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ON GENERALIZED SEMICONTINUITY-PRESERVING MULTIFUNCTIONS

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The basic problem of mathematical programming is to find the supremum v of the so-called objective function f: $Y \to R$ over some set F of constraints. One of the most important questions is the question of stability of this optimal value v. This question can be formulated as follows.

Let F: $X \to Y$ be a multifunction, f: $X \times Y \to R$ be a function and let v: $X \to R$ be defined as

$$v(x) = \sup \{f(x, y): \ y \in F(x)\}. \tag{*}$$

Then we can ask under what conditions given on F the continuity, resp. semicontinuity, of f is preserved in a certain way by v.

This question was mostly solved for continual perturbations of v (see [1], [3], [4], [5], [7]). But we can also obtain similar results for quasicontinuity (see [6]), almost continuity, e.t.c. In order to unify these results we use the concept of the so-called $\mathcal{S}$-continuity, which includes these types of generalized continuity. This concept was introduced in [2] for functions and is applicable also for multifunctions.

In the whole paper we suppose the objective function f and the function v to be finite. Note that all definitions can be modified and all theorems are valid also in the case when the values $+\infty$ or $-\infty$ are admitted.

If not specified, X, Y denote general topological spaces and R denotes the set of reals with the usual topology.

1. Local sieves and $\mathcal{S}$-semicontinuity

In [2] the following concepts are introduced.

Definition 1. *A family $\mathcal{S}_{x_0}$ of subsets of X is called a local sieve at a point $x_0 \in X$ if:*

1. *$x_0 \in A$ for any $A \in \mathcal{S}_{x_0}$,*
2. *$A \subset B$ and $A \in \mathcal{S}_{x_0}$ implies $B \in \mathcal{S}_{x_0}$,*
3. *$\mathcal{U}_{x_0} \subset \mathcal{S}_{x_0}$, where $\mathcal{U}_{x_0}$ denotes the system of all neighbourhoods of a point x_0.*

Definition 2. *A local sieve $\mathcal{S}_{x_0}$ is called strongly local if $A \cap U \in \mathcal{S}_{x_0}$ for any $A \in \mathcal{S}_{x_0}$ and any $U \in \mathcal{U}_{x_0}$.*

In everything that follows we shall consider only strongly local sieves. Examples of the sieves, which are not strongly local can be found in [2], where also the following concept is introduced.

Definition 3. *If $\mathcal{S}_{x_0}$ is a local sieve at a point $x_0 \in X$, we say the function f from X to Y is $\mathcal{S}$-continuous at x_0 if $f^{-1}(V) \in \mathcal{S}_{x_0}$ for any neighbourhood V of the point $f(x_0)$.*

If we consider real valued functions, we can introduce the concept of $\mathcal{S}$ -semicontinuity which we shall call $\mathcal{S}$-order semicontinuity to distinguish it from the $\mathcal{S}$-semicontinuity of multifunctions. In the following definitions we suppose that a local sieve $\mathcal{S}_{x_0}$ at a point x_0 is given.

Definition 4. *A function f: $X \to R$ is said to be $\mathcal{S}$-order upper (lower) semicontinuous at a point $x_0 \in X$ if for any $\varepsilon > 0$ there exists a set $A \in \mathcal{S}_{x_0}$ such that $f(x) < f(x_0) + \varepsilon$ ($f(x) > f(x_0) - \varepsilon$)* for any $x \in A$.

Definition 5. *A multifunction F: $X \to Y$ is said to be $\mathcal{S}$-upper (lower) semicontinuous at a point $x_0 \in X$ if for any open set V such that $V \supset F(x_0)$ ($F(x_0) \cap V \neq \emptyset$) there exists a set $A \in \mathcal{S}_{x_0}$ such that $F(x) \subset V$ ($F(x) \cap V \neq \emptyset$) for any $x \in A$.*

We shall denote by $\mathcal{S}$-o.u.s.c., $\mathcal{S}$-o.l.s.c., $\mathcal{S}$-u.s.c., $\mathcal{S}$-l.s.c. the $\mathcal{S}$-order upper semicontinuity, the $\mathcal{S}$-order lower semicontinuity, the $\mathcal{S}$-upper semicontinuity, the $\mathcal{S}$-lower semicontinuity respectively.

