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3-DIMENSIONAL EUCLIDEAN MANIFOLDS REPRESENTED BY LOCALLY REGULAR COLOURED GRAPHS

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(*Communicated by Martin Škoviera*)

ABSTRACT. Locally regular coloured graphs are a generalization to higher dimensions of the concept of valency regular maps (or hypermaps) on surfaces. In this paper, we study which of the closed Euclidean 3-manifolds admit a locally regular graph. We prove that there is only one manifold which cannot admit a locally regular graph and we give examples of such graphs for the other nine closed Euclidean manifolds.

Introduction

The use of coloured graphs has proved to be an important combinatorial method of representation of P. L. manifolds. It was originally introduced by M. Pezzana and it has been developed by several authors, see [FGG] as a survey of the techniques of this method. It is based on the possibility of any n-manifold to admit a triangulation which can be represented by a coloured graph. Independently, A. Vince has introduced in [V1] an equivalent concept called combinatorial map. In this sense, combinatorial maps (or coloured graphs which will be the term used in this paper) are a generalization to higher dimensions of the concept of a map on a surface. In [V2], A. Vince defines the generalization of the notion of regularity: a coloured graph Γ is said to be regular if its automorphism group $\mathrm{Aut}(\Gamma)$ acts transitively on the set of vertices of Γ. A weaker concept of regularity for maps on surfaces is the concept of valency regular maps or maps of type (m,n), where each polygonal cell has exactly n edges, and there are exactly m faces meeting at each vertex. This concept of regularity has been adapted by A. F. Costa to coloured graphs

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in [C1] and called locally regular coloured graphs. Since regularity is a very strong requirement for coloured graphs, it is natural to ask which manifolds can be represented by locally regular coloured graphs. For instance, in the case of Euclidean surfaces, there are only regular maps (regular coloured graphs) on the torus, however the Klein bottle admits valency regular maps (locally regular graphs). A. Costa in [C1] has classified and described the locally regular graphs for the 3-manifolds with spherical geometry. He also states in [C2] without proof that, for the case of Euclidean 3-manifolds, there is one orientable case which cannot be represented by a locally regular graph. In Section 3, we present a proof of this assertion and we give in Section 4.1 and 4.2 examples of locally regular coloured graphs representing the remaining orientable and non-orientable Euclidean manifolds respectively. With these examples we also prove that these nine Euclidean manifolds can be obtained by identification of the faces of a cube.

1. Coloured graphs and Coxeter representation

In this section, we present some of the basic definitions and well-known results. (See [FGG] and the references, [BM], [V1], and [V2]).

Definition 1.1. An $(n+1)$-*coloured graph* is a pair (Γ, γ) where $\Gamma = \big(V(\Gamma), E(\Gamma)\big)$ is a connected graph ($V(\Gamma)$ and $E(\Gamma)$ denote the sets of vertices and edges of Γ respectively) with all vertices of degree $n+1$, and $\gamma\colon E(\Gamma) \to I = \{0, 1, \dots, n\}$ is a map from $E(\Gamma)$ to the *colour set* I, called *edge-coloration*, such that $\gamma(e_1) \neq \gamma(e_2)$ for any two adjacent edges $e_1, e_2 \in E(\Gamma)$.

If (Γ, γ) is an $(n+1)$-coloured graph, and J is any subset of the colour set I, Γ_J will denote the subgraph $\big(V(\Gamma), \gamma^{-1}(J)\big)$ with the induced coloration. Each connected component of Γ_J is called a J-*residue* or a *residue of rank* $\sharp J$.

For the sake of conciseness, we shall omit the edge-coloration and write Γ instead of (Γ, γ).

Associated with each $(n+1)$-coloured graph Γ there is an n-dimensional pseudocomplex $K(\Gamma)$, whose k-simplexes are in $1-1$ correspondence with the residues of rank $n-k$. $K(\Gamma)$ is constructed in the following way: take an n-dimensional simplex $\sigma^n(x)$ for each $x \in V(\Gamma)$ and label its vertices with the elements of I; this induces a coloring of the $(n-1)$-faces in $\sigma^n(x)$ giving to each face the colour of the opposite vertex. Then we identify the $(n-1)$-faces of $\sigma^n(x)$ and $\sigma^n(y)$ with colour i if and only if $x, y \in V(\Gamma)$ are joined in Γ by an edge e and $\gamma(e) = i$. $K(\Gamma)$ is said to be *represented* by Γ.

