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ON A CERTAIN LATTICE OF TOPOLOGIES ON A PRODUCT OF METRIC SPACES

JOZEF DOBOS

Introduction

Let $T$ be a nonempty set. Denote by $R$ the real line and by $T^*$ the set of all non-negative functions $a: T \rightarrow R$. Denote by $\mathcal{M}(T)$ the set of all functions $f: T^* \rightarrow R$ such that

$$d(x, y) = f(\{d_i(x(t), y(t))\},_{i \in T})$$

is a metric on the set $\prod_{i \in T} M_i$ for every collection of metric spaces $\{(M_i, d_i)\}_{i \in T}$.

In [1] we investigate the metrizability by the metric $d$ of the product topology on $\prod_{i \in T} M_i$.

In the present paper we extend some results of [1]. In the special case of the set $T$ being finite, the paper presents a complete characterization of the lattice of topologies on the set $\prod_{i \in T} M_i$ generated by the set $\mathcal{M}(T)$.

1. Preliminaries

1.1. Notation. If $\delta$ is a binary relation on $R$, define the binary relation $\delta_T$ on $R^T$ as follows: $x \delta_T y$ if and only if $x(t) \delta y(t)$ for each $t \in T$. Define the function $\theta_T$: $T \rightarrow R$ by $\theta_T(t) = 0$ for each $t \in T$. If $T = \{t\}$, we write $\delta_t$ and $\theta_t$.

In paper [1] the following results (1.2—1.6) are proved.

1.2. Lemma. Let $f \in \mathcal{M}(T)$. Then

$$\forall a, b \in T^*: f(a + b) \leq f(a) + f(b),$$

$$\forall a, b \in T^*: a \leq \tau 2b \Rightarrow f(a) \leq 2f(b).$$

1.3. Theorem. Let $f: T^* \rightarrow R$. Then $f \in \mathcal{M}(T)$ if and only if
\[ \forall x \in T^*: f(x) = 0 \Leftrightarrow x = \theta_T, \quad (4) \]
\[ \forall x, y, z \in T^*: (x \leq_T y + z & y \leq_T z + x & z \leq_T x + y) \Rightarrow \] \[ f(x) \leq f(y) + f(z). \quad (5) \]

1.4. **Proposition.** Let the set \( T \) be finite. Let \( f \in \mathcal{M}(T) \). Then \( f \) is continuous (we consider \( T^* \) by subspace of the topological product \( R^T \)) if and only if
\[ \forall \varepsilon > 0 \exists x \in T^*, x > r\theta_T: f(x) < \varepsilon. \]

1.5. **Notation.** Let \( S \subset T \) be a nonempty set. Define a mapping \( i_s \tau: S^* \to T^* \) as follows
\[ (i_s \tau(a))(t) = \begin{cases} a(t) & \text{for } t \in S, \\ 0 & \text{for } t \in T - S, \end{cases} \]
for each \( a \in S^* \). If \( S = \{s\} \) we write \( i_s \tau \).

1.6. **Lemma.** Let \( f \in \mathcal{M}(T) \). Then \( (f \circ i_s \tau) \in \mathcal{M}(S) \).

1.7. **Remark.** The mapping \( i_s \tau \) is continuous (see [2], p. 59, Theorem 1).

1.8. **Proposition.** Let \( Q \) be a nonempty finite set. Let \( f \in \mathcal{M}(Q) \). Then \( f \) is continuous if and only if \( f \circ i_q \circ \) is continuous for each \( q \in Q \).

**Proof.** \( \Rightarrow \): By 1.7.
\[ \Leftarrow: \] Let \( \varepsilon > 0, q \in Q \). Since by 1.6 we have
\[ (f \circ i_q \circ) \in \mathcal{M}(\{q\}), \]
by 1.4 we obtain
\[ \exists x_q \in \{q\}^*, x_q > q\theta_q: (f \circ i_q \circ)(x_q) < \varepsilon/(\text{card } Q). \]
We put
\[ a = \sum_{q \in Q} i_q \circ(x_q). \]
Thus \( a \in Q^*, a > q\theta_q \) and by 1.2 we have
\[ f(a) \leq \sum_{q \in Q} (f \circ i_q \circ)(x_q) < \varepsilon. \]
Then by 1.4 the function \( f \) is continuous.