Suppose that a local sieve $\mathcal{S}_x$ is given for any $x \in X$. Then a set $G \subset X$ is said to be $\mathcal{S}$-open if G belongs to $\mathcal{S}_x$ for any $x \in G$. The $\mathcal{S}$-closure of a set H can be defined as the set of all $x \in X$ such that $H \cap A \neq \emptyset$ for any $A \in \mathcal{S}_x$. Let the $\mathcal{S}$-closure be denoted by $\mathcal{S}$-cl H and a set H will be called $\mathcal{S}$-closed if $\mathcal{S}\text{-cl}H = H$. It is evident that a set G is $\mathcal{S}$-open iff a set $X \setminus G$ is $\mathcal{S}$-closed.

If we denote

$$F^+(V) = \{x\colon\ F(x) \subset V\},$$

$$F^-(V) = \{x\colon\ F(x) \cap V \neq \emptyset\},$$

we can characterize the $\mathcal{S}$-semicontinuity in this way:

A multifunction F: $X \to Y$ is $\mathcal{S}$-u.s.c. ($\mathcal{S}$-l.s.c.) at a point $x \in X$ iff $F^+(V) \in \mathcal{S}_x$ ($F^-(V) \in \mathcal{S}_x$) for any open set V such that $F(x) \subset V$ ($F(x) \cap V \neq \emptyset$). A multifunction F is $\mathcal{S}$-u.s.c. ($\mathcal{S}$-l.s.c.) at $x \in X$ iff $x \in F^-(H)$ ($x \in F^+(H)$) for any closed set $H \subset Y$ such that $x \in \mathcal{S}\text{-cl}\, F^-(H)$ ($x \in \mathcal{S}\text{-cl}\, F^+(H)$).

By means of special selection of a local sieve we can obtain some known types of generalized continuity resp. semicontinuity.

If $\mathcal{S}_x = \mathcal{U}_x$, we obtain the continuity with respect to the topology given on X.

If $\mathcal{S}_x = \{A\colon\ x \in A,\ x \in \overline{A^\circ}\}$, we obtain the quasicontinuity. Here the symbols A°, $\bar{A}$ are used for the interior, the closure of the set A respectively.

If $\mathcal{S}_x = \{A\colon\ x \in A,\ x \in (\bar{A})^\circ\}$, we obtain the almost continuity.

If $X=R^n$, then the approximate continuity can be defined as the $\mathcal{S}$-continuity, where the local sieve at a point x is formed by all the sets which contain x as a density point.

For definitions of the above mentioned concepts see [2], where all these sieves are proved to be strongly local, too.

2. Preservation of $\mathcal{S}$-semicontinuity

In this Section we want to find a class of multifunctions for which the $\mathcal{S}$-order semicontinuity of an objective function in (*) is preserved. First we shall examine the $\mathcal{S}$-o.u.s.c.. We shall introduce similar notations as in [4], where this question is solved, but only for the objective function of one variable $y \in Y$.

More precisely it will be as follows. Denote

$$\mathcal{S}.\mathcal{O}.\mathcal{U}.(x)=\{v\colon\ X \to R\colon\ v \text{ is } \mathcal{S}\text{-o.u.s.c. at } x\},$$

$$\mathcal{S}.\mathcal{U}.(x)=\{F\colon\ X \to Y\colon\ F \text{ is } \mathcal{S}\text{-u.s.c. at } x\}.$$

For any $(x, y) \in X \times Y$ denote by $\mathcal{F}(x, y)$ an arbitrary subset of the set of all functions $f\colon X \times Y \to R$ which are order upper semicontinuous at a point (x, y) and further denote

$$\mathcal{F}.\mathcal{S}.\mathcal{O}.\mathcal{U}.(x)=\{F\colon\ X \to Y\colon\ v \in \mathcal{S}.\mathcal{O}.\mathcal{U}.(x) \text{ for any } f \text{ belonging to } \mathcal{F}(x, y) \text{ for any } y \in F(x)\}.$$

Our aim is to investigate a connection between $\mathcal{S}.\mathcal{U}.(x)$ and $\mathcal{F}.\mathcal{S}.\mathcal{O}.\mathcal{U}.(x)$. For this purpose we need another concept already introduced in [4].