Some combinatorial properties of the graph Γ are related to topological properties of the pseudocomplex $K(\Gamma)$. For instance, Γ is a bipartite graph (i.e., there is a coloration by black and white of the vertices of Γ such that any two adjacent vertices have different colours) if and only if $K(\Gamma)$ is orientable (see [FGG] and [V1]).

DEFINITION 1.2. Let (Γ, γ) and (Γ', γ') be two coloured graphs over I. An *m-covering* $(\varphi, \tilde{\varphi})\colon (\Gamma, \gamma) \to (\Gamma', \gamma')$ is a function $\varphi\colon V(\Gamma) \to V(\Gamma')$ and a bijection $\tilde{\varphi}\colon I \to I$ such that:

1. if $x, y \in V(\Gamma)$ are i-adjacent and $\tilde{\varphi}(i) = j$, then $\varphi(x), \varphi(y) \in V(\Gamma')$ are j-adjacent in (Γ', γ');
2. φ is a bijection when restricted to the rank-m residues.

Observe that an m-covering is automatically an $(m-1)$-covering. An $\sharp I$-covering is called an *isomorphism*, and if $(\Gamma, \gamma) = (\Gamma', \gamma')$, it is an *automorphism.* A covering naturally induces a branched covering $f\colon K(\Gamma) \to K(\Gamma')$ of the underlying pseudocomplexes, and if $K(\Gamma)$ and $K(\Gamma')$ are manifolds, then, for each $m \geq 2$, an m-covering induces an unbranched covering (see [V1]).

The following definitions are the generalization of the concepts of regular and valency regular maps on a surface and are given in [V1] and [C1] respectively.

DEFINITION 1.3. A coloured graph Γ is *regular* if the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ acts transitively on $V(\Gamma)$.

Γ is said to be a *locally regular coloured graph* if for each $\{i, j\} \subset I$ all the $\{i, j\}$-residues have the same number of vertices.

If Γ is regular, then it is also locally regular, but the converse is not true. The following result, given in [C1; Theorem 3.4], relates crystallographic groups with locally regular coloured graphs:

THEOREM 1.4. *Let Γ be a locally regular coloured graph such that $K(\Gamma)$ is a closed n-manifold. Then there is a regular graph $\tilde{\Gamma}$ such that $K(\tilde{\Gamma})$ is isomorphic to a tessellation of X by geometric n-simplexes, where X is the hyperbolic, spherical or Euclidean n-dimensional space, and there is a subgroup $G \leq \mathrm{Aut}(\tilde{\Gamma})$, $G \cong \pi_1(K(\Gamma))$, acting freely on X such that $\tilde{\Gamma}/G$ is isomorphic to Γ.*

In [V1], Vince obtained a group theoretic representation of a coloured graph called the Coxeter representation (a generalization of the relation between maps and transitive permutation representations of the extended triangle groups, see [JS]).

Let Γ be a coloured graph over I, and let denote by $2m_{ij}$ the lcm of the number of vertices of all $\{i, j\}$-residues. The *Coxeter Diagram* $D(\Gamma)$ of Γ is obtained by representing each $i \in I$ as a node labelled i, and connecting nodes

i and j by a line labelled m_{ij}. By convention, if $m_{ij} = 2$, the line is omitted, and if $m_{ij} = 3$, the line label is omitted.

If W is a group generated by involutions $\{r_i \mid i \in I\}$, and G is a subgroup of W, $\Gamma(W, G)$ will denote the I-coloured graph defined as follows: the vertices of $\Gamma(W, G)$ are the right cosets of W/G, and two vertices v and v' are i-adjacent if and only if $v' = vr_i$. Now let Γ be a coloured graph over I with Coxeter diagram $D(\Gamma)$. For each $i \in I$ define a permutation ρ_i on $V(\Gamma)$ by $\rho_i v = v'$ if v is i-adjacent to v'. Let P be the permutation group on $V(\Gamma)$ generated by the $\{\rho_i\}$, and let P_v be the stabilizer of a vertex v of Γ. Now let $\Sigma = \langle r_i \mid i \in I, r_i^2 = (r_i r_j)^{m_{ij}} = 1\rangle$ be the Coxeter group with diagram $D(\Gamma)$. Then we can define a homomorphism $\phi\colon \Sigma \to P$ such that $r_i \mapsto \rho_i$. If we denote by G the preimage of the stabilizer of a vertex, i.e., $G = \phi^{-1}(P_v)$, then $\Gamma = \Gamma(\Sigma, G)$, and it is called the *Coxeter representation* of Γ.

