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P-COMPATIBLE IDENTITIES AND THEIR APPLICATIONS TO CLASSICAL ALGEBRAS

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Abstract. Let τ: $F \to \mathbb{N}$ be a type of algebras, i.e. F is a set of fundamental operation symbols and $\mathbb{N}$ is the set of non-negative integers. Let P be a partition of F. We say that an identity $\varphi = \psi$ of type τ is P-compatible (see [7]) if it is of the form $x = x$ or of the form $f(\varphi_0, \ldots, \varphi_{\tau(f)-1}) = g(\psi_0, \ldots, \psi_{\tau(g)-1})$, where f and g belong to the same block of P.

For a variety K of type τ we denote by K_P the variety of the same type defined by all P-compatible identities of type τ satisfied in K.

In this paper we define a construction, called the P-dispersion of an algebra and we prove a general theorem which allows to represent algebras from K_P by means of P-dispersions of algebras from K when K is a variety of groups, rings, lattices, Boolean Algebras, linear spaces, etc. The results of this paper were announced in [8].

0. We shall consider algebras of a given type τ (see [3]). For a variety K we denote by Id(K) the set of all identities of type τ satisfied in all algebras from K. If E is a set of identities of type τ, we denote by $V(E)$ the variety defined by E. In [7] the notion of P-compatible identity was defined, namely:

Let P be a partition of the set F. The block of P containing $f \in F$ will be denoted by $[f]_P$. An identity $\varphi = \psi$ of type τ is called P-compatible if it is of the form

$$x = x \tag{0.1}$$

or of the form

$$f(\varphi_0, \ldots, \varphi_{\tau(f)-1}) = g(\psi_0, \ldots, \psi_{\tau((g)-1}), \tag{0.2}$$

where $g \in [f]_P$, $\varphi_0, \ldots, \varphi_{\tau(f)-1}$, $\psi_0, \ldots, \psi_{\tau(g)-1}$ are terms of type τ.
So $\varphi = \psi$ is P-compatible if the most external fundamental operation symbols in φ and ψ belong to the same block.

This notion is a generalization of some others, namely: An identity $\varphi = \psi$ is called externally compatible if it is of the form (0.1) or of the form (0.2), where the symbols f and g are identical (see [2]).

If we denote by P_0 the partition of F consisting of singletons only, then obviously $\varphi = \psi$ is externally compatible iff it is P_0-compatible.

An identity $\varphi = \psi$ is called non-trivializing if it is of the form (0.1) or neither φ nor ψ is a single variable (see [6]). So $\varphi = \psi$ is non-trivializing iff it is $\{F\}$-compatible.

For terms φ and ψ we shall write $\varphi \equiv \psi$ if φ is identical with ψ (they have the same structure). If φ is a term different from a variable, then the most external operation symbol of φ will be denoted by $\mathrm{ex}(\varphi)$. For example $\mathrm{ex}((x.y) + z) = +$.

For a variety K of type τ we shall denote by $P(K)$ the set of all P-compatible identities from $\mathrm{Id}(K)$. We denote $K_P = V(P(K))$. We shall also write $\mathrm{Ex}(K)$ instead of $P_0(K)$ and K_{Ex} instead of K_{P_0}.

In [7] some properties of P-compatible identities were considered, in particular:

(i) *If E is a set of P-compatible identities of type τ, then every identity provable from E by means of Birkhoff's deriviation rules is P-compatible.*

It means that every set $P(K)$ is an equational theory (see [1]).

Saying that $\varphi(x)$ is a non-trivial unary term we mean that $\{x\}$ is the set of all variables occurring in $\varphi(x)$ and $\varphi(x) \not\equiv x$.

1. The P-dispersion of an algebra by a P-dispersing system.

In [5] a construction $\underset{\mathfrak{A}_0}{\mathscr{S}} \mathfrak{A}_i$ was defined. Here we give a generalization of this notion.