Definition 6. *A multifunction F: $X \to Y$ is said to be $\mathcal{F}.\mathcal{S}.\mathcal{U}.$-stable at a point $x_0 \in X$ if for any $\varepsilon > 0$ and for any f belonging to $\mathcal{F}(x_0, y_0)$ for any $y_0 \in F(x_0)$ there is a set $A \in \mathcal{S}_{x_0}$ such that*

$$F(x) \subset \{y\colon\ f(x, y) < v(x_0)+\varepsilon\}$$

for any $x \in A$.

Evidently it is the same as

$$A \subset F^+(\{y\colon\ f(x, y) < v(x_0)+\varepsilon\}).$$

The ideas of the proofs of the next theorem and of the propositions following it are not very different from the ideas used in [4]. Thus we shall introduce them without proofs, later we shall give the proofs of an analoguous theorem and propositions for the $\mathcal{S}$-order lower semicontinuity.

Theorem 1. *The $\mathcal{F}.\mathcal{S}.\mathcal{U}.$-stable multifunctions are precisely those that preserve the $\mathcal{S}$-o.u.s.c. of $\mathcal{F}$. It means that*

$$F \text{ is } \mathcal{F}.\mathcal{S}.\mathcal{U}.\text{-stable at } x_0 \text{ iff } F \in \mathcal{F}.\mathcal{S}.\mathcal{O}.\mathcal{U}.(x_0).$$

Proposition 1. *The multifunction F is $\mathcal{F}.\mathcal{S}.\mathcal{U}.$-stable at x_0 iff for each f belonging to $\mathcal{F}(x_0, y_0)$ for any $y_0 \in F(x_0)$ and for any $r \in R$*

$$x_0 \in \bigcup_{\varepsilon>0} F^+(\{y: \ f(x_0, y) < r - \varepsilon\})$$

implies the existence of a set $A \in \mathcal{S}_{x_0}$ such that for any $x \in A$ there holds

$$x \in F^+(\{y: \ f(x, y) < r\}).$$

Denote $B(f, x, r) = \{y: \ f(x, y) \geq r\}$.

Proposition 2. *A multifunction F is $\mathcal{F}.\mathcal{S}.\mathcal{U}.$-stable at x_0 iff for each f belonging to $\mathcal{F}(x_0, y_0)$ for any $y_0 \in F(x_0)$ and for any $r \in R$*

$$x_0 \in \mathcal{S}\text{-cl}\,\{x: \ x \in F^-(B(f, x, r))\}$$

implies

$$x_0 \in \bigcap_{\varepsilon>0} F^-(B(f, x_0, r - \varepsilon)).$$

Proposition 3. *A multifunction F is $\mathcal{F}.\mathcal{S}.\mathcal{U}.$-stable at any $x \in X$ iff*

$$\bigcap_{\varepsilon>0} \mathcal{S}\text{-cl}\,\{x: \ x \in F^-(B(f, x, r - \varepsilon))\} = \bigcap_{\delta>0} \{x: \ x \in F^-(B(f, x, r - \delta))\}$$

for any $r \in R$ and for any f belonging to $\mathcal{F}(x, y)$ for any $y \in F(x)$.

For any $(x, y) \in X \times Y$ denote by $\mathcal{O}.\mathcal{U}.(x, y)$ the set of all functions $f: \ X \times Y \to R$ which are o.u.s.c. at (x, y). If $\mathcal{F}(x, y) = \mathcal{O}.\mathcal{U}.(x, y)$, then we speak about multifunctions $\mathcal{S}.\mathcal{U}.$-stable at a point x and the following characterization of such multifunctions is possible.