Notice that if Γ is locally regular, all $\{i, j\}$-residues in Γ have the same number of vertices $2m_{ij}$, and furthermore, if Γ represents a manifold, then G is a fixed-point-free subgroup of Σ. Thus the locally regular 4-coloured graphs representing Euclidean manifolds have Coxeter representation $\Gamma = \Gamma(\Sigma, G)$, where Σ is a rank 4 Euclidean Coxeter group (that is, Σ has a geometric realization as a subgroup of isometries of $\mathbb{E}^3$), and $G \leq \Sigma$ is a crystallographic group without fixed points, so the graph Γ has the same Coxeter diagram as Σ. Therefore, $\Gamma(\Sigma, \{1\})$ is the Cayley graph of Σ, and it is isomorphic to the universal coloured graph $\tilde{\Gamma}$ of Γ of Theorem 1.4, therefore $\Sigma \cong \mathrm{Aut}(\tilde{\Gamma})$, where $K(\tilde{\Gamma})$ is isomorphic to a tessellation of $\mathbb{E}^3$ by Euclidean 3-simplexes and $\Gamma(\Sigma, G) \cong \Gamma(\Sigma, \{1\})/G$.

2. Rank 4 Euclidean Coxeter groups

The possibilities for $D(\Sigma)$, where Σ belongs to the rank 4 Euclidean Coxeter groups, are limited to the three following cases in Figure 1, where each node $k \in \{0, 1, 2, 3\}$ of $D(\Sigma_i)$ represents a generator $r_k \in \Sigma_i$, $i = 1, 2, 3$.

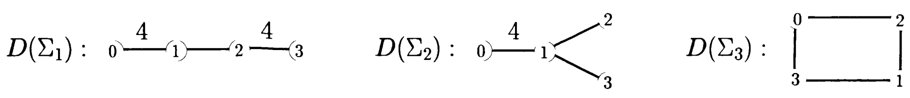


FIGURE 1. Coxeter diagrams of Σ_i, $i = 1, 2, 3$.

We have the following presentations of the associated Coxeter groups Σ_i:

$$\Sigma_1 = \langle r_0, r_1, r_2, r_3 \mid r_i^2 = (r_0r_1)^4 = (r_0r_2)^2 = (r_0r_3)^2 = (r_1r_2)^3 = (r_1r_3)^2 = (r_2r_3)^4 = 1\rangle;$$

$$\Sigma_2 = \langle r_0, r_1, r_2, r_3 \mid r_i^2 = (r_0r_1)^4 = (r_0r_2)^2 = (r_0r_3)^2 = (r_1r_2)^3 = (r_1r_3)^3 = (r_2r_3)^2 = 1\rangle;$$

$$\Sigma_3 = \langle r_0, r_1, r_2, r_3 \mid r_i^2 = (r_0r_1)^2 = (r_0r_2)^3 = (r_0r_3)^3 = (r_1r_2)^3 = (r_1r_3)^3 = (r_2r_3)^2 = 1\rangle.$$

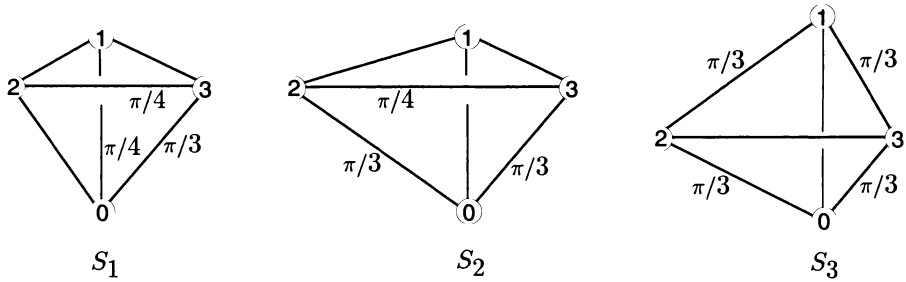


FIGURE 2. Fundamental tetrahedra S_i of Σ_i, $i = 1,2,3$.

