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THE CSÁKÁNY THEORY OF REGULARITY FOR FINITE ALGEBRAS

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ABSTRACT. If an algebra \( A \) has at most five elements, then \( A \) is congruence regular if and only if there exists a ternary functions compatible with \( \text{Con} A \) such that \( p(x, y, z) = z \) if and only if \( x = y \). If \( A \) has six elements, the assertion does not hold.

A. P i x l e y [5] posed the following problem: If some congruence property is characterized by a Mal'cev condition in varieties of algebras, can this Mal'cev condition (modified in a natural way) be used also for characterizing this congruence property in the case of a single algebra? For arithmeticity, he solved himself this problem affirmatively in [5]. Since every congruence identity can be characterized in varieties by a Mal'cev condition (see [6]), H.-P. G u m m asked for which other congruence identity there exists a Mal'cev theory in the case of a single algebra. The answer is “for none” in a general case, see [4]. However, for small algebras, permutability of congruences can be characterized by a Mal'cev theory, see e.g. [1] for at most four-element algebras, and [2] for at most eight-element algebras (the answer is negative for at least 25-element algebra). This motivated our effort to proceed similar investigations for congruence regularity (which is not a congruence identity). Although some Mal'cev-type characterizations of regular varieties are known, see e.g. [7], we prefer another but more simple term condition given by B. C s á k á n y in [3]. At first we recall:

DEFINITION. An algebra \( A \) is regular if \( \theta = \phi \) for \( \theta, \phi \in \text{Con} A \) whenever they have a congruence class in common. A variety \( \mathcal{V} \) is regular if each \( A \in \mathcal{V} \) has this property.

CSÁKÁNY'S THEOREM. ([3]) For a variety \( \mathcal{V} \), the following conditions are equivalent:

(1) \( \mathcal{V} \) is regular;

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(2) there exist ternary terms \( p_1(x, y, z), \ldots, p_n(x, y, z) \) such that
\[
[p_1(x, y, z) = z \land \cdots \land p_n(x, y, z) = z] \iff x - y.
\]

We are going to investigate if such Csákány-type conditions can characterize regularity of a single algebra.

Let \( A \) be an algebra, \( \theta \in \text{Con} A \) and \( f: A^n \to A \) be an \( n \)-ary function. We say that \( f \) is compatible with \( \theta \) if \( \langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \rangle \in \theta \) whenever \( \langle a_i, b_i \rangle \in \theta \) for \( i = 1, \ldots, n \).

Denote by \( \omega \) the least and by \( \iota \) the greatest congruence of \( A \).

**Lemma 1.** Let \( A \) be an at least two-element algebra, and \( \theta \in \text{Con} A, \theta \neq \omega \). If \( \theta \) has a one-element congruence class, then there does not exist a ternary function \( p: A^3 \to A \) compatible with \( \theta \) such that
\[
p(x, y, z) = z \iff x - y.
\]

**Proof.** Suppose \( [c]_\theta = \{c\} \) for some \( c \in A \). Let \( p(x, y, z) \) be a ternary function compatible with \( \theta \) such that
\[
p(x, y, z) = z \iff x - y.
\]
Since \( \theta \neq \omega \), there exists a congruence class \( B \) of \( \theta \) containing at least two different elements, say \( a \) and \( b \). Since \( p(a, a, c) = c \), we have
\[
\langle p(a, b, c), c \rangle = \langle p(a, b, c), p(a, a, c) \rangle \in \theta,
\]
\[\text{hence} p(a, b, c) = c, \text{ which is a contradiction.}\]

**I 0 R 3 M.** Let \( A \) be a finite algebra with \( \text{card} A \leq 5 \). The following conditions are equivalent:

1. \( A \) is regular,
2. there exists a ternary function \( p: A^3 \to A \) compatible with every congruence of \( A \) such that
\[
p(x, y, z) = z \iff x = y.
\]

**Proof.** For \( A \) with \( \text{card} A \leq 2 \), the assertion is trivial.

(a) Suppose \( \text{card} A = 3 \), i.e. \( A = \{a, b, c\} \). If \( A \) is regular, then evidently \( \text{Con} A = \{\omega, \iota\} \). Define \( p: A^3 \to A \) by the rules
\[
p(x, y, z) = \begin{cases} 
z & \text{if } x = y, \\
x & \text{if } x \neq y \text{ and } x \neq z, \\
y & \text{if } x \neq y \text{ otherwise}.
\end{cases}
\]
Trivially, \( p \) is compatible with every congruence of \( \text{Con} \ A \) and satisfies (2).

Conversely, let \( A \) fail to be regular. Without loss of generality, suppose the existence of \( \theta \in \text{Con} \ A \) such that \( \theta \) has two classes, namely \( \{c\} \) and \( \{a, b\} \). By Lemma 1, we obtain a contradiction with (2).