Theorem 2. *If $F(x_0)$ is compact, then F is $\mathcal{S}.\mathcal{U}.$-stable at x_0 iff $F \in \mathcal{S}.\mathcal{U}.(x_0)$. It means that in the class of compact valued multifunctions the $\mathcal{S}.\mathcal{U}.$-stable multifunctions are precisely the $\mathcal{S}$-u.s.c. multifunctions.*

Proof. Suppose F to be $\mathcal{S}.\mathcal{U}.$-stable at x_0. Let K be a closed set in Y and $x_0 \in \mathcal{S}\text{-cl}\, F^-(K)$. The function $f: \ X \times Y \to R$ defined by $f(x, y) = \chi_K(y)$, where χ_K is the characteristic function of the set K, is o.u.s.c. on $X \times Y$ and $K = B(f, x, 1)$. Therefore for any $x \in F^-(K)$ there also holds $x \in F^-(B(f, x, 1))$. Thus we have $x_0 \in \mathcal{S}\text{-cl}\,\{x: \ x \in F^-(B(f, x, 1))\}$ and according to Proposition 2 we have

$$x_0 \in \bigcap_{\varepsilon>0} F^-(\{y: \ f(x_0, y) \geq 1 - \varepsilon\}) =$$
$$= \bigcap_{\varepsilon>0} \{x: \ \exists y \in F(x), \ f(x_0, y) \geq 1 - \varepsilon\}.$$

With respect to our definition of the function f we obtain

$$x_0 \in \{x: \ \exists y \in F(x) \cap K\} = F^-(K)$$

and therefore F is $\mathcal{S}$-u.s.c. at x_0.

Let now $F\in\mathcal{S}.\mathcal{U}.(x_0)$, $f\in\mathcal{O}.\mathcal{U}.(x_0, y_0)$ for any $y_0\in F(x_0)$, $r\in R$ and $x_0\in\bigcup_{\varepsilon>0} F^+(\{y\colon\ f(x_0, y)<r-\varepsilon\})$.

Then there exists $\varepsilon_0>0$ such that the set $\{x_0\}\times F(x_0)$, which is compact, is a subset of the set $W=\{(x, y)\colon\ f(x, y)<r-\varepsilon_0\}$, which is open. Thus we can use the Wallace lemma and find two open sets U, V such that $\{x_0\}\subset U$, $F(x_0)\subset V$ and $U\times V\subset W$.

From the $\mathcal{S}$-u.s.c. of F it follows that a set $A\in\mathcal{S}_{x_0}$ exists such that $F(A)\subset V$. Since $\mathcal{S}_{x_0}$ is a strongly local sieve the set $A_0=A\cap U$ belongs to $\mathcal{S}_{x_0}$ and for any $x\in A_0$ we have $F(x)\subset V$. Therefore $f(x, y)<r-\varepsilon_0<r$ for any $y\in F(x)$. Thus $x\in F^+(\{y\colon\ f(x, y)<r\})$ and F is $\mathcal{S}.\mathcal{U}.$-stable at x_0 because of Proposition 1.

If we denote $\mathcal{Q}_1(x, y)=\{f\in\mathcal{O}.\mathcal{U}.(x, y)\colon\ f$ is quasiconcave on $X\times Y\}$, then the following characterization of $\mathcal{Q}_1.\mathcal{S}.\mathcal{U}.$-stable multifunctions is possible.

Theorem 3. *Let X, Y be locally convex topological vector spaces and let $F(x_0)$ be compact and convex. Then a multifunction F is $\mathcal{Q}_1.\mathcal{S}.\mathcal{U}.$-stable at x_0 iff $x_0\in\mathcal{S}$-cl $F^-(K)$ implies $x_0\in F^-(K)$ for any closed, convex set K.*

Proof. For necessity take a closed convex set K such that $x_0\in\mathcal{S}$-cl $F^-(K)$, consider the function $f(x, y)=\chi_K(y)$ and follow the proof of the previous theorem. Note that f is quasiconcave if the set $\{z\colon\ f(z)\geq r\}$ is convex for any $r\in R$.

To prove suffiency suppose that $f\in\mathcal{Q}_1(x_0, y)$ for any $y\in F(x_0)$, $r\in R$ and

$$x_0\in\bigcup_{\varepsilon>0} F^+(\{y\colon\ f(x_0, y)<r-\varepsilon\}).$$

It means there exists $\varepsilon_0>0$ such that $F(x_0)\subset\{y\colon\ f(x_0, y)<r-\varepsilon_0\}$ or it is the same as $\{x_0\}\times F(x_0)\cap B=\emptyset$, where $B=\{(x, y)\colon\ f(x, y)\geq r-\varepsilon_0\}$.

