Christoph Baxa

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DIOPHANTINE REPRESENTATION OF THE DECIMAL EXPANSIONS OF e AND π

CHRISTOPH BAXA

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ABSTRACT. Let $\alpha \in \{e, \pi\}$, $\alpha = [\alpha] + \sum_{\kappa=1}^{\infty} \alpha_\kappa(\beta) \cdot \beta^{-\kappa}$ (where $\beta \in \mathbb{N} \setminus \{1\}$ and $\alpha_\kappa(\beta) \in \{0, 1, \dots, \beta-1\}$) and $\zeta \in \{0, 1, \dots, \beta-1\}$. We describe short Diophantine representations for the predicate $\alpha_\kappa(\beta) = \zeta$. The proofs use methods which were developed for the solution of Hilbert's Tenth Problem.

Hilbert's Tenth Problem was solved in 1970 by Yu. V. Matijasevič [12] relying heavily on results by M. Davis, H. Putnam and J. Robinson [5]. Already in 1960 H. Putnam [15] had pointed out a surprising consequence of this result: Any recursively enumerable set of positive integers equals the set of positive values of a certain polynomial whose variables range over the nonnegative integers. Yu. V. Matijasevič [13] described such a polynomial for the primes. A very short polynomial for the primes was constructed by J. P. Jones, D. Sato, H. Wada and D. Wiens [10]. Subsets of the primes which have been treated are the Fermat-, Mersenne- and twin-primes ([6], [2]). Further examples of predicates from number theory which have been tackled — including the Riemann hypothesis — can be found in [4] and [14; Section 6.4]. In the present note we apply these techniques to describe such a representation for the digits in the decimal expansion of the constants e and π. Although π especially has received a lot of attention and surprising new facts about its digits have been found recently ([1]), these seem to be the first results of this kind. A reader who wants to learn more about Hilbert's Tenth Problem is referred to [3], [4], [9], [11; Chapter 6], [14] and [16]. Unless stated otherwise all occurring quantities are integers.

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Definition. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. A sequence of intervals $([p_n, q_n])_{n\geq 1}$ will be called a *rational nest of intervals for* α if:

(1) $p_n, q_n \in \mathbb{Q}$ for all $n \geq 1$,
(2) $\lim\limits_{n\to\infty} p_n = \lim\limits_{n\to\infty} q_n = \alpha$,
(3) $(p_n)_{n\geq 1}$ is monotonically increasing and $(q_n)_{n\geq 1}$ is monotonically decreasing.

Lemma 1.
(1) $\left(\left[\left(1+\frac{1}{n}\right)^n, \left(1+\frac{1}{n}\right)^{n+1}\right]\right)_{n\geq 1}$ *is a rational nest of intervals for* e.

(2) $\left(\left[\frac{1}{2n+1}\binom{2n}{n}^{-2}2^{4n+1}, \frac{1}{n}\binom{2n}{n}^{-2}2^{4n}\right]\right)_{n\geq 1}$ *is a rational nest of intervals for* π.

Proof. These are basic facts from calculus. Part (2) is a reformulation of the Wallis product formula. □

Lemma 2. *Let* $\beta \in \mathbb{N} \setminus \{1\}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *and* $\alpha = [\alpha] + \sum\limits_{\kappa=1}^{\infty} \alpha_\kappa(\beta)\beta^{-\kappa}$, *where* $0 \leq \alpha_\kappa(\beta) < \beta$ *for* $\kappa \geq 1$. *Furthermore, let* $([p_n, q_n])_{n\geq 1}$ *be a rational nest of intervals for* α, $0 \leq \zeta < \beta$ *and* $k \geq 1$. *Then the following are equivalent:*

(1) $\alpha_k(\beta) = \zeta$.
(2) *There exists* $n \in \mathbb{N}$ *such that* $[\beta^k p_n] = [\beta^k q_n] \equiv \zeta \pmod{\beta}$.