If $\mathfrak{A} = (A; F^{\mathfrak{A}})$ is an algebra and $f^{\mathfrak{A}} \in F^{\mathfrak{A}}$, then we denote by $f^{\mathfrak{A}}(A)$ the set of all $a \in A$ such that

$$a = f^{\mathfrak{A}}(a_0, \ldots, a_{\tau(f)-1}) \quad \text{for some} \quad a_0, \ldots, a_{\tau(f)-1} \in A.$$

Let $D = (P, \mathfrak{I}, \{A_i\}_{i \in I}, \{o_{[f]_P}\}_{f \in F})$ be a quadruple satisfying the following conditions (1°) – (4°):

(1°) P is a partition of F.

(2°) $\mathfrak{I}$ is an algebra of type τ and $\mathfrak{I} = (I; F^{\mathfrak{I}})$.

(3°) $\{A_i\}_{i \in I}$ is a family of non-empty pairwise disjoint sets.

(4°) $\{o_{[f]_P}\}_{f \in F}$ is a family of mappings $o_{[f]_P}\colon I \to \bigcup_{i \in I} A_i$ such that for every $i \in I$ we have $o_{[f]_P}(i) \in A_i$ and $o_{[f]_P} = o_{[g]_P}$, if $g \in [f]_P$.

The quadruple D will be called a P-dispersing system.

We define a new algebra $\mathfrak{I}_D$ of type τ putting $\mathfrak{I}_D = (A; F^{\mathfrak{I}_D})$, where $A = \bigcup_{i \in I} A_i$

and for each $f \in F$, $a_k \in A_{i_k}$ $(k = 0, \ldots, \tau(f) - 1)$ we define

$$f^{\mathfrak{J}_D}(a_0, \ldots, a_{\tau(f)-1}) = o_{[f]_P}(f^{\mathfrak{J}}(i_0, \ldots, i_{\tau(f)-1})).$$

The algebra $\mathfrak{J}_D$ will be called the P-dispersion of $\mathfrak{J}$ by the P-dispersing system D or briefly the P-dispersion of $\mathfrak{J}$. If $P = P_0$, then we shall say "the dispersion" instead of "the P_0-dispersion".

If $\mathfrak{J}$ is an idempotent algebra and $P = P_0$, then we obtain the construction from [5] as a particular case.

(ii) *The equivalence relation* $\sim$ *induced on* A *by the partition* $\{A_i\}_{i \in I}$ *is a congruence on* $\mathfrak{J}_D$ *and* $\mathfrak{J}_{D/\sim}$ *is isomorphic to* $\mathfrak{J}$.

(iii) *If* $\mathfrak{J} = (J, F^{\mathfrak{J}})$ *is an algebra isomorphic to* $\mathfrak{J}$ *and* $\varphi\colon J \to I$ *is the isomorphism, then* $\mathfrak{J}_D$ *is a* P*-dispersion of* $\mathfrak{J}$.

In fact, $\mathfrak{J}_D = \mathfrak{J}_{D'}$, where $D' = (P, \mathfrak{J}, \{A_{\varphi(j)}\}_{j \in J}, \{o_{[f]_P} \circ \varphi\}_{f \in F})$.

From (ii) and (iii) we get

(iv) *The algebra* $\mathfrak{J}_D$ *is a* P*-dispersion of the algebra* $\mathfrak{J}_{D/\sim}$.

If K is a class of algebras of type τ, we shall denote by K_{Pd} the class of all P-dispersions of algebras from K.

(v) *For each class* K *of algebras of type* τ *we have* $K \subseteq K_{Pd}$. In fact, each algebra $\mathfrak{A} = (A, F^{\mathfrak{A}})$ is the P-dispersion by a system $(P, \mathfrak{A}, \{\{a\}\}_{a \in A}, \{o_{[f]_P}\}_{f \in F})$, where each $o_{[f]_P}$ is the identity map.