With respect to the assumptions given on the multifunction F and the function f the set $\{x_0\}\times F(x_0)$ is convex and compact and the set B is convex and closed. Thus we can separate these two sets by a closed hyperplane $\varrho=\{(x, y)\colon\ h(x, y)=c\}$ in this way

$$\{x_0\}\times F(x_0)\subset H_\varrho^+=\{(x, y)\colon\ h(x, y)>c\},$$
$$B\subset H_\varrho^-=\{(x, y)\colon\ h(x, y)<c\}.$$

Since the function h is continuous it attains its minimum in the set $\{x_0\}\times F(x_0)$, e. g. at the point (x_0, y_0). Denote $h(x_0, y_0)=c_0>c$.

Consider now the hyperplane $\varrho_0=\left\{(x, y)\colon\ h(x, y)=\frac{c_0+c}{2}\right\}$. Denote

$$H_{\varrho_0}^+=\left\{(x, y)\colon\ h(x, y)>\frac{c_0+c}{2}\right\},$$
$$V_0=\{y\in Y\colon\ (x_0, y)\in H_{\varrho_0}^+\}.$$

It is obvious that $h(x_0, y) > \frac{c_0+c}{2}$ for any $y \in V_0$ and since h is continuous in linear space there exists a neighbourhood U_0 of x_0 such that

$$h(x, y) > \frac{c_0+c}{2} - \frac{c_0-c}{4} = \frac{c_0+3c}{4} > c$$

for any $(x, y) \in U_0 \times V_0$. Thus $U_0 \times V_0 \cap B = \emptyset$.

On the other hand $\{x_0\} \times F(x_0) \subset H^+_{\varphi 0}$ and so $F(x_0) \subset V_0$. The set V_0 is open and its complement $Y \setminus V_0$ is convex. Thus the assumption laid upon F provides the existence of a set $A \in \mathscr{S}_{x_0}$ such that $F(x) \subset V_0$ for any $x \in A$.

Now if we take $A_0 = A \cap U_0 \in \mathscr{S}_{x_0}$, then $\{x\} \times F(x) \subset U_0 \times V_0$ and therefore $f(x, y) < r - \varepsilon_0 < r$ for any $x \in A_0$ and $y \in F(x)$. Thus $x \in F^+(\{y: f(x, y) < r\})$ and from Proposition 1 we have F is $\mathscr{Q}_1.\mathscr{S}.\mathscr{U}.$-stable at x_0.

The following simple examples show that the compactness of $F(x_0)$ is not a necessary condition for v to be o.u.s.c., but it cannot be omitted. In these examples X, Y are equal to the set of reals with the usual topology.

Example 1. Let
$$F(0) = R,$$
$$F(x) = \{0\} \quad \text{if} \quad x \neq 0$$

and f: $R \times R \to R$ be an arbitrary function. Then $v(0) \geq f(0, 0)$ and $v(x) = f(x, 0)$ if $x \neq 0$. Now if f is o.u.s.c., then there exists a neighbourhood U of the point 0 such that $f(0, 0) + \varepsilon > f(x, 0)$ for any $x \in U$ and therefore $v(0) + \varepsilon > v(x)$.

Example 2. Let $F(x) = R$ for any $x \in R$ and let $f(x, y) = xy$. Then $v(0) = 0$ and $v(x) = +\infty$ for any $x \neq 0$. We see v is not o.u.s.c. at 0.

In the case when an objective function of only one variable $y \in Y$ is considered the compactness of $F(x_0)$ in the two previous theorems can be omitted as it was done for the order upper semicontinuity in [4].