1.9. **Notation.** For each \( f \in \mathcal{M}(T) \) we put
\[ F(f) = \{t \in T; f \circ i \tau \text{ is continuous}\}. \]
Define a function \( j_\tau: T^* \to R \) as follows
\[ j_\tau(x) = \begin{cases} 0 & \text{for } x = \theta_T, \\ 1 & \text{for } x \neq \theta_T. \end{cases} \]
The following Example shows that the condition “finite” in Proposition 1.8 cannot be omitted.

1.10. Example. Let $P$ be a nonempty set. Define a mapping $f: P^* \to R$ as follows

$$f(x) = \sup \{ \min (1, x_t); t \in P \}.$$ 

Then $f \in \mathcal{M}(P)$, $F(f) = P$ and $f$ is continuous if and only if $P$ is finite.

**1.11. Corollary.** Let the set $T$ be finite. Let $f \in \mathcal{M}(T)$. Then $f \circ i_{s,T}$ is continuous if and only if $S \subseteq F(f)$.

**Proof.** By 1.6 we have $(f \circ i_{s,T}) \in \mathcal{M}(S)$. Then by 1.8 we obtain that $f \circ i_{s,T}$ is continuous if and only if $f \circ i_{s,T} = (f \circ i_{s,T}) \circ i_{s}$ is continuous for each $s \in S$.

**1.12. Proposition.** Let the set $T$ be finite, $\emptyset \neq S \subseteq T$. Let $h \in \mathcal{M}(T)$ be continuous. Define a mapping $h_s: T^* \to R$ as follows

$$h_s(x) = \begin{cases} 
\frac{h(x)}{(1 + h(x))} & \text{for } x \in \text{Im } (i_{s,T}), \\
1 & \text{otherwise.}
\end{cases}$$

Then $h_s \in \mathcal{M}(T)$ and $F(h_s) = S$.

**Proof.** Let $x \in T^*$. Then $h_s(x) = 0 \iff h(x) = 0 \iff x = \emptyset_T$.

Let $x, y, z \in T^*$, $x \leq_T y + z$, $y \leq_T x + z$, $z \leq_T x + y$. Since $h \in \mathcal{M}(T)$, by 1.3 we have

$$h(x) \leq h(y) + h(z).$$

If $h_s(y) + h_s(z) < 1$, then $x, y, z \in \text{Im } (i_{s,T})$, thus $h_s(x) = h(x)/(1 + h(x)) \leq h(y)/(1 + h(y)) + h(z)/(1 + h(z)) = h_s(y) + h_s(z)$. If $h_s(y) + h_s(z) \geq 1$, then $h_s(x) \leq 1 \leq h_s(y) + h_s(z)$. Then by 1.3 we have $h_s \in \mathcal{M}(T)$.

Since $h$ is continuous, by 1.7 we obtain $h_s \circ i_{s,T} = (h/(1 + h)) \circ i_{s,T}$ is continuous. Thus by 1.11 we get $S \subseteq F(h_s)$.

Let $t \in T - S$. Since $h_s \circ i_{s,T} = j_T \circ i_{s,T}$ is not continuous, by 1.11 we have $t \in T - F(h_s)$. Thus $T - S \subseteq T - F(h_s)$.

### 2. Lattice of topologies generated by the set $\mathcal{M}(T)$

**2.1. Notation.** Let $T$ be a nonempty set. Let $\{(M_i, d_i)_{i \in T}\}$ be a collection of metric spaces. We put $M = \prod_{i \in T} M_i$. For each $f \in \mathcal{M}(T)$ denote by $\mathcal{T}_f$ the topology on the set $M$ derived from the metric (1). We put

$$\mathcal{L} = \{ \mathcal{T}_f: f \in \mathcal{M}(T) \},$$

$$H = \{ t \in T: M_t \neq \emptyset \}$$

(where $A'$ is the derived set of $A$).
2.2. Proposition. Let $f, g \in \mathcal{M}(T)$. Let $\mathcal{F}_f \subseteq \mathcal{F}_g$. Then

$$F(f) \supseteq F(g) \cap H.$$ 

Proof. Let $t \in F(g) \cap H$. Let $\varepsilon > 0$. Select $a \in M$ such that $a(t) \in M'$. Since $\mathcal{F}_f \subseteq \mathcal{F}_a$, there exists $\delta > 0$ such that