(vi) *For each class* K *of algebras of type* τ *the class* K_{Pd} *is closed under isomorphic images.*

In fact, if $\mathfrak{B} = (B; F^{\mathfrak{B}})$ is an isomorphic image of $\mathfrak{J}_D$ and φ is the corresponding isomorphism, then $\mathfrak{B} = \mathfrak{J}_{D'}$, where

$$D' = (P, \mathfrak{J}, \{\varphi(A_i)\}_{i \in I}, \{\varphi \circ o_{[f]_P}\}_{f \in F}).$$

(vii) *If* $\varphi(x_0, \ldots, x_{n-1})$ *is an* n*-ary term of type* τ *different from a variable,* $a_k \in A_{i_k}$ $(k = 0, \ldots, n - 1)$, *then*

$$\varphi^{\mathfrak{J}_D}(a_0, \ldots, a_{n-1}) = o_{[\mathrm{ex}(\varphi)]_P}(\varphi^{\mathfrak{J}}(i_0, \ldots, i_{n-1})).$$

In fact, the statement is true for fundamental operation symbols. Further, we use induction on the complexity of φ.

(viii) *The algebra* $\mathfrak{J}_D$ *satisfies all* P*-compatible identities satisfied in* $\mathfrak{J}$.

In fact, let

$$\varphi = \psi \tag{1.1}$$

be a P-compatible identity satisfied in $\mathfrak{J}$, where φ and ψ are n-ary terms. If (1.1) is of the form (0.1), then it is satisfied in $\mathfrak{J}_D$. Let (1.1) be of the form (0.2) and let $a_k \in A_{i_k}$ $(k = 0, \ldots, n - 1)$. Since (1.1) is satisfied in $\mathfrak{J}$ and $[\mathrm{ex}(\varphi)]_P = [\mathrm{ex}(\psi)]_P$, we have by (vii):

$$\varphi^{\mathfrak{J}_D}(a_0, \ldots, a_{n-1}) = o_{[\mathrm{ex}(\varphi)]_P}(\varphi^{\mathfrak{J}}(i_0, \ldots, i_{n-1})) =$$

$$= o_{[\mathrm{ex}(\psi)]_P}(\psi^{\mathfrak{J}}(i_0, \ldots, i_{n-1})) = \psi^{\mathfrak{J}_D}(a_0, \ldots, a_{n-1}).$$

Let us denote by V^P the variety of algebras of type τ defined by all identities:

$$f(x_0, \ldots, x_{\tau(f)-1}) = g(y_0, \ldots, y_{\tau(g)-1}), \qquad (1.2)$$

$$f, g \in F \quad \text{and} \quad g \in [f]_P.$$

Let $\mathfrak{A} = (A; F^{\mathfrak{A}})$ be an arbitrary algebra of type τ and $\mathfrak{B} = (B; F^{\mathfrak{B}}) \in V^P$.

(ix) *Every subdirect product of algebras* $\mathfrak{A}$ *and* $\mathfrak{B}$ *is a P-dispersion of* $\mathfrak{A}$.

In fact, let $\mathfrak{S} = (S; F^{\mathfrak{S}})$ be a subdirect product of $\mathfrak{A}$ and $\mathfrak{B}$. For each $a \in A$ we define $S_a = \{\langle a, x\rangle : \langle a, x\rangle \in S\}$, $\Pi = \{S_a\}_{a \in A}$. For $a \in A$ we put

$$o_{[f]_P}(a) = \begin{cases} \langle a, f^{\mathfrak{B}}(b, \ldots, b)\rangle \text{ for some } b \in B, \text{ if } a \in f^{\mathfrak{A}}(A) \\ \langle a, c\rangle \text{ for some } \langle a, c\rangle \in S_a, \text{ otherwise.} \end{cases}$$

Then $\mathfrak{S} = \mathfrak{A}_D$, where $D = (P, \mathfrak{A}, \Pi, \{o_{[f]_P}\}_{f \in F})$.

However, the algebra $\mathfrak{J}_D$ is not in general isomorphic to a subdirect product of $\mathfrak{J}$ and some $\mathfrak{B} \in V^P$ (see Example 12).

Theorem 1. *A veriety K is defined only by P-compatible identities iff it is closed under P-dispersions of algebras from K.*

Proof. ($\Rightarrow$) Follows from (viii).