Since Σ_i are crystallographic groups acting on $\mathbb{E}^3$, there is an exact sequence for each $i = 1,2,3$:

$$1 \longrightarrow T_{\Sigma_i} \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \longrightarrow \Sigma_i \longrightarrow \Sigma_i{}^* \longrightarrow 1,$$

where T_{Σ_i} is a free abelian normal subgroup of rank 3 of finite index, which is maximal abelian in Σ_i and corresponds to the translation subgroup, and where $\Sigma_i{}^* = \Sigma_i/T_{\Sigma_i}$ is a finite subgroup of $O(3)$. Each Σ_i has an Euclidean simplex S_i for its fundamental domain. If we label the vertices $v_i(k)$ of S_i with colours $k \in \{0,1,2,3\}$, then the geometric realization of Σ_i corresponds to considering the generator $r_k \in \Sigma_i$ as the reflection in the face of S_i opposite to the vertex $v_i(k)$ with label k. Figure 2 shows the fundamental tetrahedron S_i of each Σ_i, where the dihedral angle between the faces is indicated when it is different from $\pi/2$.

Observe (Figure 2) that S_2 can be obtained as a double cover of S_1 reflecting along the face containing the vertices of S_1 with colours 0, 1 and 2. Also S_3 can be obtained from S_2 after a reflection along the face of S_2 whose vertices are coloured by 0, 1, and 2. Therefore we can define the following inclusions:

$$\begin{aligned} \rho_2 : \ & \Sigma_3 \hookrightarrow \Sigma_2 & \qquad \rho_1 : \ & \Sigma_2 \hookrightarrow \Sigma_1 \\ & r_0 \mapsto r_0 r_1 r_0 & & r_0 \mapsto r_0 \\ & r_1 \mapsto r_1 & & r_1 \mapsto r_1 \\ & r_2 \mapsto r_2 & & r_2 \mapsto r_2 \\ & r_3 \mapsto r_3 , & & r_3 \mapsto r_3 r_2 r_3 . \end{aligned}$$

Thus $\Sigma_3 \leq \Sigma_2 \leq \Sigma_1$ as crystallographic groups with $|\Sigma_1 : \Sigma_2| = |\Sigma_2 : \Sigma_3| = 2$ (see [Cox2; p. 84]). Moreover, the translation subgroups satisfy: $T_{\Sigma_3} = T_{\Sigma_2} \leq T_{\Sigma_1}$ with $|T_{\Sigma_1} : T_{\Sigma_2}| = 2$, thus $\Sigma_3^* \leq \Sigma_2^* \cong \Sigma_1^*$ as finite subgroups of $O(3)$, and Σ_3^* is a subgroup of Σ_2^* of index two.

3. Euclidean manifolds admitting locally regular coloured graphs

Let $M = \mathbb{E}^3/G$ be a 3-dimensional closed Euclidean manifold with associated crystallographic group G. If there is a locally regular graph Γ representing M, then the Coxeter representation of Γ is of the form $\Gamma(\Sigma_i, G)$ with $i = 1, 2$ or 3, where $G \leq \Sigma_i$ is a fixed-point-free subgroup. Therefore, finding the manifolds admitting a locally regular graph is equivalent to studying which of their space groups admits an inclusion as a subgroup of one of the three given Euclidean Coxeter groups. There are altogether ten closed Euclidean manifolds, six being orientable and four non-orientable ([S; p. 447], [W; pp. 117–121]). Denote the orientable ones by $M_i = \mathbb{E}^3/G_i$, $i = 1, \ldots, 6$, and the non-orientable ones by $N_j = \mathbb{E}^3/B_j$, $j = 1, \ldots, 4$. The presentations of the fundamental groups will follow those given by Wolf in [W], we also present in brackets the corresponding notation of the International Tables of Crystallography [IT].

