 \oplus \cdots \oplus a_n.$$ 

We remark that the preceding $\oplus$-sum is independent of any permutation of elements.

**Definition 2.8.** A subset $G$ of $L$ is $\oplus$-orthogonal if every finite subset $F$ of $G$ is $\oplus$-orthogonal.

We say that an $\oplus$-orthogonal subset $G = \{a_i : i \in I\}$ of $L$ has an $\oplus$-sum in $L$, $\bigoplus_{i \in I} a_i$, if in $L$ there exists the join

$$\bigoplus_{i \in I} a_i := \sup \left\{ \bigoplus_{i \in F} a_i : F \text{ finite subset of } I \right\}.$$ 

Any subset of a $\oplus$-orthogonal set is again $\oplus$-orthogonal.
**Definition 2.9.** A D-poset $L$ is a complete D-poset ($\sigma(\oplus)$-D-poset) if, for every $\oplus$-orthogonal subset (every countable $\oplus$-orthogonal subset) $G$ of $L$, there exists the $\oplus$-sum in $L$.

**Definition 2.10.** A D-poset $L$ is quasi-$\sigma$-complete if for every $\oplus$-orthogonal sequence $(a_i)$ in $L$ there exists a subsequence $(a_i)_{i \in M}$ such that $\bigoplus_{i \in I} a_i \in L$ for each $I \subseteq M$.

**Remark 2.11.** The notion of quasi-$\sigma$-ring is introduced by C. Constantinescu [4], [3].

We shall give now an example of a $\sigma(\oplus)$-D-poset.

**Example 2.12.** Let $S$ be any set of real numbers between 0 and 1, where $S$ satisfies the following conditions

(i) $0 \in S$ and $1 \in S$;
(ii) if $x, y \in S$, then $\min(1, x+y) \in S$;
(iii) if $x, y \in S$, then $\max(0, x+y-1) \in S$;
(iv) if $x \in S$, then $1 - x \in S$.

The operations $\oplus$, $\odot$ and $*$ are defined as follows:

\[
\begin{align*}
x \oplus y & := \min(1, x+y), \\
x \odot y & := \max(0, x+y-1), \\
x^* & := 1 - x.
\end{align*}
\]

The system $(S, \oplus, \odot, *, 0, 1)$ is an MV-algebra. If we take $S = [0, 1]$, we obtain a $\sigma$-MV-algebra with respect to the operation $\oplus$ and, in this way, also a $\sigma(\oplus)$-D-poset, since for $x \leq y$ we have that the operation $\odot$ defined by

\[
x \odot y := (x \odot y^*)^*
\]
gives a $\sigma$-D-poset with respect to the operation $\oplus_D$ defined by

\[
x \oplus_D y = (y^* \ominus x)^*,
\]
which coincides with the operation $\oplus$, i.e.,

\[
x \oplus_D y = (y^* \ominus x)^* = \left( (x \oplus (y^*))^* \right)^* = x \oplus y.
\]

We remark that for $S = \{0, 1\}$ we trivially obtain also a $\sigma(\oplus)$-D-poset. But if $S$ is the set of all rational numbers between 0 and 1, then this is a MV-algebra, and so also a D-poset, which is not $\sigma(\oplus)$-MV-algebra, and so also not a $\sigma(\oplus)$-D-poset.
3. Nikodým convergence theorem

Let $Y$ be a uniform space with the uniformity $U$. We denote by $D$ the family of all uniformly continuous pseudometrics defined on $(Y, U)$.

Let $L$ be a $D$-poset.

**Definition 3.1.** For $d \in D$ the $d$-semivariation of a function $\mu : L \to Y$ with respect to a point $x_0 \in Y$ is

$$\tilde{\mu}^{x_0}_d(b) := \sup \{d(\mu(c), x_0) : c \leq b, c \in L\} \quad (b \in L).$$

We define for $d \in D$, $x_0 \in Y$ and a function $\mu : L \to Y$

$$\alpha^{x_0}_d(a, \mu) := \limsup_{n \to \infty} \{d(\mu(a \oplus b), x_0) : \tilde{\mu}^{x_0}_d(b) < \frac{1}{n}, b \in L\} \quad (a \in L).$$

**Remark 3.2.** For a set function $\mu$ defined on a quasi-$\sigma$-ring $\Sigma$, the previous definition of $\alpha^{x_0}_d$ coincides with that given in the paper of E. Pap [23].