(b) Let \( \text{card} \ A = 4, \ A = \{a, b, c, d\} \). If \( A \) is regular, the desired compatible function can be defined by the rule

\[
p(x, y, z) \begin{cases} 
  = z & \text{for } x = y, \\
  \in [z]_{\theta(x, y)} - \{z\} & \text{otherwise}, 
\end{cases}
\]

since \( \text{Con} \ A \subseteq \{\omega, \iota, \theta_1, \theta_2, \theta_3\} \), where

\[
\begin{align*}
\theta_1 & \text{ has classes } \{a, b\}, \{c, d\}, \\
\theta_2 & \text{ has classes } \{a, c\}, \{b, d\}, \\
\theta_3 & \text{ has classes } \{a, d\}, \{b, c\}.
\end{align*}
\]

It is easy to show that \( p \) is compatible with every congruence of \( \text{Con} \ A \).

If \( A \) fails to be regular, then there exists \( \theta \in \text{Con} \ A \) such that \( \theta \neq \omega \), and \( \theta \) has a one-element class. By Lemma 1, we obtain a contradiction.

(c) Let \( \text{card} \ A = 5, \ A = \{a, b, c, d, e\} \). If \( A \) is regular, then the lattice \( \text{Con} \ A \) cannot include any congruence \( \theta, \theta \neq \omega \), having a one-element class, i.e. \( \text{Con} \ A \subseteq \{\omega, \iota, \theta_i\} \), where every \( \theta_i \) has one two-element and one three-element class. There exist 10 of such \( \theta_i \) on the underlying set of \( A \), however, since \( A \) is regular, \( \text{Con} \ A \) contains at most one of them (because for \( i \neq j, \theta_i \cap \theta_j \neq \omega \), and \( \theta_i \cap \theta_j \) contains a one-element class). Suppose that \( \theta_1 \) has classes \( C = \{a, b, c\} \) and \( D = \{d, e\} \). Define \( p_1: A^3 \to A \) by the rules

\[
\begin{align*}
p_1(x, x, z) &= z, \\
p_1(x, y, d) &= e, \\
p_1(x, y, e) &= d \\
p_1(x_1, x_2, x_3) &= p_1(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) & \text{for } x_1, x_2 \in C, \ x_3 \in D, \\
p_1(x, y, a) &= b, \\
p_1(x, y, b) &= c, \\
p_1(x, y, c) &= a \\
\end{align*}
\]

For \( x, y, z \in C \) we put

\[
\begin{align*}
p_1(x, y, z) &= x & \text{for } z = y, \\
p_1(x, y, z) &= v & \text{for } z \neq y, \\
\end{align*}
\]

where \( v \in C, \ z \neq v \neq y \).
Since $D$ has only two elements, the case $x, y, z \in D$ yields $p(x, y, z) = p(x, x, z)$, which was solved before.

It is a routine calculation to verify that $p_1$ is compatible with $\theta_1$ (and, trivially, also with $\omega$, $\iota$). Permuting the elements $a, b, c, d, e$, we obtain the functions $p_i$ for each $\theta_i$ ($i = 1, \ldots, 10$).

If $A$ is not regular, then again Con $A$ has to contain a congruence $\theta$, $\theta \neq \omega$, with a one-element class; thus we obtain a contradiction by Lemma 1. \hfill $\Box$

For algebras with more than 5 elements, the conditions (1), (2) of our Theorem need not be equivalent. The essential part of this statement is contained in the following:

**Lemma 2.** There exists a six-element non-regular algebra with a ternary function $p: A^3 \to A$ satisfying (2) of Theorem.

**Proof.** Let $A = \{a, b, c, d, e, f\}$ and $p$ be a ternary operation on $A$ as follows:

$$p(x, x, z) = z \quad \text{for each} \quad x, z \in A,$$

and for each $x, y \in A$, $x \neq y$, we put

$$p(x, y, a) = b, \quad p(x, y, c) = d, \quad p(x, y, e) = f,$$
$$p(x, y, b) = a, \quad p(x, y, d) = c, \quad p(x, y, f) = e.$$

Let $\theta, \phi$ be equivalences on $A$ determined by their partitions:

$$\theta \quad \text{has classes} \quad \{a, b\}, \{c, d\}, \{e, f\},$$
$$\phi \quad \text{has classes} \quad \{a, b\}, \{c, d, e, f\}.$$

Then $\theta, \phi$ are congruences on the algebra $(A, p)$, and $p(x, y, z)$ satisfies (2) of Theorem (trivially, $p$ is compatible with every congruence on $(A, p)$ because it is the operation of this algebra). Moreover, $(A, p)$ is not regular because two different congruences $\theta, \omega$ have a common class $\{a, b\}$. \hfill $\Box$

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