Let $\mathrm{Isom}(\mathbb{E}^3)$ denote the group of isometries of $\mathbb{E}^3$. Any isometry α of $\mathbb{E}^3$ can be expressed as a composition $\alpha = (A, t_a)$, where $\alpha(x) = A(x) + a$, $A \in O(3)$ is an orthogonal transformation, and t_a is the translation of vector $a \in \mathbb{R}^3$, having multiplication given by $(A, t_a)(B, t_b) = (AB, t_{A(b)+a})$. Thus $\mathrm{Isom}(\mathbb{E}^3)$ is the semidirect product $O(3) \cdot \mathbb{R}^3$ with exact sequence:

$$0 \longrightarrow \mathbb{R}^3 \longrightarrow \mathrm{Isom}(\mathbb{E}^3) \longrightarrow O(3) \longrightarrow 1 .$$

A crystallographic group (also called space group) G is a discrete subgroup of $\mathrm{Isom}(\mathbb{E}^3)$ such that G is an extension of its translation subgroup $T_G \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ by a finite subgroup G^* of $O(3)$, $G^* \cong G/T_G$. Thus we have the exact sequence:

$$1 \longrightarrow T_G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \longrightarrow G \longrightarrow G^* \longrightarrow 1 .$$

If G is non-orientable, that is, if $G^* \not\subset SO(3)$, then $G^+ = G \cap SO(3) \cdot \mathbb{R}^3$ is the orientable subgroup of G, so that $\mathbb{E}^3/G^+ \to \mathbb{E}^3/G$ is the 2-sheeted orientable covering.

Let $M_6 = \mathbb{E}^3/G_6$ denote the orientable Euclidean manifold, where $[G_6 = \mathbf{P6_1}]$ is generated by:

$$G_6 = \{\alpha = (A, t_{u_1/6}), t_{u_2}, t_{u_3}\},$$

where α is a screw motion composed by a rotation A by angle $2\pi/6$ with rotation axis parallel to u_1, $A(u_1) = u_1$, $A(u_2) = u_3$, $A(u_3) = u_3 - u_2$, and a translation $t_{u_1/6}$ of vector $u_1/6$. Thus, $\alpha^6 = t_{u_1} \in T_{G_6}$. Moreover, u_1 is orthogonal to u_2 and u_3, $\|u_2\| = \|u_3\|$ and $\{u_2, u_3\}$ forms a hexagonal plane lattice. The translation subgroup T_{G_6} is generated by $\{t_{u_1}, t_{u_2}, t_{u_3}\}$, and the finite group $G_6^* = G_6/T_{G_6}$ is isomorphic to $\mathbb{Z}_6$ ([W; p. 117]).

Since G_6 contains screw motions along lines parallel to u_1 with rotation angle $2\pi/6$, the direction of u_1 is the only one direction in $\mathbb{E}^3$ which is left invariant by the action of G_6. Therefore, there is only one foliation of $\mathbb{E}^3$ preserved by G_6, and M_6 has a unique Seifert bundle structure over the orbifold S^2_{236} (the sphere with three cone points of orders 2, 3, 6 respectively) corresponding to foliations of $\mathbb{E}^3$ by lines parallel to u_1.

THEOREM 3.1. *The orientable Euclidean manifold M_6 cannot be represented by a locally regular coloured graph.*

Proof. Suppose there is a locally regular graph Γ representing $M_6 = \mathbb{E}^3/G_6$. By Theorem 2.4, Γ has Coxeter representation $\Gamma = \Gamma(\Sigma, G)$, where $G \cong \pi_1(M_6) = G_6$. We write $\Gamma = \Gamma(\Sigma, G_6)$, where, by abuse of language, G_6 refers to the inclusion $G_6 \hookrightarrow \Sigma$ as well as the crystallographic group, and $\Sigma = \Sigma_i$ for $i = 1, 2$ or 3.

