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TRIPLE POSITIVE SOLUTIONS FOR \((k, n - k)\) CONJUGATE BOUNDARY VALUE PROBLEMS

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ABSTRACT. For the \(n\)th order differential equation,

\[ (-1)^{n-k} y^{(n)} - f(y) = 0, \quad t \in [0,1], \]

satisfying the boundary conditions, \(y^{(i)}(0) = 0, \ 0 \leq i \leq k - 1,\) and \(y^{(j)}(1) = 0, \ 0 \leq j \leq n - k - 1,\) where \(f: \mathbb{R} \to [0,\infty),\) growth conditions are imposed on \(f\) which yield the existence of at least three positive solutions.

1. Introduction

Let \(n \geq 2\) and \(1 \leq k \leq n - 1\) be given. We are concerned with the existence of multiple solutions for the \(n\)th order boundary value problem

\[ (-1)^{n-k} y^{(n)} - f(y) = 0, \quad t \in [0,1], \quad (1.1) \]
\[ y^{(i)}(0) = 0, \quad 0 \leq i \leq k - 1, \quad (1.2) \]
\[ y^{(j)}(1) = 0, \quad 0 \leq j \leq n - k - 1, \]

where \(f: \mathbb{R} \to [0,\infty)\) is continuous. It is fairly standard to refer to the boundary value problem (1.1), (1.2) as a \((k, n - k)\) conjugate boundary value problem. We will impose growth conditions on \(f\) which insure the existence of at least three positive solutions of (1.1), (1.2).

A good deal of recent attention has been directed toward obtaining triple solutions for boundary value problems for ordinary differential equations. This paper can be considered as a generalization of previous work on triple solutions for special cases of two-point boundary value problems by Avery [2],

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Chyan, Davis and Yin [5], Henderson and Thompson [9], and Wong and Agarwal [11]. Other papers on triple solutions for boundary value problems for ordinary differential equations have been written by Anderson [1], Chyan and Davis [4], and Guo and Lakshmikantham [7], and the recent papers [3], [8] and [12] were devoted to triple solutions for boundary value problems for finite difference equations.

For the most part, each of the above cited papers makes an application of a fixed point theorem by Leggett and Williams [10], which they developed using the fixed point index in ordered Banach spaces. Leggett and Williams [10] applied their fixed point theorem to prove the existence of three positive solutions for Hammerstein integral equations of the form

$$y(x) = \int_{\Omega} G(x, s)g(s, y(s)) \, ds, \quad \Omega \subset \mathbb{R}^N,$$

by making use of suitable inequalities they imposed on the kernel $G$ and $g$.

In Section 2, we provide some definitions and background results, and we state the Leggett-Williams fixed point theorem. Then in Section 3, we impose growth conditions on $f$ which allow us to apply the Leggett-Williams fixed point theorem in obtaining three positive solutions of (1.1), (1.2).

### 2. Background definitions and results

In this section, we provide some background material from the theory of cones in Banach spaces. We also state a fixed point theorem due to Leggett and Williams [10] for multiple fixed points of a cone preserving operator.

Let $B$ be a real Banach space equipped with a norm, $\| \cdot \|$. If $\mathcal{P} \subset B$ is a cone, we denote the order induced by $\mathcal{P}$ on $B$ by $\leq_P$.

**Definition 2.1.** A map $\alpha$ is said to be a nonnegative continuous concave functional on $\mathcal{P}$ if $\alpha : \mathcal{P} \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in \mathcal{P}$ and $t \in [0,1]$.

**Definition 2.2.** For numbers $0 < a < b$ and $\alpha$, a nonnegative continuous concave functional on $\mathcal{P}$, define convex sets $\mathcal{P}_r$ and $\mathcal{P}(\alpha, a, b)$ by

$$\mathcal{P}_r = \{ y \in \mathcal{P} : \| y \| < r \} \quad \text{and} \quad \mathcal{P}(\alpha, a, b) = \{ y \in \mathcal{P} : a \leq \alpha(y), \| y \| \leq b \}.$$

In obtaining multiple positive solutions of (1.1), (1.2), the following fixed point theorem due to Leggett and Williams [10] will be fundamental.
THEOREM 2.1 (LEGGETT-WILLIAMS FIXED POINT THEOREM). Let $A: \mathcal{P}_c \to \mathcal{P}_c$ be a completely continuous operator and let $\alpha$ be a nonnegative continuous concave functional on $\mathcal{P}$ such that $\alpha(y) \leq \|y\|$ for all $y \in \mathcal{P}_c$. Suppose there exist $0 < a < b < d \leq c$ such that

(C1) $\{y \in \mathcal{P}(\alpha, b, d) : \alpha(y) > b\} \neq \emptyset$ and $\alpha(Ay) > b$ for $y \in \mathcal{P}(\alpha, b, d)$,

(C2) $\|Ay\| < a$ for $y \in \mathcal{P}_a$,

(C3) $\alpha(Ay) > b$ for $y \in \mathcal{P}(\alpha, b, c)$ with $\|Ay\| > d$.

