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DOMATIC NUMBER AND LINEAR ARBORICITY OF CACTI

BOHDAN ZELINKA

A cactus is a connected undirected graph with at least two vertices and with the property that each of its edges is contained in at most one of its circuits. Every tree is a cactus, but not conversely. In this paper we shall prove two theorems on numerical invariants of cacti.

The domatic number of a graph was introduced by E. J. Cockayne and S. T. Hedetniemi [2]. A dominating set in an undirected graph \( G \) is a subset \( D \) of the vertex set \( V(G) \) of \( G \) with the property that to each vertex \( x \in V(G) - D \) there exists a vertex \( y \in D \) adjacent to \( x \). A domatic partition of \( G \) is a partition of \( V(G) \), all of whose classes are dominating sets in \( G \). The maximal number of classes of a domatic partition of \( G \) is called the domatic number of \( G \) and denoted by \( d(G) \).

The domatic number of \( G \) may be defined also as the maximal number of colours of a domatic colouring of \( G \). A domatic colouring of \( G \) is a colouring of vertices of \( G \) with the property that to each vertex \( x \) of \( G \) and to each colour \( c \) distinct from the colour of \( x \) there exists a vertex of \( G \) having the colour \( c \) and adjacent to \( x \). (Two vertices of the same colour may be adjacent.) Evidently both definitions are equivalent. The set of all vertices coloured by a given colour is a class of the corresponding domatic partition of \( G \).

The linear arboricity of a graph was introduced by J. Akiyama, G. Exoo and F. Harary [1]. A linear forest is an undirected graph, all of whose connected components are paths. The linear arboricity \( \xi(G) \) of a graph \( G \) is the minimum number of edge-disjoint subgraphs of \( G \) which are linear forests and whose union is whole the graph \( G \).

Before formulating theorems, we shall explain briefly the structure of cacti. Each block of a cactus is either a circuit, or an edge with its end vertices. A cactus consisting of one block will be called trivial; other cacti will be called non-trivial. A block of a non-trivial cactus \( G \) which contains only one articulation of \( G \) will be called a terminal block of \( G \). Evidently each finite non-trivial cactus has at least two terminal blocks.

A path in a graph \( G \) whose inner vertices (if any) have degree 2 and whose terminal vertices have degrees different from 2 will be called a simple path. If
a circuit $C$ in a non-trivial cactus $G$ does not form a terminal block, then it contains at least two articulations and is the union of at least two edge-disjoint simple paths; each of these paths connects two articulations of $G$. The set of these paths will be denoted by $\mathcal{F}(G)$.

For trivial cacti the domatic number is well known. If such a cactus consists of one edge with its end vertices, then evidently its domatic number is 2. If such a cactus $G$ is a circuit, then $d(G) = 3$ if and only if the length of this circuit is divisible by 3, otherwise $d(G) = 2$; this was proved by E. J. Cockayne and S. T. Hedetniemi. Thus it remains to consider non-trivial cacti.

**Theorem 1.** Let $G$ be a finite non-trivial cactus. Then the following two assertions are equivalent:

(i) Each terminal block of $G$ is a circuit of a length divisible by 3 and for any circuit $C$ in $G$ not forming a terminal block the set $\mathcal{F}(C)$ contains either at least one path of length 1, or the number of paths of $\mathcal{F}(C)$ with lengths non-divisible by 3 is different from 1.

(ii) $d(G) = 3$ and there exists a domatic partition of $G$ with 3 classes such that each vertex is adjacent to at most one vertex of the same class and any edge joining two vertices of the same class belongs to a circuit.

If (i) does not hold, then $d(G) = 2$.

**Proof.** First we prove that $2 \leq d(G) \leq 3$ for any finite cactus $G$. The inequality $2 \leq d(G)$ follows from the fact that a cactus has no isolated vertices [2]. The inequality $d(G) \leq 3$ follows from the fact that any cactus contains at least one vertex of degree 1 or 2 (in a non-trivial cactus such a vertex is in its terminal block) and thus the minimal degree $\delta(G) \geq 2$; in [2] it was proved that $d(G) \leq \delta(G) + 1$. Now we prove the equivalence of (i) and (ii).