Proof.
($1 \Longrightarrow 2$) Let $l := \min\{\kappa \in \mathbb{N} \mid \kappa > k,\ \alpha_\kappa(\beta) \neq \beta - 1\}$. Then

$$[\alpha] + \sum_{\kappa=1}^{k} \alpha_\kappa(\beta)\beta^{-\kappa} < p_n < \alpha < q_n < [\alpha] + \sum_{\kappa=1}^{l} \alpha_\kappa(\beta)\beta^{-\kappa} + \beta^{-l}$$

for sufficiently large n and thus

$$[\beta^k p_n] = [\beta^k q_n] = [\beta^k \alpha] = [\alpha]\beta^k + \sum_{\kappa=1}^{k} \alpha_\kappa(\beta)\beta^{k-\kappa} \equiv \alpha_k(\beta) = \zeta \pmod{\beta}.$$

($2 \Longrightarrow 1$) As $p_n < \alpha < q_n$ we can deduce

$$\zeta \equiv [\beta^k p_n] = [\beta^k q_n] = [\beta^k \alpha] = \beta^k[\alpha] + \sum_{\kappa=1}^{k} \alpha_\kappa(\beta)\beta^{k-\kappa} \equiv \alpha_k(\beta) \pmod{\beta}$$

and therefore $\alpha_k(\beta) = \zeta$. □

Remarks.

(1) The existence of one n which satisfies condition (2) implies that there are infinitely many and we may assume $n > k$.

(2) If $\alpha > 0$ condition (2) can be replaced by:

$$(\exists\, n \geq 1)(\exists\, t \geq 0)\big([\beta^k p_n] = [\beta^k q_n] = \zeta + t\beta\big).$$

Next we introduce some notations: Let $a \geq 2$. For $n \geq 0$ we denote by $\big(x_n(a), y_n(a)\big)$ the solution of the Pell equation $x^2 - (a^2-1)y^2 = 1$ defined by the relation $x_n(a) + y_n(a)\sqrt{a^2-1} = \big(a + \sqrt{a^2-1}\,\big)^n$. All nonnegative solutions (x, y) of this Pell equation are of this shape, see [3; Lemmata 2.1–2.4]. We use $Z = \square$ as a shorthand notation for $(\exists\, X \geq 0)(Z = X^2)$.

Lemma 3. *$y_n(a) \equiv n \pmod{a-1}$ for $n \geq 0$.*

Proof. See [3; Lemma 2.14] and [10; Lemma 2.2]. $\square$

Lemma 4. *$n + y_{n-1}(a) \leq y_n(a)$ for $n \geq 1$ which implies that the sequence $\big(y_n(a)\big)_{n\geq 0}$ is strictly monotonically increasing and that $y_n(a) \geq n$ for $n \geq 0$.*

Proof. By [3; Lemmata 2.5, 2.19]

$$y_n(a) = x_1(a)y_{n-1}(a) + x_{n-1}(a)y_1(a) \geq y_{n-1}(a) + a^{n-1} \geq y_{n-1}(a) + n.$$

$\square$

Lemma 5. *Let $a \geq 2$ and $P, n \geq 0$. Then $x_n(a) \equiv P^n + y_n(a)(a-P) \pmod{2aP - P^2 - 1}$. If $0 < P^n < a$, then $P^n + y_n(a)(a-P) \leq x_n(a)$.*

Proof. This is [10; Lemma 2.4]. $\square$

Lemma 6. *Let $a \geq 2$, $n \geq 1$ and $y \geq 0$. Then the following are equivalent:*

(1) *$y = y_n(a)$.*

(2) *There exist $c, d, r, u, x \geq 0$ such that*

(i) $x^2 = (a^2-1)y^2 + 1$,

(ii) $u^2 = 16(a^2-1)r^2y^4 + 1$,

(iii) $(x + cu)^2 = \big((a + u^2(u^2-a))^2 - 1\big)(n + 4dy)^2 + 1$,

(iv) $n \leq y$.

Proof. This is [10; Corollary 2.6]. $\square$

Lemma 7. *Let $e \geq 2$. If $e^3(e+2)(N+1)^2 + 1 = \square$ for some $N \geq 0$, then $e - 1 + e^{e-2} \leq N$. Furthermore, for any $T > 0$ there is a $N \geq 0$ such that $e^3(e+2)(N+1)^2 + 1 = \square$ and $T \mid N + 1$.*

Proof. This is [10; Lemma 2.3]. $\square$

Remark. This e is a positive integer and not $\exp(1)$.

Lemma 8. *Let $\beta \geq 2$, $k,n,A,B \geq 1$ and $b,g,h \geq 0$. The following are equivalent:*

(1) $b = \beta^k \wedge g = A^n \wedge h = B^n \wedge n > k$.