Let Σ^+ denote the orientation preserving subgroup of Σ, and $\Sigma^{+*} = \Sigma^+/T_\Sigma$. Since M_6 is orientable, we have $G_6 \le \Sigma^+$, and since $T_{G_6} = G_6 \cap T_\Sigma$, we can define the following monomorphism:

$$\begin{aligned} i:\ G_6^* &\longrightarrow \Sigma^{+*} \\ gT_{G_6} &\longrightarrow gT_\Sigma , \end{aligned}$$

where $G_6^* \cong \mathbb{Z}_6$. Then we shall prove the result if we see that Σ_i^{+*} does not contain elements of order 6 for any $i = 1, 2, 3$. However, since $\Sigma_3^{+*} \le \Sigma_2^{+*} \cong \Sigma_1^{+*}$, we only need to show that Σ_1^{+*} does not contain elements of order 6. Let $g_{ij} = r_i r_j$, $i, j \in \{0, 1, 2, 3\}$, denote the generators of the orientation preserving subgroup. Σ_1^+ admits the following presentation:

$$\Sigma_1^+ = \langle g_{01}, g_{12}, g_{23} \mid g_{01}^4 = g_{12}^3 = g_{23}^4 = 1,\ (g_{01}g_{12})^2 = 1,\ (g_{01}g_{12}g_{23})^2 = 1,\ (g_{12}g_{23})^2 = 1 \rangle .$$

To obtain a presentation of Σ_1^{+*}, we must add the following relations associated to T_{Σ_1}:

$$g_{01}^{-1}g_{12}g_{23}^{-1} = 1\,, \qquad g_{12}^{-1}g_{01}g_{23} = 1\,, \qquad g_{01}g_{23}g_{12}^{-1} = 1\,,$$

giving rise to the presentation:

$$\Sigma_1^{+*} = \langle g_{01}, g_{12} \mid g_{01}^4 = g_{12}^3 = 1\,,\ (g_{01}g_{12})^2 = 1\,,\ (g_{01}g_{12}^2)^4 = 1\rangle\,.$$

Thus Σ_1^{+*} is isomorphic to the *octahedral group* $\mathbf{O}$, which does not contain elements of order 6. □

The remaining nine Euclidean manifolds admit a triangulation which can be represented by a locally regular graph. We present in the next section examples of these graphs with Coxeter diagram $D(\Sigma_1)$.

4. Examples of locally regular graphs on Euclidean manifolds with $D(\Gamma) = D(\Sigma_1)$

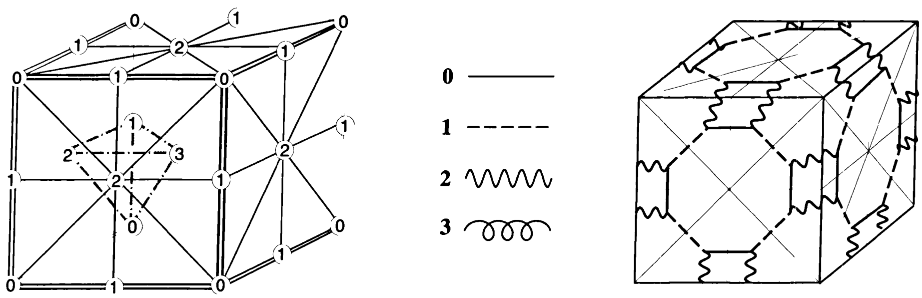
If $\Gamma = \Gamma(\Sigma_1, G)$ is a locally regular graph representing a closed Euclidean manifold M with diagram $D(\Gamma) = D(\Sigma_1)$, then $G \leq \Sigma_1$ is the fixed-point-free crystallographic subgroup associated to $M = \mathbb{E}^3/G$. Moreover, Γ represents a simplicial decomposition of M by $\sharp V(\Gamma) = |\Sigma_1 : G|$ tetrahedra S_1 (the Euclidean simplex which tessellates $\mathbb{E}^3$ by the action of Σ_1). Now, since M is a manifold, at each vertex of the triangulation, there is a solid angle of 2π, therefore $|\Sigma_1 : G|$ must be a multiple of the lcm of the orders of the isotropy groups of the vertices of S_1. For each vertex $v(k)$ with colour $k \in \{0,1,2,3\}$ of S_1, its isotropy group is the spherical Coxeter subgroup H_k generated by the reflections along the faces of S_1 incident with $v(k)$. Thus $H_k \leq \Sigma_1$ has finite order $|H_k|$, and its Coxeter diagram $D(H_k)$ is obtained from $D(\Sigma_1)$ deleting the node with label k and the edges incident with it (see Figure 3).