$$S_{\delta}(a, 2\delta) \subseteq S(a, \varepsilon). \quad (7)$$

Since $g \circ i_T$ is continuous, by 1.4 we have

$$\exists y \in \{t\}^*, y > \varepsilon : (g \circ i_T)(y) < \delta.$$ 

Let $q \in M$, such that $0 < d(a(t), q) < y(t)$. Define a mapping $b : T \to \bigcup_{t \in T} M_t$ as follows

$$b(s) = \begin{cases} 
q & \text{for } s = t, \\
 a(s) & \text{otherwise.}
\end{cases}$$

Define a mapping $x : \{t\} \to R$ as follows

$$x(t) = d(a(t), b(t)).$$

Then obviously $x \in \{t\}^*$, $x > \varepsilon$. Since $(g \circ i_T) \in \mathcal{M}(\{t\})$ and $x \leq y$, by 1.2 (3) we obtain

$$g((d, a(t), b(t)))_{t \in T} = (g \circ i_T)(x) \leq 2 \cdot (g \circ i_T)(y) < 2\delta.$$ 

Thus $b \in S_{\delta}(a, 2\delta)$. Then by (7) we have

$$(f \circ i_T)(x) = f((d, a(t), b(t)))_{t \in T} < \varepsilon.$$ 

Hence by 1.4 the function $f \circ i_T$ is continuous.

In the following it will be proved that if $T$ is finite, then the topologies of metrics which are generated by functions from $\mathcal{M}(T)$ are determined by subsets of the set of all indices $t$, so that $d_t$ is not discrete.

2.3. Theorem. Let the set $T$ be finite. Let $f, g \in \mathcal{M}(T)$. Then $\mathcal{F}_f \subseteq \mathcal{F}_g$ if and only if

$$F(f) \supseteq F(g) \cap H.$$ 

Proof. $\Rightarrow$: By 2.2.

$\Leftarrow$: Let $a \in M$, $\varepsilon > 0$. We show that

$$\exists \delta > 0: S_{\delta}(a, \delta) \subseteq S(a, \varepsilon).$$

Let $\gamma > 0$ such that

$$\forall t \in T - H \forall b \in T^*: (d(a(t), b(t)) < \gamma) \Rightarrow a(t) = b(t). \quad (8)$$

Let $\eta > 0$ such that

$$\forall t \in T - F(g) \forall x \in \{t\}^*, x > \varepsilon : (g \circ i_T)(x) < \eta. \quad (9)$$

Let $t \in F(f)$. Since $f \circ i_T$ is continuous, there exists $x, \in \{t\}^*$, $x > \varepsilon$, such that

$$(f \circ i_T)(x) < \varepsilon / (2 \text{ card } T). \quad (10)$$
We put 
\[ \delta_i = g(i, \tau(x_i))/2. \]
For each \( t \in T - F(f) \) we put \( x_i = \theta_i \). For each \( t \in T \) define a function \( y_i : \{t\} \to \mathbb{R} \) by 
\[ y_i(t) = \gamma \] and put 
\[ \gamma_i = g(i, \tau(y_i))/2. \]
We put \( \delta = \min \{ \{ \delta_i : t \in F(f) \} \cup \{ \gamma_i : t \in T \} \cup \{ \eta/2 \} \} \). Let \( b \in S_\delta(a, \delta), t \in F(f) \). Since \( 2g(\{ d_n(a(u), b(u)) \}_{u \in T}) < 2\delta \leq g(i, \tau(x_i)) \), by 1.2 (3) we obtain 
\[ 2d(a(t), b(t)) < x_i(t). \]
Let \( t \in T - H \). Since \( 2g(\{ d_n(a(u), b(u)) \}_{u \in T}) \leq 2\delta \leq g(i, \tau(y_i)) \), by 1.2 (3) we have \( 2d(a(t), b(t)) < \gamma \). Then by (8) we get \( a(t) = b(t) \).
Let \( t \in T - F(g) \). Define a function \( u : \{t\} \to \mathbb{R} \) by 
\[ u(t) = d(a(t), b(t)). \]
Since \( i, \tau(u) \leq \tau \cdot \{ d_n(a(v), b(v)) \}_{v \in T} \), by 1.2 (3) we have 
\[ (g \circ i, \tau)(u) \leq 2 \cdot g(\{ d_n(a(v), b(v)) \}_{v \in T}) < 2\delta \leq \eta. \]
Thus by (9) we set \( a(t) = b(t) \).
Then by 1.2 and (10) we obtain 
\[ f(\{ d_n(a(t), b(t)) \}_{t \in T}) \leq 2 \cdot f \left( \sum_{t \in T} i_t \tau(x_t) \right) \leq 2 \cdot \sum_{t \in T} (f \circ i_t \tau)(x_t) < \varepsilon. \]
Hence \( b \in S_f(a, \varepsilon) \).