We shall need, in the proof of the main theorem, the following:

**Definition 3.3.** A function $\mu : L \to Y$ is said to be $x_0$-exhaustive for $x_0 \in Y$ if for each $d \in D$

$$\lim_{n \to \infty} d(\mu(a_n), x_0) = 0$$

for each $\oplus$-orthogonal sequence $(a_n)$ of elements from $L$.

**Lemma 3.4.** Let $Y$ be a uniform space and $L$ a quasi-$\sigma$-complete $D$-poset. If $\mu : L \to Y$ is an $x_0$-exhaustive function and $(a_n)$ a sequence of $\oplus$-orthogonal elements from $L$, then, for each $d \in D$ and each $\varepsilon > 0$, there exists a $\oplus$-orthogonal subsequence $(a_{n_i})$ of $(a_n)$ such that

$$\tilde{\mu}^{x_0}_d\left(\bigoplus_{i \in I} a_{n_i}\right) < \varepsilon$$

for any $I \subset \mathbb{N}$.

The proof goes taking to a contradiction with the $x_0$-exhaustivity of the function $\mu$.

**Theorem 3.5.** Let $Y$ be a uniform space and $L$ a quasi-$\sigma$-complete $D$-poset. Let $(\mu_n)$ be a sequence of functions $\mu_n, \mu_n : L \to Y$, such that each $\mu_n$ is $x_0$-exhaustive, and they satisfy the following conditions for the element $x_0$ from $Y$:

(i) for each $d \in D$ and for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(\mu_n(a), x_0) < \delta \quad \text{and} \quad d(\mu_n(b), x_0) < \delta$$
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for $a \leq b$, $a, b \in \mathbb{L}$ ($n \in \mathbb{N}$) implies

$$d(\mu_n(b \ominus a), x_0) < \varepsilon;$$

(ii) for each $d \in \mathcal{D}$ and for each $\delta > 0$, there exists $\gamma > 0$ such that

$$d(\mu_n(a), x_0) < \gamma, a \in \mathbb{L} \ (n \in \mathbb{N}) \implies \alpha_{d}^{x_0}(a, \mu_n) < \delta \ (n \in \mathbb{N});$$

(iii) for each $d \in \mathcal{D}$

$$\lim_{n \to \infty} d(\mu_n(a), \mu(a)) = 0$$

for each $a \in \mathbb{L}$.

Then $\mu$ is $x_0$-exhaustive if and only if $\mu_n$ ($n \in \mathbb{N}$) are uniformly $x_0$-exhaustive.

Proof. Let us suppose that $\mu$ is $x_0$-exhaustive, but $(\mu_n)$ is not uniformly $x_0$-exhaustive. Hence there exist $\varepsilon > 0$, $d$ from $\mathcal{D}$ and a $\oplus$-orthogonal sequence $(a_k)$ of elements from $\mathbb{L}$ and a subsequence $(\mu_{n_k})$ such that

$$d(\mu_{n_k}(a_k), x_0) > \varepsilon \quad (1)$$

for each $k \in \mathbb{N}$. By (i), we choose $\delta > 0$ corresponding to $\varepsilon > 0$. By (ii), we choose $\gamma > 0$ corresponding to $\delta > 0$. Since $\mu$ is $x_0$-exhaustive, by Lemma 3.4, there exists a $\oplus$-orthogonal subsequence $(a_{k_i})$ of $(a_k)$ such that

$$\tilde{\mu}_d^{x_0} \left( \bigoplus_{i \in I} a_{k_i} \right) < \frac{\gamma}{2} \quad (2)$$

for each $I \subset \mathbb{N}$. Now, let us denote $m_i := \mu_{n_{k_i}}$ and $b_i := a_{k_i} \ (i \in \mathbb{N})$ and $i_1 = 1$. By (iii), there exists an index $i_2$ such that