($\Leftarrow$) Consider an algebra $\mathfrak{B}_P = (B_P; F^{\mathfrak{B}_P})$,

where $B_P = \{k_1, k_2\} \cup \{w_{[f]_P}\}_{f \in F}$, $\{k_1, k_2\} \cap \{w_{[f]_P}\}_{f \in F} = \emptyset$, $w_{[f]_P} \neq w_{[g]_P}$ for $[f]_P \neq \neq [g]_P$ and for each $x_0, \ldots, x_{\tau(f)-1} \in B_P$ we have $f(x_0, \ldots, x_{\tau(f)-1}) = w_{[f]_P}$. This algebra is a P-dispersion of a 1-element algebra from K. It was shown in [7] that $\mathfrak{B}_P$ satisfies all P-compatible identities of type τ and only them. Thus $\mathfrak{B}_P \in K$. But each identity from $\mathrm{Id}(K)$ must be satisfied in $\mathfrak{B}_P$, so K satisfies only some P-compatible identities and no others.

Remark 1. Since the identity $x = y$ is not P-compatible we need k_1 and k_2 in B_P to avoid degenerate algebras when $|F| \leq 1$.

2. A Representation Theorem of Algebras from K_P.

A block $[f]_P$ of a partition P of F will be called nullary if $\tau(g) = 0$ for each $g \in [f]_P$; a block $[f]_P$ will be called non-nullary if it is not nullary.

Let P be a partition of F and let K be a variety of type τ satisfying the following three conditions:

(5°) There exists a non-trivial unary term $q(x)$ such that for each $f \in F$ the identity

$$q(f(x_0, \dots, x_{\tau(f)-1})) = q(f(q(x_0), \dots, q(x_{\tau(f)-1}))) \tag{2.1}$$

belongs to $\mathrm{Id}(K)$.

(6°) If $[f]_P$ is a non-nullary block and $g, h \in [f]_P$, then there exists a non-trivial unary term $q_{g,h}(x)$ such that $\mathrm{ex}(q_{g,h}(x)) \in [f]_P$ and the identities

$$\begin{aligned} g(x_0, \dots, x_{\tau(g)-1}) &= q_{g,h}(q(g(x_0, \dots, x_{\tau(g)-1}))), \\ h(x_0, \dots, x_{\tau(h)-1}) &= q_{g,h}(q(h(x_0, \dots, x_{\tau(h)-1}))) \end{aligned} \tag{2.2}$$

belong to $\mathrm{Id}(K)$.

(7°) If $[f]_P$ is a nullary block of P, then for each $g \in [f]_P$ the identity

$$f = g \tag{2.3}$$

belongs to $\mathrm{Id}(K)$.

Let us fix $q(x)$ under conditions (5°) and (6°) and let us fix $q_{g,h}(x)$ under condition (6°) for every g, h.

Let B be an equational base of K. We define a set B^* of identities of type τ by the following three conditions:

(b_1) The identities (2.1), (2.2) and (2.3) belong to B^*.

(b_2) If $\varphi = \psi$ belongs to B, then the identity

$$q(\varphi) = q(\psi) \tag{2.4}$$

belongs to B^*.

(b_3) B^* contains only identities described in (b_1) and (b_2).

Let $\mathfrak{A} = (A; F^{\mathfrak{A}})$ be an algebra of type τ.

Theorem 2. *If P is a partition of F and K is a variety of type τ satisfying conditions* (5°), (6°) *and* (7°), *then $\mathfrak{A}$ belongs to K_P iff $\mathfrak{A}$ is a P-dispersion of an algebra from K by a P-dispersing system D. Moreover, if B is an equational base of K, then B^* is an equational base of K_P.*

Proof. By (viii) we have $K_{Pd} \subseteq K_P$. Further, $B^* \subset P(K)$ since (2.1), (2.2), (2.3) are P-compatible and belong to $\mathrm{Id}(K)$. So $K_P \subseteq V(B^*)$. To complete the proof it is enough to show that any algebra $\mathfrak{A} = (A; F^{\mathfrak{A}})$ from $V(B^*)$ is a P-dispersion of an algebra from K. We define in $\mathfrak{A}$ a relation $\sim$ putting for $a, b \in A$:

$$a \sim b \Leftrightarrow q(a) = q(b).$$

By (b_1) and (2.1), $\sim$ is a congruence on $\mathfrak{A}$. By (b_2) the algebra $\mathfrak{A}|\sim$ belongs to K.