Then $A$ has at least three fixed points $y_1$, $y_2$, and $y_3$ such that $\|y_1\| < a$, $b < \alpha(y_2)$, and $\|y_3\| > a$ with $\alpha(y_3) < b$.

3. Multiple positive solutions

In this section, we will impose growth conditions on $f$ which allow us to apply Theorem 2.1 in regard to obtaining three positive solutions of (1.1), (1.2). We will apply Theorem 2.1 in conjunction with a completely continuous operator whose kernel is the Green's function $G(t, s)$ for

$$(-1)^{n-k}y^{(n)} = 0$$

satisfying the boundary conditions (1.2). It is fairly well-known that

$$G(t, s) > 0, \quad (t, s) \in (0, 1) \times (0, 1).$$

Also, for $s \in (0, 1)$, there exists $\tau(s) \in (0, 1)$ such that

$$G(t, s) \leq G(\tau(s), s), \quad t \in (0, 1),$$

and it is shown in [6] that

$$G(t, s) \geq \frac{1}{4m} G(\tau(s), s), \quad t \in [1/4, 3/4], \quad s \in [0, 1],$$

where $m = \max\{k, n - k\}$.

Next, we note

$$\int_0^1 G(t, s) \, ds = \frac{t^k (1-t)^{n-k}}{n!}, \quad t \in [0, 1],$$

and as a result, we see

$$\max_{t \in [0, 1]} \int_0^1 G(t, s) \, ds = \int_0^1 G(k/n, s) \, ds = \frac{(k)^k (1 - \frac{k}{n})^{n-k}}{n!},$$

$$\min_{t \in [0, 1]} \int_0^1 G(t, s) \, ds = \min \left\{ \frac{3^{n-k}}{4^n n!}, \frac{3^k}{4^n n!} \right\} = \frac{3^{n-m}}{4^n n!}. \quad (3.5)$$
For notational convenience we define the following constants involving the quantities above

\[ K = \left( \max_{t \in [0,1]} \int_{0}^{1} G(t, s) \, ds \right)^{-1} = \frac{n!}{(\frac{k}{n})^k (1 - \frac{k}{n})^{n-k}}, \quad (3.6) \]

\[ L = \left( \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} \int_{0}^{1} G(t, s) \, ds \right)^{-1} = \frac{4^n n!}{3n-m}. \quad (3.7) \]

By (3.3) and (3.2) we obtain

\[ \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} \int_{0}^{1} G(t, s) \, ds \geq \frac{1}{4^m} \int_{0}^{1} G(\tau(s), s) \, ds \]

\[ \quad \geq \frac{1}{4^m} \int_{0}^{1} G(t, s) \, ds \quad \text{for all} \quad t \in (0, 1). \]

Hence,

\[ \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} \int_{0}^{1} G(t, s) \, ds \geq \frac{1}{4^m} \int_{0}^{1} dt \int_{0}^{1} G(t, s) \, ds \]

\[ = \frac{1}{4^m} \int_{0}^{1} ds \int_{0}^{1} G(t, s) \, dt = \frac{1}{4^m} \int_{0}^{1} ds \int_{0}^{1} G^*(s, t) \, dt, \]

where \( G^* \) is the Green’s function for the boundary value problem

\[ (-1)^k z^{(n)}(t) = 0, \quad t \in [0, 1], \]

\[ z^{(i)}(0) = 0, \quad 0 \leq i \leq n-k, \]

\[ z^{(j)}(1) = 0, \quad 0 \leq j \leq k-1. \quad (3.8) \]

This is due to the fact that

\[ \int_{0}^{1} G^*(s, t) \, dt = \frac{s^{n-k}(1-s)^k}{n!}, \quad s \in [0, 1], \]

and

\[ \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} \int_{0}^{1} G(t, s) \, ds \geq \frac{1}{4^m} \int_{0}^{1} s^{n-k}(1-s)^k \, ds \]

\[ > \frac{1}{4^m \cdot 2L}. \quad (3.10) \]
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Let \(B\) denote the Banach space \(C[0,1]\) endowed with the norm \(\|x\| = \max_{t \in [0,1]} |x(t)|\), and let the cone \(\mathcal{P} \subset B\) be defined by

\[
\mathcal{P} = \{x \in B : x(t) \geq 0, \ t \in [0,1]\},
\]

and finally let the nonnegative continuous concave functional \(\alpha : \mathcal{P} \to [0,\infty)\) be defined by

\[
\alpha(x) = \min_{t \in \left\{\frac{1}{4}, \frac{3}{4}\right\}} x(t), \quad x \in \mathcal{P}.
\]

We note that, for each \(x \in \mathcal{P}\), \(\alpha(x) \leq \|x\|\), and also that \(x \in B\) is a solution of (1.1), (1.2) if and only if

\[
x(t) = \int_0^1 G(t,s)f(x(s)) \, ds, \quad t \in [0,1].
\]

We now present the main result of the paper.