(2) *There exist $a,c,d,e,i,l,m,p,q,r,u,v,x,y \geq 0$ such that:*

(i) $x^2 = (a^2-1)y^2+1$,

(ii) $u^2 = 16(a^2-1)r^2y^4+1$,

(iii) $(x+cu)^2 = \big((a+u^2(u^2-a))^2-1\big)(n+4dy)^2+1$,

(iv) $m^2 = (a^2-1)l^2+1$,

(v) $l = k+i(a-1)$,

(vi) $n+l \leq y$,

(vii) $e = n+k+b+g+h+\beta+A+B+2$,

(viii) $e^3(e+2)(a+1)^2+1 = \square$,

(ix) $x = g+y(a-A)+p(2aA-A^2-1)$,

(x) $x = h+y(a-B)+q(2aB-B^2-1)$,

(xi) $m = b+l(a-\beta)+v(2a\beta-\beta^2-1)$.

Remark. This lemma is modelled on [10; Theorem 2.12] and has a very similar proof. For the reader's convenience we include the proof instead of just giving a reference.

Proof.

($1 \implies 2$) Define e according to (vii). Due to Lemma 7, there is a $a \geq 2$ satisfying (viii). Put $y := y_n(a)$. By Lemma 6, there are $c,d,r,u,x \geq 0$ such that (i), (ii) and (iii) are fulfilled, where $x = x_n(a)$. Put $m := x_k(a)$ and $l := y_k(a)$. Then (iv) is fulfilled. Because of Lemmata 3 and 4 we get $k \equiv l \pmod{a-1}$ and $k \leq l$ and there is a $i \geq 0$ such that (v) holds. Lemma 4 implies $n + l \leq n + y_{n-1}(a) \leq y$, i.e. (vi) is satisfied. Lemma 5 yields $x \equiv g + y(a-A) \pmod{2aA - A^2 - 1}$. Conditions (vii) and (viii) and Lemma 7 imply

$$\begin{aligned} &n+k+b+g+h+\beta+A+B+1 \\ &\qquad + (n+k+b+g+h+\beta+A+B+2)^{n+k+b+g+h+\beta+A+B} \leq a. \end{aligned} \tag{$*$}$$

Therefore, it holds that $0 < A^n < a$ and Lemma 5 implies $g + y(a-A) \leq x$. This proves that there is a $p \geq 0$ such that (ix) is true. It is proved analogously that $q,v \geq 0$ exist such that (x) and (xi) are satisfied.

($2 \implies 1$) As in the first part of the proof we see that ($*$) holds and thus $a \geq 2$. Because of (i), (ii), (iii), (vi) and Lemma 6 we get $y = y_n(a)$ and $x = x_n(a)$. Equation (iv) implies that $m = x_{k'}(a)$ and $l = y_{k'}(a)$ for some $k' \geq 0$. Due to (vi), $l < y$ and therefore $k' < n$ by Lemma 4. It follows from ($*$) that $k < a-1$ and $n < a-1$ and thus $k' < a-1$. Using (v) and Lemma 3 we get $k \equiv l \equiv k'$ (mod $a-1$), thus $k = k'$, $n > k$, $m = x_k(a)$ and $l = y_k(a)$. Furthermore, $(*)$ implies $g < a \leq 2aA - A^2 - 1$ and $A^n < a \leq 2aA - A^2 - 1$. Because of (ix) and Lemma 5, $g \equiv x - y(a-A) \equiv A^n$ (mod $2aA - A^2 - 1$) and therefore $g = A^n$. In the same way it is proved that $b = \beta^k$ and $h = B^n$. □

LEMMA 9. *Let $n \geq 1$, $f \geq 0$ and $g = 2^{4n}$. Then*

$$\binom{2n}{n} = f \iff$$
$$(\exists\, w \geq 0)\left((2gw + f)^2(g-2) \leq g(4g^2)^n < (2gw + f + 1)^2(g-2) \wedge f < 2g\right).$$

P r o o f. It is proved in [8] that for $U > 4^{n+1} + 4$

$$\binom{2n}{n} = f \iff (\exists\, w \geq 0)\left(\left[U^n \big/ \sqrt{1 - 4/U}\,\right] = wU + f \wedge f < U\right).$$

(This can also be found as [16; Chapter I, Lemma 10.17].) The equation

$$\left[\frac{U^n}{\sqrt{1 - \frac{4}{U}}}\right] = wU + f$$

is equivalent to $(wU + f)^2(U-4) \leq U^{2n+1} < (wU + f + 1)^2(U-4)$. Finally set $U := 2g = 2^{4n+1} > 4^{n+1} + 4$. □

Now we are able to state exponential Diophantine representations for the predicate $\alpha_k(\beta) = \zeta$ for e and π:

LEMMA 10A. *Let $\beta \in \mathbb{N} \setminus \{1\}$, $\mathrm{e} = 2 + \sum_{\kappa=1}^{\infty} \alpha_\kappa(\beta)\beta^{-\kappa}$, where $0 \leq \alpha_\kappa(\beta) < \beta$ for $\kappa \geq 1$, $0 \leq \zeta < \beta$ and $k \geq 1$. The following are equivalent:*

(1) $\alpha_k(\beta) = \zeta$.