We shall show that $\mathfrak{A}$ is a P-dispersion of $\mathfrak{A}|\sim$.

Let $[a]_\sim = g^{\mathfrak{A}|\sim}([a_0]_\sim, \dots, [a_{\tau(g)-1}]_\sim)$ for some $g \in [f]_P$ and $a_0, \dots, a_{\tau(g)-1} \in A$. Put

$$o_{[f]_P}([a])_\sim) = g^{\mathfrak{A}}(a_0, \dots, a_{\tau(g)-1}).$$

If $[f]_P$ is nullary, then $o_{[f]_P}$ is well defined by (2.3).

Assume that $[f]_P$ is non-nullary and for some $h \in [f]_P$ and $b_0, \ldots, b_{\tau(h)-1} \in A$ we have $h(b_0, \ldots, b_{\tau(h)-1}) \in [a]_\sim$. Then by (2.2) we get

$$g^{\mathfrak{A}}(a_0, \ldots, a_{\tau(g)-1}) = q_{g,h}(q(g^{\mathfrak{A}}(a_0, \ldots, a_{\tau(g)-1}))) =$$
$$= q_{g,h}(q(h^{\mathfrak{A}}(b_0, \ldots, b_{\tau(h)-1}))) = h^{\mathfrak{A}}(b_0, \ldots, b_{\tau(h)-1}).$$

So $o_{[f]_P}$ is well defined again, i.e. it does not depend on the choice of g and on the choice of arguments.

If $[a]_\sim$ is the value of no $g^{\mathfrak{A}|\sim}$ for $g \in [f]_P$, then put $o_{[f]_P}([a]_\sim) = b$ for fixed $b \in [a]_\sim$.

Consequently $\mathfrak{A} = (\mathfrak{A}|\sim)_D$, where $D = (P, \mathfrak{A}|\sim, \{[a]_\sim\}_{a \in A}, \{o_{[f]_P}\}_{f \in F})$.

Corollary 1. *If P is a partition of F and K is a variety of type τ satisfying* (5°), (6°) *and* (7°), *K is finitely based and F is finite, then K_P is finitely based.*

Corollary 2. *Let P be a partition of F and K satisfy* (5°), (7°) *and*

(8°) *For every non-nullary block $[f]_P$ there exists a non-trivial unary term $q_{[f]_P}(x)$ such that* $\mathrm{ex}(q_{[f]_P}(x)) \in [f]_P$ *and for each $g \in [f]_P$ the identity*

$$g(x_0, \ldots, x_{\tau(g)-1}) = q_{[f]_P}(q(g(x_0, \ldots, x_{\tau(g)-1})))$$

belongs to $\mathrm{Id}(K)$.

Then $K_P = K_{Pd}$. Moreover, if *K is finitely based and F is finite, then K_P is finitely based.*

In fact, the condition (8°) implies (6°).

Remark 2. If there exists a non-trivial unary term $r(x)$ of type τ such that the identity $r(x) = x$ belongs to $\mathrm{Id}(K)$, then putting $q(x) \equiv r(x)$ we get (5°).

Corollary 3. *If $\tau(F) \setminus \{0\} \neq \emptyset$, K satisfies* (7°) *and for each non-nullary block $[f]_P$, K satisfies*

(9°) *There exists a non-trivial unary term $q_h(x)$ with* $\mathrm{ex}(q_h(x)) = h \in [f]_P$ *and the identity $q_h(x) = x$ belongs to* $\mathrm{Id}(K)$,

then $K_P = K_{Pd}$. Moreover, if F is finite and K is finitely based, then K_P is finitely based.

In fact by assumption there exists a non-nullary block $[f]_P$ of F. Let us fix h in (9°) and put $q(x) \equiv q_h(x)$. Then corollary 3 follows from remark 2 and corollary 2.