Now we shall study the $\mathscr{S}$-order lower semicontinuity of a function v. Again some notations and new notions are needed. Denote

$$\mathscr{S}.\mathscr{O}.\mathscr{L}.(x) = \{v\colon\ X \to R\colon\ v \text{ is } \mathscr{S}\text{-o.l.s.c. at } y\},$$

$$\mathscr{S}.\mathscr{L}.(x) = \{F\colon\ X \to Y\colon\ F \text{ is } \mathscr{S}\text{-l.s.c. at } x\}.$$

For any $(x, y) \in X \times Y$ denote by $\mathscr{G}(x, y)$ an arbitrary subset of the set of all functions f: $X \times Y \to R$ which are order lower semicontinuous at a point (x, y) and further denote

$$\mathscr{G}.\mathscr{S}.\mathscr{O}.\mathscr{L}.(x) = \{F\colon\ X \to Y\colon\ v \in \mathscr{S}.\mathscr{O}.\mathscr{L}.(x) \text{ for any } f \text{ belonging to } \mathscr{G}(x, y) \text{ for any } y \in F(x)\}.$$

As we did in the first part of this section we shall characterize the set $\mathscr{G}.\mathscr{S}.\mathscr{O}.\mathscr{L}.(x)$. The first characterization uses the following concept of a stable multifunction.

Definition 7. *A multifunction F: $X \to Y$ is said to be $\mathcal{G}.\mathcal{S}.\mathcal{L}.$-stable at a point $x_0 \in X$ if for any $\varepsilon > 0$ and for any f belonging to $\mathcal{G}(x_0, y_0)$ for any $y_0 \in F(x_0)$, there exists a set $A \in \mathcal{S}_{x_0}$ such that*

$$F(x) \cap \{y\colon\ f(x, y) > v(x_0) - \varepsilon\} \neq \emptyset$$

for any $x \in A$.

Evidently it means that for any $x \in A$ there holds

$$x \in F^{-}(\{y\colon\ f(x, y) > v(x_0) - \varepsilon\}).$$

Theorem 4. *The $\mathcal{G}.\mathcal{S}.\mathcal{L}.$-stable multifunctions are precisely those that preserve the $\mathcal{S}$-o.l.s.c. of the family $\mathcal{G}$. Thus*

$$F \text{ is } \mathcal{G}.\mathcal{S}.\mathcal{L}.\text{-stable at } x_0 \text{ iff } F \in \mathcal{G}.\mathcal{S}.\mathcal{O}.\mathcal{L}.(x_0).$$

Proof. Suppose F to be $\mathcal{G}.\mathcal{S}.\mathcal{L}.$-stable at x_0. Let f be from the set $\mathcal{G}(x_0, y_0)$ for any $y_0 \in F(x_0)$ and let $\varepsilon > 0$. Then there is a set $A \in \mathcal{S}_{x_0}$ such that for any $x \in A$ there exists $y_x \in F(x)$ satisfying

$$f(x, y_x) > v(x_0) - \varepsilon.$$

From the definition of v we have $v(x) \geq f(x, y_x) > v(x_0) - \varepsilon$.

Now if $F \in \mathcal{G}.\mathcal{S}.\mathcal{O}.\mathcal{L}.(x_0)$ and $f \in \mathcal{G}(x_0, y_0)$ for any $y_0 \in F(x_0)$, then $v \in \mathcal{S}.\mathcal{O}.\mathcal{L}.(x_0)$ and so for any $\varepsilon > 0$ we have a set $A \in \mathcal{S}_{x_0}$ such that

$$v(x) > v(x_0) - \varepsilon$$

for any $x \in A$.

From the property of the supremum there exists $y_x \in F(x)$ such that

$$f(x, y_x) > v(x_0) - \varepsilon.$$

Therefore for any $x \in A$ we obtain $x \in F^{-}(\{y\colon\ f(x, y) > v(x_0) - \varepsilon\})$.

Proposition 4. *A multifunction F is $\mathcal{G}.\mathcal{S}.\mathcal{L}.$-stable at x_0 iff for any $r \in R$* and *for any f belonging to $\mathcal{G}(x_0, y_0)$ for any $y_0 \in F(x_0)$*

$$x_0 \in \bigcup_{\varepsilon > 0} F^{-}(\{y\colon\ f(x_0, y) > r + \varepsilon\})$$

implies the existence of a set $A \in \mathcal{S}_{x_0}$ such that

$$x \in F^{-}(\{y\colon\ f(x, y) > r\})$$

for any $x \in A$.