(2) *There exist $n \geq 1$ and $b, g, h, s, t, z \geq 0$ such that:*

(i) – (xi) $b = \beta^k \wedge g = (n+1)^n \wedge h = n^n \wedge n > k$,

(xii) $bg = (\zeta + t\beta)h + s$,

(xiii) $s < h$,

(xiv) $b(n+1)g = (\zeta + t\beta)nh + z$,

(xv) $z < nh$.

(*Numbers* (i) – (xv) *are for later reference only.*)

P r o o f . Because of Lemmata 1 and 2 we have

$$\begin{aligned}
&\alpha_k(\beta) = \zeta \\
\iff &(\exists\, n > k)(\exists\, t \ge 0)\Big(\big[\beta^k(n+1)^n/n^n\big] = \big[\beta^k(n+1)^{n+1}/n^{n+1}\big] = \zeta + t\beta\Big) \\
\iff &(\exists\, n > k)(\exists\, s, t, z \ge 0)\Big(\beta^k(n+1)^n = (\zeta + t\beta)n^n + s \ \wedge\ s < n^n \\
&\qquad \wedge\ \beta^k(n+1)(n+1)^n = (\zeta + t\beta)nn^n + z \\
&\qquad \wedge\ z < nn^n\Big)
\end{aligned}$$

and this is equivalent to (2). □

LEMMA 10B. *Let* $\beta \in \mathbb{N} \setminus \{1\}$, $\pi = 3 + \sum_{\kappa=1}^{\infty} \alpha_\kappa(\beta)\beta^{-\kappa}$, *where* $0 \le \alpha_\kappa(\beta) < \beta$ *for* $\kappa \ge 1$, $0 \le \zeta < \beta$ *and* $k \ge 1$. *The following are equivalent:*

(1) $\alpha_k(\beta) = \zeta$.

(2) *There exist* $n \ge 1$ *and* $b, f, g, h, s, t, w, z \ge 0$ *such that:*

- (i)–(xi) $b = \beta^k \ \wedge\ g = 2^{4n} \ \wedge\ h = (4g^2)^n \ \wedge\ n > k$,
- (xii) $2bg = (\zeta + t\beta)f^2(2n+1) + s$,
- (xiii) $s < f^2(2n+1)$,
- (xiv) $bg = (\zeta + t\beta)f^2 n + z$,
- (xv) $z < f^2 n$,
- (xvi) $(2gw + f)^2(g-2) \le gh$,
- (xvii) $gh < (2gw + f + 1)^2(g-2)$,
- (xviii) $f < 2g$.

(*Again the numbers are for later reference only.*)

P r o o f . As in the proof of Lemma 10A we see that

$$\begin{aligned}
&\alpha_k(\beta) = \zeta \\
\iff &(\exists\, n > k)(\exists\, b, f, g, s, t, z \ge 0)\Big(b = \beta^k \ \wedge\ f = \tbinom{2n}{n} \ \wedge\ g = 2^{4n} \\
&\qquad \wedge\ 2bg = (\zeta + t\beta)f^2(2n+1) + s \\
&\qquad \wedge\ s < f^2(2n+1) \\
&\qquad \wedge\ bg = (\zeta + t\beta)f^2 n + z \ \wedge\ z < f^2 n\Big)
\end{aligned}$$

and the proof is completed by using Lemma 9. □

THEOREM 11A. *Under the assumptions of Lemma* 10A *the following are equivalent:*

(1) $\alpha_k(\beta) = \zeta$.

(2) *There are* $a, b, c, d, e, g, h, i, l, m, p, q, r, s, t, u, v, x, y, z \ge 0$ *and* $n \ge 1$ *such that conditions* (i)–(xv) *are fulfilled, where* (i)–(xi) *are taken from Lemma* 8 *with* $A = n+1$ *and* $B = n$, *and* (xii)–(xv) *are identical with those in Lemma* 10A.

THEOREM 11B. *Under the assumptions of Lemma* 10B *the following are equivalent:*

(1) $\alpha_k(\beta) = \zeta$.
(2) *There are* $a, b, c, d, e, f, g, h, i, l, m, p, q, r, s, t, u, v, w, x, y, z \geq 0$ *and* $n \geq 1$ *such that conditions* (i) – (xviii) *are fulfilled, where* (i) – (xi) *are taken from Lemma* 8 *with* $A = 16$ *and* $B = 4g^2$, *and* (xii) – (xviii) *are identical with those in Lemma* 10B.