Corollary 4. *Let K be a variety of type τ satisfying* (5°) *and*

(10°) *For each $f \in F$ such that $\tau(f) > 0$ there exists a non-trivial unary term $q_f(x)$ such that* $\mathrm{ex}(q_f(x)) = f$ *and the identity*

$$f(x_0, \ldots, x_{\tau(f)-1}) = q_f(q(f(x_0, \ldots, x_{\tau(f)-1})))$$

belongs to $\mathrm{Id}(K)$.

Then $K_{\mathrm{Ex}} = K_{P_0 d}$. *Moreover, if F is finite and K is finitely based, then* K_{Ex} *is finitely based.*

In fact, this follows from Corollary 2 since (7°) for P_0 is always satisfied.

Example 1. Let K be a variety of groups with fundamental operation symbols $\cdot$, $^{-1}$, 1. Then for each partition P of the set $F = \{\cdot, {}^{-1}, 1\}$ we have $K_P = K_{Pd}$ and K_P is finitely based if K is.

In fact, put $q_{\cdot}(x) \equiv x \cdot (x \cdot x^{-1})$, $q_{-1}(x) \equiv (x^{-1})^{-1} \equiv q(x)$ and use Corollary 3.

Example 2. The statements of Example 1 hold if we consider groups with fundamental operation symbols $\cdot$, $^{-1}$, i.e. $F = \{\cdot, {}^{-1}\}$.

Example 3. Let K be a variety of rings with fundamental operations $+$, $-$, $\cdot$, where $+$ and $\cdot$ are binary, $-$ is unary and K satisfies an identity $x^n = x$ for some $n > 1$; then for each partition P of $F = \{+, -, \cdot\}$ we have $K_P = K_{Pd}$ and K_P is finitely based if K is.

In fact, put $q(x) \equiv q_+(x) \equiv x + (x + (-x))$, $q_-(x) \equiv -(-x)$, $q_{\cdot}(x) \equiv x^n$ and use Corollary 3.

Example 4. Let K be a variety of type τ such that for each $f \in F$ we have $\tau(f) > 0$ and the identity $f(x, \ldots, x) = x$ belongs to $\mathrm{Id}(K)$. Then for each partition P of F we get $K_P = K_{Pd}$ and K_P is finitely based if K is finitely based and F is finite.

This follows from Corollary 3.

Example 5. Let K be a variety of lattices with fundamental operations $\vee$ and $\wedge$. Then for each partition P of $\{\vee, \wedge\}$ we have $K_P = K_{Pd}$ and K_P is finitely based if K is.

This follows from Corollary 3.

Example 6. Let K be the variety of Boolean Algebras with fundamental operations $+$, $\cdot$, $'$, 0, 1. Then for each partition P of the set $\{+, \cdot, ', 0, 1\}$ such that $[0]_P \neq \{0, 1\}$ we have $K_P = K_{Pd}$ and K_P is finitely based.

In fact, put $q_+(x) \equiv x + x$, $q_{\cdot}(x) \equiv x \cdot x$, $q_{'}(x) \equiv (x')' \equiv q(x)$ and use Corollary 3.

Example 7. It is known that quasi-groups are algebras with three binary operations $\backslash$, $\cdot$, $/$ satisfying the identities $x \backslash (x \cdot y) = y$, $(x \cdot y)/y = x$, $x \cdot (x \backslash y) = y$, $(x/y) \cdot y = x$ (see [1]). If K is a variety of quasi-groups, then for each partition P of $\{\backslash, \cdot, /\}$ we have $K_{Pd} = K_P$ and K_P is finitely based if K is.

Example 8. Let K be a variety of pseudocomplemented distributive lattices (see [1]) with fundamental operation symbols $\vee$, $\wedge$, $'$. Then for each partition P of $\{\vee, \wedge, '\}$ we have $K_P = K_{Pd}$ and K_P is finitely based.

In fact, if $[']_P \neq \{'\}$, then put $q_\vee(x) \equiv x \vee x$, $q_\wedge(x) \equiv x \wedge x$ and use Corollary 3. If $[']_P = \{'\}$, then put $q_{[']_P}(x) \equiv (x')'$ and use Corollary 2.