(i) $\Rightarrow$ (ii). The proof will be done by induction according to the number $k$ of non-terminal blocks of $G$. Let $k = 0$. Then $G$ has only one articulation $a$ which is common to all blocks of $G$. We shall construct a domatic colouring of $G$. The vertex $a$ will be coloured by the colour 1. Now let $B$ be a block of $G$; the block $B$ is a terminal one, hence (if we suppose (i)) it is a circuit of a length divisible by 3. We colour its vertices subsequently by 1, 2, 3, 1, 2, 3, ..., starting at $a$. If we do this with each block of $G$, we obtain a domatic colouring by 3 colours and $d(G) = 3$.

Now let $k = k_0 \geq 1$ and suppose that the assertion is true for $k = k_0 - 1$. Choose a non-terminal block $B_0$ of $G$ and contract all its vertices (the obtained loops are omitted). We obtain a cactus $G_0$ with $k = k_0 - 1$. The contraction of $B_0$ does not change other blocks; thus if $G$ satisfies (i), so does $G_0$. Then there exists a domatic colouring of $G_0$ by three colours satisfying (ii). Let $w$ be the vertex of $G_0$ obtained by contracting the block $B_0$; without loss of generality we may suppose that it is coloured by the colour 1. Suppose that $B_0$ consists of one edge with its end vertices $u$, $v$. At least one of them, say $u$, is adjacent in $G$ to a vertex coloured in the mentioned colouring of $G_0$ by 2. Then $u$ will be coloured by 1 and $v$ by 3. All
Now suppose that \( B \) is a circuit. Then either \( \mathcal{G}(B) \) contains at least one path of the length 1, or the number of paths of lengths non-divisible by 3 is greater than the number of paths of lengths divisible by 3. In the second case we colour first the vertices in \( B \) sequentially by 1, 2, 3. The obtained colouring is a domatic colouring of \( G \) by three colours.

Now let \( G \) be a graph with three colours. Then each vertex of \( G \) of degree 2 must be adjacent to two vertices of different colours. If there are two vertices of degree 2 and thus its three colours, then those vertices are mutually different from its own two colours. If there are more than two vertices of degree 2, then those vertices are mutually different from its own two colours.

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a length at least 2 have the same colour if and only if the length of this path is divisible by 3. Suppose that a circuit $C$ of $G$ which does not form a terminal block has the property that $\mathcal{F}(C)$ contains no path of the length 1 and exactly one path of a length non-divisible by 3. Denote by $a_1, ..., a_m$ the articulations in $C$ in such a way that the pairs $a_i, a_{i+1}$ for $i = 1, ..., m - 1$ are connected by simple paths of lengths divisible by 3 and $a_m, a_0$ are connected by a simple path of a length non-divisible by 3. Then according to the above mentioned assertion we obtain (inductively) that all the vertices $a_1, ..., a_m$ have the same colour, but on the other hand $a_m$ and $a_0$ have different colours, which is a contradiction. Thus (i) must hold.

A numerical invariant of a graph which is closely related to the domatic number is the idomatic number of $G$. An idomatic partition of $G$ is a partition of $V(G)$, each of whose classes is a set which is simultaneously dominating and independent in $G$. If there exists at least one idomatic partition of $G$, then the maximal number of classes of such a partition is called the idomatic number of $G$ and denoted by $id(G)$. If no idomatic partition of $G$ exists, then we put $id(G) = 0$.

**Theorem 2.** Let $G$ be a finite non-trivial cactus. Then the following two assertions are equivalent:

(i) Each terminal block of $G$ is a circuit of a length divisible by 3 and for any circuit $C$ in $G$ not forming a terminal block the number of paths of $S(C)$ with lengths non-divisible by 3 is different from 1.

(ii) $id(G) = 3$.

**Proof.** (i) $\Rightarrow$ (ii). The domatic partition constructed in the first part of the proof of Theorem 1 without using the assumption that for a circuit $C$ in $G$ not forming a terminal block the set $\mathcal{F}(C)$ contains at least one path of length 1 is in tact an idomatic partition. This implies the assertion.

(ii) $\Rightarrow$ (i). If $id(G) = 3$, then evidently also $d(G) = 3$ and (ii) from Theorem 1 is satisfied. Hence (i) from Theorem 1 holds. If a circuit $C$ in $G$ not forming a terminal block has the property that in the set $\mathcal{F}(C)$ there exists exactly one path of a length non-divisible by 3, then this length must be 1. By the consideration from the end of the proof of Theorem 1 we prove that the terminal vertices of such a path must have the same colour in any domatic colouring of $G$ with three colours and thus no domatic partition of $G$ with three classes is idomatic.