P r o o f. Theorems 11A and 11B follow from Lemmata 8, 10A and 10B. □

COROLLARY 12A. *Let* $\beta \in \mathbb{N} \setminus \{1\}$, $\mathrm{e} = 2 + \sum_{\kappa=1}^{\infty} \alpha_\kappa(\beta)\beta^{-\kappa}$, *where* $0 \leq \alpha_\kappa(\beta) < \beta$ *for* $\kappa \geq 1$ *and* $0 \leq \zeta < \beta$. *Then* $\{\kappa \in \mathbb{N} \mid \alpha_\kappa(\beta) = \zeta\} = P(\mathbb{N}_0^{26}) \cap \mathbb{N}$, *where*

$$
\begin{aligned}
&P(a, \ldots, z) = \\
&= (k+1)\Big(1 - \big((a^2-1)y^2 + 1 - x^2\big)^2 - \big(16(a^2-1)r^2y^4 + 1 - u^2\big)^2 \\
&\qquad - \Big(\big((a + u^2(u^2-a)\big)^2 - 1)(n+1+4dy)^2 + 1 - (x+cu)^2\Big)^2 \\
&\qquad - \big((a^2-1)l^2 + 1 - m^2\big)^2 - \big(k+1+i(a-1)-l\big)^2 \\
&\qquad - (n+1+l+j-y)^2 \\
&\qquad - (3n+k+b+g+h+\beta+7-e)^2 - \big(e^3(e+2)(a+1)^2 + 1 - o^2\big)^2 \\
&\qquad - \big(g + y(a-n-2) + p\big(2a(n+2) - (n+2)^2 - 1\big) - x\big)^2 \\
&\qquad - \big(h + y(a-n-1) + q\big(2a(n+1) - (n+1)^2 - 1\big) - x\big)^2 \\
&\qquad - \big(b + l(a-\beta) + v(2a\beta - \beta^2 - 1) - m\big)^2 \\
&\qquad - \big((\zeta + t\beta)h + s - bg\big)^2 - \big(s+f+1-h\big)^2 \\
&\qquad - \big((\zeta + t\beta)(n+1)h + z - b(n+2)g\big)^2 - \big(z + w + 1 - (n+1)h\big)^2\Big).
\end{aligned}
$$

P r o o f. This follows from Theorem 11A by the usual construction. Note that k and n have been replaced by $k+1$ and $n+1$ to allow k and n to range over the nonnegative integers. □

Remarks.

(1) In a similar way Theorem 11B implies the existence of a like polynomial for π. Counting the additions and multiplications occurring in it one finds that the relation $\alpha_k(\beta) = \zeta$ can be proved by less than 200 additions and multiplications regardless of the values of β, ζ and k. However, a universal bound of 100 operations has been established by J. P. J o n e s [7].

(2) If α and β are positive irrationals with rational nests of intervals $([p_n(\alpha), q_n(\alpha)])_{n\geq 1}$ and $([p_n(\beta), q_n(\beta)])_{n\geq 1}$ (and $p_1(\alpha), p_1(\beta) > 0$), then $([p_n(\alpha) + p_n(\beta), q_n(\alpha) + q_n(\beta)])_{n\geq 1}$, $([p_n(\alpha)p_n(\beta), q_n(\alpha)q_n(\beta)])_{n\geq 1}$ and $([q_n(\alpha)^{-1}, p_n(\alpha)^{-1}])_{n\geq 1}$ are rational nests of intervals for $\alpha+\beta$, $\alpha\beta$ and α^{-1} respectively. This means that the results above could be used to construct Diophantine representations, e.g. for the digits of $e+\pi$ or $e\cdot\pi$. Furthermore, if $\sigma \in \mathbb{N}$, then $\left(\left[\left(1+(\sigma n)^{-1}\right)^n, \left(1+(\sigma n)^{-1}\right)^{n+1}\right]\right)_{n\geq 1}$ is a rational nest of intervals for $\sqrt[\sigma]{e}$.

(3) There are only countably many irrationals for which Lemma 2 can be used to construct such Diophantine representations as there are only countably many Diophantine representations.

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Department of Mathematics
University of Vienna
Strudlhofgasse 4
A–1090 Wien
AUSTRIA
E-mail: baxa@ap.univie.ac.at