Proof. Let F be $\mathcal{G}.\mathcal{S}.\mathcal{L}.$-stable at x_0 and the first part of the implication holds. Then $\varepsilon_0 > 0$ exists such that $x_0 \in F^{-}(\{y\colon\ f(x_0, y) > r + \varepsilon_0\})$. Thus we have

$$v(x_0) > r + \varepsilon_0.$$

From the $\mathscr{G}.\mathscr{S}.\mathscr{L}.$-stability of F there is a set $A \in \mathscr{S}_{x_0}$ such that

$$x \in F^-(\{y: f(x, y) > v(x_0) - \varepsilon_0\})$$

for any $x \in A$. Since $v(x_0) - \varepsilon_0 > r$ we obtain

$$x \in F^-(\{y: f(x, y) > r\}).$$

Now let $f \in \mathscr{G}(x_0, y_0)$ for any $y_0 \in F(x_0)$, $\varepsilon_0 > 0$ and let the implication be valid. Put $r_0 = v(x_0) - \varepsilon_0$. Evidently

$$x_0 \in \bigcup_{\varepsilon > 0} F^-(\{y: f(x_0, y) > r_0 + \varepsilon\})$$

and therefore a set $A \in \mathscr{S}_{x_0}$ must exist such that

$$x \in F^-(\{y: f(x, y) > r_0 = v(x_0) - \varepsilon_0\})$$

for any $x \in A$.

In the following two propositions we use the notation $D(f, x, r) = \{y: f(x, y) \leqslant r\}$.

Proposition 5. *A multifunction F is $\mathscr{G}.\mathscr{S}.\mathscr{L}.$-stable at a point x_0 iff for any $r \in R$ and for any f belonging to $\mathscr{G}(x_0, y_0)$ for any $y_0 \in F(x_0)$*

$$x_0 \in \mathscr{S}\text{-cl}\{x: x \in F^+(D(f, x, r))\}$$

implies

$$x_0 \in \bigcap_{\varepsilon > 0} F^+(D(f, x_0, r + \varepsilon)).$$

Proof. However, we must only notice that if we denote the implication in Proposition 4 as

$$\mathbf{P} \Rightarrow \mathbf{Q},$$

then in this proposition we have an implication

$$\text{non } \mathbf{Q} \Rightarrow \text{non } \mathbf{P}.$$

Proposition 6. *A multifunction F is $\mathscr{G}.\mathscr{S}.\mathscr{L}.$-stable at any $x \in X$ iff for any $r \in R$ and for any f belonging to $\mathscr{G}(x, y)$ for any $x \in X$ and any $y \in F(x)$*

$$\bigcap_{\varepsilon > 0} \mathscr{S}\text{-cl}\{x: x \in F^+(D(f, x, r + \varepsilon))\} = \bigcap_{\delta > 0} \{x: x \in F^+(D(f, x, r + \delta))\}.$$

Proof. Suppose the $\mathscr{G}.\mathscr{S}.\mathscr{L}.$-stability of F and let us prove the equality. Since one inclusion is evident we need only to prove that

$$\bigcap_{\varepsilon > 0} \mathscr{S}\text{-cl}\{x: x \in F^+(D(f, x, r + \varepsilon))\} \subset \bigcap_{\delta > 0} \{x: x \in F^+(D(f, x, r + \delta))\}.$$

Let x_0 belong to the left set and suppose there is $\delta_0 > 0$ such that

$x_0 \notin \{x\colon\ x \in F^+(D(f, x, r+\delta_0))\}$. Thus $v(x_0) > r+\delta_0$ and the $\mathcal{G}.\mathcal{S}.\mathcal{L}.$-stability of F provides the existence of a set $A \in \mathcal{S}_{x_0}$ such that

$$x \in F^-\left(\left\{y\colon\ f(x, y) > v(x_0) - \frac{\delta_0}{2} > r + \frac{\delta_0}{2}\right\}\right)$$

for any $x \in A$. Therefore $x_0 \notin \mathcal{S}\text{-cl}\left\{x\colon\ x \in F^+\left(D\left(f, x, r+\frac{\delta_0}{2}\right)\right)\right\}$, which yields a contradiction.