2.4. Corollary. Let the set \( T \) be finite. Let \( f, g \in M(T) \). Then \( T_f = T_g \) if and only if \( H_f \cap F(f) = H_g \cap F(g) \).

The following Example shows that the condition "finite" in Theorem 2.3 cannot be omitted.

2.5. Example. Let \( W \) be a infinite set. Let \( a : W \to \mathbb{N} \) be a surjection (when \( \mathbb{N} \) denotes the set of all natural numbers). Define a mapping \( g : W^* \to \mathbb{R} \) as follows 
\[ g(x) = \sup \{ \min (1, a \cdot x_t) ; t \in W \}. \]
Then \( g \in M(W) \). Let \( f \in M(W) \) be the function from Example 1.10. Consider the collection of metric spaces \( \{ (M_t, d_t) \}_{t \in W} \) given by \( M_t = \mathbb{R}, d_t(x, y) = |x - y| \) for each \( t \in W \). Evidently \( S_\delta(\theta_w, 1) \in T_f \). We prove that \( S_\delta(\theta_w, 1) \notin T_f \). Since for every constant function \( u \in W^* \), \( u \neq \theta_w \) we have \( g(u) = 1 \), for every \( \varepsilon > 0 \) we obtain 
\[ S_f(\theta_w, \varepsilon) \notin S_\delta(\theta_w, 1). \]
Thus $S_u(\theta_w, 1)$ is not the neighbourhood of $\theta_w$ in $J$. Hence $S_u(\theta_w, 1) \notin \mathcal{T}_i$. Then $\mathcal{T}_u \in \mathcal{T}_i$, but $F(f) = F(g) = W$.

2.6. Proposition. Let the set $T$ be finite. Let $h \in \mathcal{M}(T)$ be continuous. Put $h_n = j_T$. Then

$$\mathcal{L} = \{ \mathcal{T}_h : S \subset H \}.$$ 

Proof. Let $f \in \mathcal{M}(T)$ We put $S = H \cap F(f)$. Then by 1.12 we have $H \cap F(h_s) = H \cap S = H \cap F(f)$. Hence by 2.3 we obtain

$$\mathcal{T}_i = \mathcal{T}_{h_s}.$$ 

2.7. Remark. It is not difficult to prove that the partially ordered set $(\mathcal{L}, \subset)$ is a lattice.

2.8. Theorem. The lattice $(\mathcal{L}, \subset)$ is dually isomorphic to the lattice $(\exp H, \subset)$. 

Proof. Define a mapping $\Omega: \mathcal{L} \rightarrow \exp H$ by

$$\Omega(\mathcal{T}_i) = H \cap F(f)$$

for each $f \in \mathcal{M}(T)$. By 1.12, 2.3, 2.4 and 2.6 the mapping $\Omega$ is a dual isomorphism.

REFERENCES


ОБ ОДНОЙ СТРУКТУРЕ ТОПОЛОГИИ НА ПРОИЗВЕДЕНИИ МЕТРИЧЕСКИХ ПРОСТРАНСТВ

Иозеф Добош

Резюме

Пусть $T$ является непустым конечным множеством. Обозначим $T^*$ множество всех неотрицательных вещественных функции, определенных на множестве $T$. Обозначим $\mathcal{M}(T)$ множество всех функций $f: T^* \rightarrow R$, для которых

$$d(x, y) = f\{d(x(t), y(t)) \}$$

является метрикой на множестве

$$\prod_{x, y \in M}$$

для каждого семейства метрических пространств $((M, d)) \in T$. В настоящей работе мы предлагаем характеристику структуры топологий, порожденной множеством $\mathcal{M}(T)$.