$$d(m_{i_2}(b_{i_1}), \mu(b_{i_1})) < \frac{\gamma}{2}. \quad (3)$$

The inequality

$$d(m_{i_2}(b_{i_1}), \mu(b_{i_1})) \geq d(m_{i_2}(b_{i_1}), x_0) - d(\mu(b_{i_1}), x_0),$$

by (2) and (3), implies

$$d(m_{i_2}(b_{i_1}), x_0) < \gamma. \quad (4)$$

Since $m_{i_2}$ is $x_0$-exhaustive, we have by Lemma 3.4 that there exists a $\oplus$-orthogonal subsequence $(b_i^{2})$ of $(b_i)_{i=i_1+1}$ such that

$$(m_{i_2})^{x_0} \left( \bigoplus_{i \in I} b_i^2 \right) < \frac{\gamma}{2}$$

for each $I \subset \mathbb{N}$. This implies by (4) and (ii)

$$\alpha_{d}^{x_0}(b_{i_1} \oplus \bigoplus_{i \in I} b_i^2, m_{i_2}) < \delta$$
for each \( I \subseteq \mathbb{N} \). Using (2) we obtain
\[
d\left(\mu(b_{i_1} \oplus b_{i_2}^k), x_0\right) < \frac{\gamma}{2}
\]
for each \( k \in \mathbb{N} \), and, by (iii), there exists an index \( i_3 \) such that
\[
d\left(m_{i_3}(b_{i_1} \oplus b_{i_2}^k), \mu(b_{i_1} \oplus b_{i_2}^k)\right) < \frac{\gamma}{2}.
\]
Hence, by the inequality
\[
d\left(m_{i_3}(b_{i_1} \oplus b_{i_2}^k), \mu(b_{i_1} \oplus b_{i_2}^k)\right) \geq d\left(m_{i_3}(b_{i_1} \oplus b_{i_2}^k), x_0\right) - d\left(\mu(b_{i_1} \oplus b_{i_2}^k), x_0\right),
\]
we obtain
\[
d\left(m_{i_3}(b_{i_1} \oplus b_{i_2}^k), x_0\right) < \gamma,
\]
where \( b_{i_3} \) is chosen from the sequence \( (b_{i_k}^k) \). Since \( m_{i_3} \) is \( x_0 \)-exhaustive, by Lemma 3.4, there exists a \( \oplus \)-orthogonal subsequence \( (b_{i_3}^3) \) of \( (b_{i_1}^3)_{i=i_2+1}^\infty \) such that
\[
\left(\tilde{m}_{i_3}\right)^{x_0}_{i_3} \left(\bigoplus_{i \in I} b_{i_3}^3\right) < \frac{\gamma}{2}
\]
for each \( I \subseteq \mathbb{N} \). This implies by (5) and (ii)
\[
d\left(m_{i_3}\left(b_{i_1} \oplus b_{i_2} \oplus \bigoplus_{i \in I} b_{i_3}^3\right), x_0\right) < \delta
\]
for each \( I \subseteq \mathbb{N} \). Continuing the preceding procedure we obtain two sequences \( (m_{i_k}) \) and \( (b_{i_k}) \). Taking \( b = \bigoplus_{k=1}^\infty b_{i_k} \), we choose by (iii) an index \( k_0 \) such that
\[
d\left(m_{i_{k_0}}(b), x_0\right) < \eta < \delta.
\]
This follows by (2) and the inequality
\[
d\left(m_{i_{k_0}}(b), \mu(b)\right) \geq d\left(m_{i_{k_0}}(b), x_0\right) - d\left(\mu(b), x_0\right).
\]
Since, by \( (DP_1) \),
\[
b \ominus a \leq b,
\]
we obtain by the preceding procedure that
\[
d\left(m_{i_{k_0}}(b \ominus a), x_0\right) < \delta.
\]
This, together with (i) and (6), implies by \( (DP_2) \)
\[
\epsilon > d\left(m_{i_{k_0}}(b \ominus (b \ominus b_{i_{k_0}})), x_0\right) = d\left(m_{i_{k_0}}(b_{i_{k_0}}), x_0\right)
\]
and gives a contradiction with (1).

The opposite statement follows by (iii). \( \Box \)
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