Example 9. Let K be a variety of rings with $F = \{+, -, \cdot, 0, 1\}$. Let P be a partition of F such that $[0]_P \neq \{0, 1\}$. Then $K_P = K_{Pd}$ and K_P is finitely based if K is.

In fact, define $q_+(x)$ and $q_-(x)$ as in Example 3 and $q_\cdot(x) \equiv x \cdot 1$. Then use Corollary 3.

Example 10. Let K be a variety of linear spaces over a field M. So $F = \{+, -, \mathbf{0}, \{f_c\}_{c \in M}\}$, where $f_c(x) = c \cdot x$. Then $K_P = K_{Pd}$ for each partition P of F.

In fact, put $q(x) \equiv f_1(x)$, $q_+(x) \equiv x + (x + (-x))$, $q_{f_c}(x) = c \cdot \left(\frac{1}{c} \cdot x\right)$ for $c \in M \setminus \{0\}$, $q_{f_0}(x) = 0 \cdot x$.

Now the statement holds from Corollary 3 for all partitions P such that $\{\mathbf{0}, 0 \cdot x\} \neq [0 \cdot x]_P \neq \{0 \cdot x\}$. If $\{\mathbf{0}, 0 \cdot x\} = [0 \cdot x]_P$ or $[0 \cdot x]_P = \{0 \cdot x\}$, then put $q_{f_0}(x) = 0 \cdot x$ and use Corollary 2 together with Remark 2.

Example 11. Let K be a variety of algebras with two unary fundamental operation symbols f and g defined by the identities

$$f(x) = f(f(x)) = g(x).$$

Then $K_{\mathrm{Ex}} = K_{P_0 d}$. In fact, K_{Ex} is defined by the identities: $f(f(x)) = f(g(x)) = f(x), g(g(x)) = g(f(x)) = g(x)$. We put $q(x) \equiv f(x)$, $q_f(x) \equiv f(x)$, $q_g(x) \equiv g(x)$ and we use Corollary 4.

Remark 3. The last example shows that for the term $q(x)$ the identity $q(x) = x$ need not belong to $\mathrm{Id}(K)$.

Remark 4. The classes K_{P_0} were considered in [2] for classes of algebras in which all operations were idempotent and for Boolean algebras. In [4] the class K_{P_0} was considered if K was the class of pseudocomplemented distributive lattices. In [2] and [4] the representation was given by means of the congruence $\sim$ considered in the proof of theorem 2.

3. Comments.

Let us denote by K_0 the variety of type τ defined by all identities $f(x_0, \ldots, x_{\tau(f)-1}) = f(y_0, \ldots, y_{\tau(f)-1})$. The proposition (ix) can suggest that if an algebra $\mathfrak{A}$ belongs to a variety K of type τ, then a dispersion $\mathfrak{A}_D$ is isomorphic to a subdirect product of $\mathfrak{A}$ and $\mathfrak{B}$, where $\mathfrak{B} \in K_0$.

The following example shows that this is not the case.

Example 12. Let K be a variety of algebras with two unary fundamental operations f and g defined by the identities

$$f(x) = g(x) = x.$$

Consider an algebra $\mathfrak{A} = (\{a, b, c\}; f, g)$, where

$$f(a) = f(b) = b, \qquad g(a) = g(b) = a,$$

$$f(c) = g(c) = c.$$

Let $\sim$ be an equivalence relation induced by the partition $\{\{a, b\}, \{c\}\}$. Then $\sim$ is a congruence on $\mathfrak{A}$, $\mathfrak{A}|\sim \in K$ and $\mathfrak{A}$ is a dispersion of $\mathfrak{A}|\sim$. By (viii), $\mathfrak{A} \in K_{\mathrm{Ex}}$. However, $\mathfrak{A}$ is not decomposable into a subdirect product of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, where $\mathfrak{A}_1 \in K$ and $\mathfrak{A}_2 \in K_0$. In fact $\mathfrak{A} \notin K$, $\mathfrak{A} \notin K_0$ and the only non-trivial congruence on $\mathfrak{A}$ is the congruence $\sim$.