**Theorem 3.** Let $G$ be a finite non-trivial cactus not satisfying the condition (i) from Theorem 2. Then $id(G) = 2$ if and only if $G$ is bipartite; otherwise $id(G) = 0$.

**Proof.** is straightforward.

Now we shall prove a theorem concerning the linear arboricity of cacti. The symbol $\lceil x \rceil$ denotes the least integer greater than or equal to $x$ and $A(G)$ denotes the maximum degree of a vertex of $G$. In [1] it is proved that for every tree $T$ the equality $\Xi(T) = \lceil \frac{1}{2} \Delta(T) \rceil$ holds. Further evidently $\Xi(G) \geq \lceil \frac{1}{2} A(G) \rceil$ for every graph $G$, because each linear forest of the required decomposition can contain at most two edges incident with a given vertex. In [1] it is conjectured that for
a regular graph $G$ of the degree $r$ the equality $\Xi(G) = \frac{1}{2}(r+1)$ holds. As a non-regular cactus $G$ can be embedded into a regular graph of the degree $\Delta(G)$, the following result is related to this conjecture.

**Theorem 4.** Let $G$ be a finite non-trivial cactus, let $\Delta(G)$ be the maximum degree of a vertex of $G$. Then

$$\Xi(G) = \frac{1}{2} \Delta(G).$$

**Proof.** We shall carry out the proof by induction according to the number $b(G)$ of blocks of $G$; as $G$ is a non-trivial cactus, we have $b(G) \geq 2$. Let $b(G) = 2$. Then $G$ consists of two blocks. If both these blocks are edges with their end vertices, then $\Delta(G) = 2$ and $G$ is a path, hence $\Xi(G) = 1 = \frac{1}{2} \Delta(G)$. If at least one of the blocks is a circuit, then $G$ is the union of two edge-disjoint paths and $\Xi(G) = 2$, while $\Delta(G) = 3$ or $\Delta(G) = 4$. Now let $b(G) = k \geq 3$ and suppose that the assertion is true for $b(G) = k - 1$. Let $G$ be decomposed into edge-disjoint linear forests and let $u$ be a vertex of $G$ of degree $\Delta(G)$. Each of the forests of the decomposition can contain at most two edges incident with $u$, hence $u$ is contained in at least $\frac{1}{2} \Delta(G)$ such forests and $\Xi(G) \geq \frac{1}{2} \Delta(G)$. Let $B_0$ be a terminal block of $G$, let $a$ be the articulation of $G$ contained in $B_0$. Let $G_0$ be the graph obtained from $G$ by deleting all vertices of $B_0$ except $a$; then $G_0$ is a finite non-trivial cactus and $b(G_0) = k - 1$. According to the induction hypothesis $\Xi(G_0) = \frac{1}{2} \Delta(G_0)$. Let $\mathcal{L}$ be a decomposition of $G_0$ into $\frac{1}{2} \Delta(G_0)$ linear forests. First suppose that $B_0$ consists of one edge $e$ with its end vertices. If the degree of $a$ in $G$ is even, then in $G_0$ it is odd and there exists at least one forest from $\mathcal{L}$ which contains exactly one edge adjacent to $a$. Then we add $B_0$ to this forest and obtain a decomposition of $G$ into $\frac{1}{2} \Delta(G_0)$ edge-disjoint linear forests and evidently $\frac{1}{2} \Delta(G_0) \leq \frac{1}{2} \Delta(G)$. If the degree of $a$ in $G$ is odd, then in $G_0$ it is even. Let $\delta(a)$ be the degree of $a$ in $G_0$. If there is no forest from $\mathcal{L}$ containing exactly one edge adjacent to $a$, then there are $\frac{1}{2} \delta(a)$ forests from $\mathcal{L}$, each from which contains two edges adjacent to $a$. If $\delta(a) < \Delta(G_0)$, then $\frac{1}{2} \delta(a) < \frac{1}{2} \Delta(G_0)$ and there exists at least one forest from $\mathcal{L}$ not containing $a$; we add $B_0$ to this forest and again obtain a decomposition of $G$ into $\frac{1}{2} \Delta(G_0)$ linear forests. If $\delta(a) = \Delta(G_0)$, then $\Delta(G) = \delta(a) + 1 = \Delta(G_0) + 1$. As $\delta(a) = \Delta(G_0)$ is even, we have $\frac{1}{2} \Delta(G) = \frac{1}{2} \Delta(G_0) + 1$. To $\mathcal{L}$ we add $B_0$ as a new forest and we obtain a decomposition of $G$ into $\frac{1}{2} \Delta(G)$ edge-disjoint linear forests.