**THEOREM 3.1.** Let \(0 < a < b < 4^m b \leq \frac{c}{4^m}\) be such that \(f\) satisfies

(i) \(f(w) < Ka\) for \(0 \leq w \leq a\),
(ii) \(f(w) \geq 4^m \cdot 2Lb\) for \(b \leq w \leq 4^m b\),
(iii) \(f(w) \leq Kc\) for \(0 \leq w \leq c\).

Then the boundary value problem (1.1), (1.2) has at least three positive solutions \(y_1, y_2,\) and \(y_3\) satisfying \(\|y_1\| < a,\ b < \alpha(y_2),\) and \(\|y_3\| > a\) with \(\alpha(y_3) < b\).

**Proof.** We first define the completely continuous operator \(A : B \to B\) by

\[
Ay(t) = \int_0^1 G(t,s)f(y(s)) \, ds,
\]

and we seek fixed points of \(A\) which satisfy the conclusion of the theorem. We observe from the positivity of \(f\) and (3.1) that \(Ay(t) \geq 0\) on \([0,1]\) for each \(y \in \mathcal{P}\). Thus, \(A : \mathcal{P} \to \mathcal{P}\).

We now show that the conditions of Theorem 2.1 are satisfied. Choose \(y \in \overline{\mathcal{P}}_c\). Then \(\|y\| \leq c\), and by assumption (iii), \(f(y(s)) \leq Kc, s \in [0,1]\). Thus, from (3.6) we have

\[
\|Ay\| = \max_{t \in [0,1]} \int_0^1 G(t,s)f(y(s)) \, ds
\leq \max_{t \in [0,1]} \int_0^1 G(t,s)Kc \, ds = c.
\]

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Hence, \( A : \overline{P}_c \to \overline{P}_c \). In a similar way, if \( y \in \overline{P}_a \), then assumption (i) yields \( f(y(s)) < Ka, \ s \in [0,1] \), and it follows as above that \( A : \overline{P}_a \to P_a \). Consequently, condition (C2) of Theorem 2.1 is fulfilled.

To verify property (C1) of Theorem 2.1, we note that \( x(t) = 4^m b, \ t \in [0,1] \), belongs to \( P(\alpha, b, 4^m b) \), and \( \alpha(x) = 4^m b > b \). So

\[
\{ y \in P(\alpha, b, 4^m b) : \alpha(y) > b \} \neq \emptyset.
\]

Furthermore, if we choose \( y \in P(\alpha, b, 4^m b) \), then

\[
\alpha(y) = \min_{t \in [\frac{1}{4}, \frac{3}{4}]} y(t) \geq b,
\]

and so \( b \leq y(s) \leq 4^m b, \ s \in [\frac{1}{4}, \frac{3}{4}] \). Thus, for any \( y \in P(\alpha, b, 4^m b) \), assumption (ii) yields \( f(y(s)) \geq 4^m \cdot 2Lb, \ s \in [\frac{1}{4}, \frac{3}{4}] \), and from (3.10) we obtain

\[
\alpha(Ay) = \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \int_{0}^{1} G(t, s) f(y(s)) \, ds \\
\geq \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \int_{\frac{1}{4}}^{\frac{3}{4}} G(t, s) 4^m \cdot 2Lb \, ds > b.
\]

Hence, condition (C1) of Theorem 2.1 is satisfied.

We finally exhibit that (C3) of Theorem 2.1 is satisfied. To this end, choose \( y \in P(\alpha, b, c) \) such that \( \| Ay \| > 4^m b \). From (3.2) and (3.3),

\[
\alpha(Ay) = \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \int_{0}^{1} G(t, s) f(y(s)) \, ds \\
\geq \frac{1}{4^m} \int_{0}^{1} G(t(s), s) f(y(s)) \, ds \\
\geq \frac{1}{4^m} \max_{t \in [0,1]} \int_{0}^{1} G(t, s) f(y(s)) \, ds = \frac{1}{4^m} \| Ay \| > b.
\]

Therefore (C3) of Theorem 2.1 is satisfied. An application of Theorem 2.1 completes the proof. \( \square \)

**Remark.** Hypothesis (iii) can be replaced by

\[
\lim_{x \to \infty} \frac{f(x)}{x} < K.
\]
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For if this is the case, then there exist \(\tau > 0\) and \(\sigma < K\) such that \(x \geq \tau\) implies \(\frac{f(x)}{x} < \sigma\). Let \(\beta = \max_{x \in [0,\tau]} f(x)\). Then

\[
f(x) \leq \sigma x + \beta, \quad x \geq 0.
\]

Define

\[
c \geq \max\left\{\frac{\beta}{K - \sigma}, 4^{2m}b\right\}.
\]

Now we observe that if \(y \in \overline{\mathcal{P}}_c\), then

\[
\|Ay\| \leq \max_{t \in [0,1]} \int_0^1 G(t, s) (\sigma y(s) + \beta) \, ds \\
\leq \max_{t \in [0,1]} \int_0^1 G(t, s) (\sigma \|y\| + \beta) \, ds \\
\leq \max_{t \in [0,1]} \int_0^1 G(t, s)(\sigma c + \beta) \, ds < c.
\]

Hence \(A: \overline{\mathcal{P}}_c \rightarrow \mathcal{P}_c\).

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