The next example shows that the assumption (6°) in Theorem 2 is essential.

Example 13. Let K be a variety of algebras with two unary fundamental operations f and g defined by the identities

$$f(x) = g(x), \qquad f(f(f(x))) = f(f(x)).$$

Then the following system of identities forms an equational base of K_{Ex}:

$$\begin{aligned} f(f(f(x))) &= f(f(x)) = f(g(x)) \\ g(g(g(x))) &= g(g(x)) = g(f(x)). \end{aligned} \tag{3.1}$$

In fact any term $\varphi(x)$ of this type can be by means of (3.1) reduced to one of the following forms:

$$x, \quad f(x), \quad f(f(x)), \quad g(x), \quad g(g(x)).$$

In the algebra of terms of our type let us denote $[\varphi(x)] = \varphi(x)/_{\mathrm{Id}(K)}$. Then the free algebra $\mathfrak{F}([x])$ in K with one free generator $[x]$ has five elements, namely:

$$[x], \quad [f(x)], \quad [f(f(x))], \quad [g(x)], \quad [g(g(x))].$$

Let us denote by Θ the equivalence relation induced on $\mathfrak{F}([x])$ by the partition $\{\{[x]\}, \{[f(x)]\}, \{[f(f(x))]\}, \{[g(x)], [g(g(x))]\}\}$. Then Θ is a congruence on $\mathfrak{F}([x])$ and consequently $\mathfrak{F}([x])/_{\Theta} \in K_{\mathrm{Ex}}$.
Putting $a = \{[x]\}$, $b = \{[f(x)]\}$, $c = \{[f(f(x))]\}$, $d = \{[g(x)], [g(g(x))]\}$ we see that $\mathfrak{F}([x])/_{\Theta}$ is isomorphic to the algebra $\mathfrak{A} = (\{a, b, c, d\}; f, g)$, where $f(a) = b$, $f(b) = f(c) = f(d) = c$ and $g(a) = g(b) = g(c) = g(d) = d$. So $\mathfrak{A} \in K_{\mathrm{Ex}}$.
However, $\mathfrak{A}$ is not of the form $\mathfrak{B}_D$ for some algebra $\mathfrak{B} \in K$. In fact, if it is, then by (iv) there exists a congruence $\sim$ on $\mathfrak{A}$ such that $\mathfrak{A}|\sim \in K$ and $\mathfrak{A} = (\mathfrak{A}|\sim)_D$. The reader can check that there are only two congruences Θ_1, Θ_2 on $\mathfrak{A}$ such that the quotient algebras belong to K. These congruences are $\Theta_1 = \iota$ (the greatest congruence) and Θ_2 induced by the partition: $\{\{a\}, \{b, c, d\}\}$. In both cases the condition (4°) is not satisfied since $f(a)$ and $f(b)$ belong to the same congruence class. So $\mathfrak{A}$ is neither a dispersion of $\mathfrak{A}|\Theta_1$ nor $\mathfrak{A}|\Theta_2$.

Problem. Does there exist a variety K of a finite type such that K is finitely based but for some partition P of F, K_P is not finitely based.

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P-СОВМЕСТНЫЕ ТОЖДЕСТВА И ИХ ПРИЛОЖЕНИЯ К КЛАССИЧЕСКИМ АЛГЕБРАМ

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Резюме

Пусть F — множество основных операционных символов многообразия K алгебр типа τ и пусть P-разбиение множества F. Тождество называется P-совместным, если оно имеет вид $x = x$ или же вид $f(\varphi_0, \ldots, \varphi_{\tau(f)-1}) = g(\psi_0, \ldots, \psi_{\tau(g)-1})$, где f и g принадлежат одному и тому же смежному классу разбиения P.

Показывается, что при некоторых предположениях всякая алгебра, удовлетворяющая всем P-совместным тождествам множества $\operatorname{Id} K$, является так называемой P-дисперсией некоторой алгебры из K.