$D(H_0) = D(H_3) =$ 

$D(H_1) = D(H_2) =$

FIGURE 3. Spherical Coxeter diagrams of H_k, $k \in \{0,1,2,3\}$.

The action of H_k tessellates the sphere (corresponding to the link of v_k) with $|H_k|$ spherical triangles. Since $|H_0| = |H_3| = 48$ and $|H_1| = |H_2| = 16$ (see [Cox1; pp. 618, 619] or [Cox2; p. 297]), the lcm of the orders of the isotropy groups of the vertices of S_1 is 48. Thus we have $|\Sigma_1 : G| = k.48$, $k \geq 1$, so $\sharp V(\Gamma) = k.48$, $k \geq 1$. If $\sharp V(\Gamma) = 48$, these 48 vertices correspond to 48 tetrahedra S_1 which can be grouped together as the barycentric subdivision of a cube (see Figure 4).



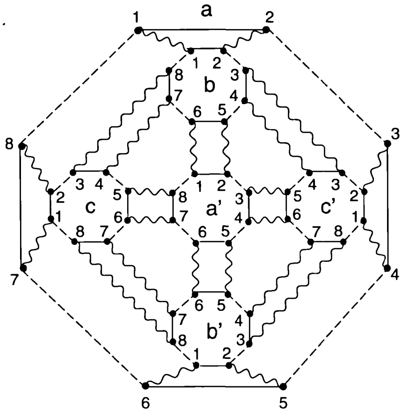
(a) (b)

FIGURE 4. (a) Barycentric subdivision of the cube;
(b) $\{0,1,2\}$-residue.

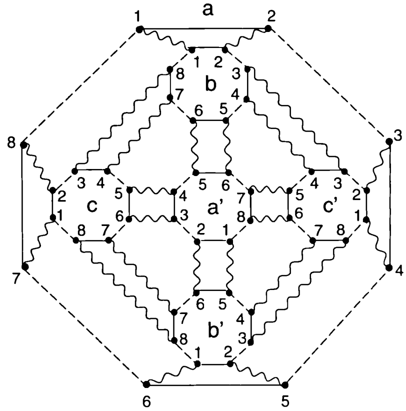
Next we present examples of locally regular coloured graphs Γ representing the nine Euclidean manifolds with $\sharp V(\Gamma) = 48$. This proves that the fundamental groups admit an inclusion as subgroups of Σ_1 with minimum index, and the manifolds are therefore obtained by identification of the faces of a cube. We shall describe these graphs Γ by adding to the $\{0,1,2\}$-coloured subgraph given in Figure 4(b) (which represents the baricentric subdivision of the cube) the 3-coloured edges of Γ (which indicate the face identifications of the cube). These edges, with colour 3, are the edges of Γ that join the vertices labelled ix with ix', where $1 \leq i \leq 8$ and $x \in \{a,b,c\}$. The figures present a caption with the manifold $M = \mathbb{E}^3/G$ represented by Γ and the notation of the fundamental group G corresponding to the International Tables of Crystallography [IT].

Remark. Although all the examples are given with Coxeter diagram $D(\Sigma_1)$, it is also possible to represent these nine manifolds by locally regular graphs with diagram $D(\Sigma_2)$, and eight of them (except the orientable manifold M_4) with diagram $D(\Sigma_3)$.

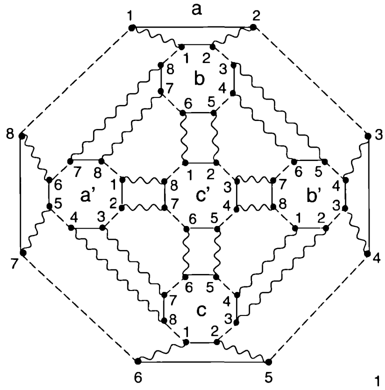
4.1. Orientable manifolds $M_i = \mathbb{E}^3/G_i$, $i = 1, \dots, 5$:



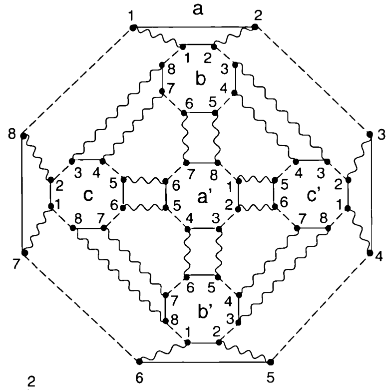
$M_1 = \mathbb{E}^3/G_1$, $[G_1 = P1]$



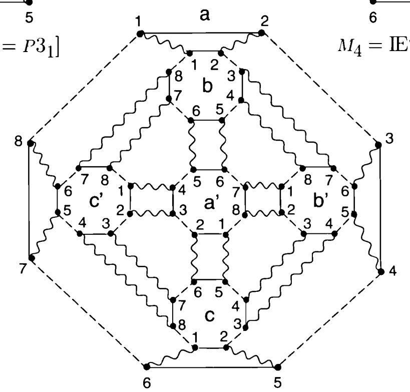
$M_2 = \mathbb{E}^3/G_2$, $[G_2 - P2_1]$



$M_3 = \mathbb{E}^3/G_3$, $[G_3 = P3_1]$

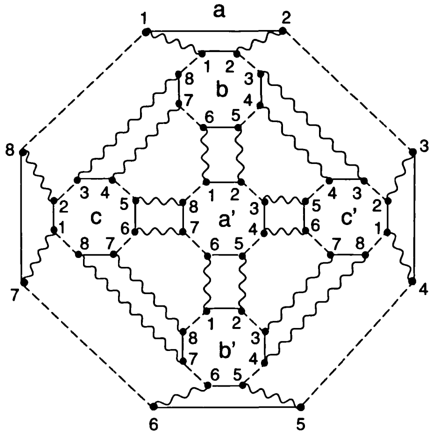


$M_4 = \mathbb{E}^3\ G_4$. $[G_4 = P4_1]$

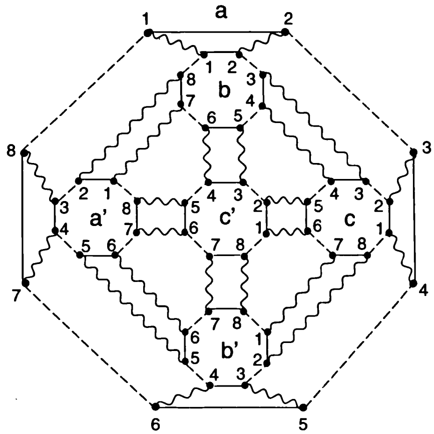


$M_5 = \mathbb{E}^3/G_5$, $[G_5 = P2_12_12_1]$

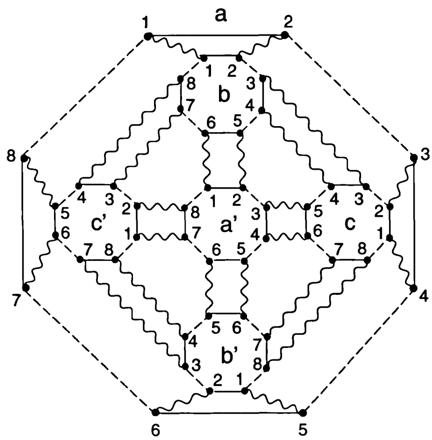
4.2. Non-orientable manifolds $N_j = \mathbb{E}^3/B_j$, $j = 1, \ldots, 4$:



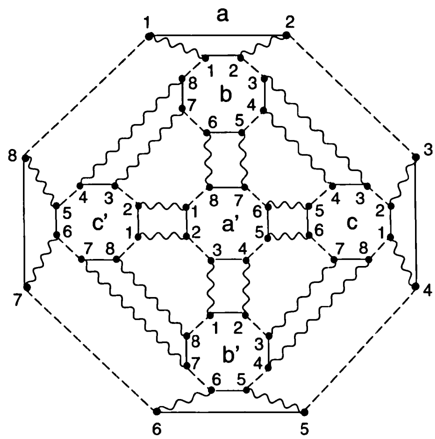
$N_1 = \mathbb{E}^3/B_1$, $[B_1 = Pb]$



$N_2 = \mathbb{E}^3/B_2$, $[B_2 = Bb]$



$N_3 = \mathbb{E}^3/B_3$, $[B_3 = Pca2_1]$



$N_4 = \mathbb{E}^3/B_4$, $[B_4 = Pna2_1]$

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