Now suppose the equality holds. Let $f \in \mathcal{G}(x_0, y_0)$ for any $y_0 \in F(x_0)$, $r \in R$ and let $x_0 \in \mathcal{S}\text{-cl}\{x\colon\ x \in F^+(D(f, x, r))\}$. Then

$$x_0 \in \bigcap_{\varepsilon>0} \mathcal{S}\text{-cl}\{x\colon\ x \in F^+(D(f, x, r+\varepsilon))\}.$$

According to our assumption $x_0 \in \bigcap_{\delta>0} \{x\colon\ x \in F^+(D(f, x, r+\delta))\}$ holds and therefore $x_0 \in F^+(D(f, x_0, r+\delta))$ for any $\delta > 0$. The $\mathcal{G}.\mathcal{S}.\mathcal{L}.$-stability of F follows from Proposition 5.

In the last two theorems we give only an outline of the proofs.

For any $(x, y) \in X \times Y$ denote by $\mathcal{O}.\mathcal{L}.(x, y)$ the set of all functions $f\colon\ X \times Y \to R$ which are o.l.s.c. at (x, y). If $\mathcal{G}(x, y) = \mathcal{O}.\mathcal{L}.(x, y)$, we speak about $\mathcal{S}.\mathcal{L}.$-stable multifunctions.

Theorem 5. *The $\mathcal{S}.\mathcal{L}.$-stable multifunctions are precisely the $\mathcal{S}$-l.s.c. ones. Thus F is $\mathcal{S}.\mathcal{L}.$-stable at x_0 iff $F \in \mathcal{S}.\mathcal{L}.(x_0)$.*

Proof. The necessity can be proved by using a characteristic function of a certain closed set and Proposition 5. However, it will not be done because a similar procedure was used in the proof of Theorem 2.

Sufficiency. Proposition 4 as a characterization of $\mathcal{S}.\mathcal{L}.$-stability is used.

If we denote $\mathcal{Q}_2(x, y) = \{f \in \mathcal{O}.\mathcal{L}.(x, y)\colon\ f$ is quasiconvex in $X \times Y\}$, then the following characterization of $\mathcal{Q}_2.\mathcal{S}.\mathcal{L}.$-stable multifunctions is possible. Note that f is quasiconvex if the set $\{z\colon\ f(z) \leq r\}$ is convex for any $r \in R$.

Theorem 6. *Let X, Y be locally convex topological vector spaces. Then a multifunction F is $\mathcal{Q}_2.\mathcal{S}.\mathcal{L}.$-stable at a point x_0 iff $x_0 \in \mathcal{S}\text{-cl}\, F^+(K)$ implies $x_0 \in F^+(K)$ for any closed convex set K.*

Proof. The proof of the necessity will be again omitted since it is analogous to the proof of the necessity in Theorem 3.

Sufficiency. We use Proposition 4 as a characterization of the $\mathcal{S}.\mathcal{L}.$-stability. Thus we shall have $\varepsilon_0 > 0$ and $y_0 \in F(x_0)$ such that

$$f(x_0, y_0) > r + \varepsilon_0$$

and we can separate a point (x_0, y_0) and the set $D = \{(x, y)\colon\ f(x, y) \leq r + \varepsilon_0\}$ by a closed hyperplane.

Using a translation of this hyperplane we obtain a neighbourhood $U \times V$ of (x_0, y_0) such that $(U \times V) \cap D = \emptyset$ and the complement of V is convex.

Provided that F is $\mathcal{Q}_2.\mathcal{S}.\mathcal{L}$.-stable at x_0 we can find a set $A \in \mathcal{S}_{x_0}$ such that

$$x \in F^-(\{y: \ f(x, y) > r + \varepsilon_0 > r\})$$

for any $x \in A$.

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О МНОГОЗНАЧНЫХ ОТОБРАЖЕНИЯХ, СОХРАНЯЮЩИХ ОБОБЩЕННУЮ НЕПРЕРЫВНОСТЬ

Ondrej Náther

Резюме

В статье вводится по образцу [2] понятие обобщенной полунепрерывности для многозначных отображений и для действительных функций. Изучается класс многозначных отображений, сохраняющих обобщенную полунепрерывность данного класса функций при операции

$$v(x) = \sup \{f(x, y): \ y \in F(x)\}.$$

Здесь полученные результаты являются обобщением результатов из [4].