Now suppose that $B_0$ is a circuit. If the degree of $a$ in $G$ is even, then it is even also in $G_0$. If there are two forests $F_1, F_2$ from $\mathcal{L}$ such that $a$ is incident at most with one edge from each of them, then we decompose $B_0$ into two edge-disjoint paths, each of which has a terminal vertex $a$, and add one of them to $F_1$ and the other to $F_2$; we obtain a decomposition of $G$ into $\frac{1}{2} \Delta(G_0)$ edge-disjoint linear forests. If there is only one such forest $F$, then (as the degree of $a$ is even) it contains no edge incident with $a$. Let $P_1$ be the path whose edges are the two edges of $B_0$ incident with $a$ and let $P_2$ be the path in $B_0$ with the same terminal vertices as $P_1$ and
edge-disjoint with $P_1$. We add $P_1$ to $F$ and $P_2$ to an arbitrary other forest from $\mathcal{L}$ and we obtain a decomposition of $G$ into $\lfloor \Delta(G_0) \rfloor$ edge-disjoint linear forests. If there is no forest with the required property, then $L$ contains $\frac{1}{2}\delta(a)$ forests and $\frac{1}{2}\delta(a) = \lfloor \Delta(G_0) \rfloor$, which implies $\delta(a) = \Delta(G_0)$. The degree of $a$ in $G$ is $\delta(a) + 2 = \Delta(G_0) + 2$ and evidently $\Delta(G) = \Delta(G_0) + 2$, which implies $\lfloor \frac{1}{2}\Delta(G) \rfloor = \lfloor \frac{1}{2}\Delta(G_0) \rfloor + 1$. We use again the paths $P_1$ and $P_2$. The path $P_2$ will be added to an arbitrary forest from $\mathcal{L}$ and the path $P_1$ will form a new forest; thus a required decomposition of $G$ is obtained. If the degree of $a$ in $G$ is odd, then it is odd also in $G$. There exists at least one forest $F$ from $\mathcal{L}$ which contains exactly one edge incident with $a$. If there is another forest $F'$ from $\mathcal{L}$ which has at most one edge incident with $a$, we choose a vertex $b \neq a$ of $B_0$ and take two edge-disjoint paths $P$, $P'$ both connecting $a$ with $b$. We add $P$ to $F$ and $P'$ to $F'$ and obtain a required decomposition of $G$. If there is no such forest $F'$, then there are $\frac{1}{2}(\delta(a) - 1)$ forests of $\mathcal{L}$ having two edges incident with $a$ and the forest $F$ and hence $\Xi(G_0) = \lfloor \frac{1}{2}\Delta(G_0) \rfloor = \frac{1}{2}(\delta(a) + 1)$. Then $\Delta(G_0) = \delta(a)$ or $\Delta(G_0) = \delta(a) + 1$. The degree of $a$ in $G$ is $\delta(a) + 2$ and this is $\Delta(G_0) + 2$ or $\Delta(G_0) + 1$. Evidently also $\Delta(G) = \delta(a) + 2$. We have $\lfloor \frac{1}{2}\Delta(G) \rfloor = \frac{1}{2}(\delta(a) + 3) = \Xi(G_0) + 1$. We add $P$ to $F$ and $P'$ will be a new forest; thus a required decomposition of $G$ is constructed. □

Remark. The assertion of Theorem 4 does not hold for trivial cacti which are circuits; for such a cactus $G$ we have $\Delta(G) = 2$ and $\Xi(G) = 2$. For other trivial cacti the assertion is true.

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ДОМАТИЧЕСКОЕ ЧИСЛО И ЛИНЕЙНАЯ ДРЕВЕСНОСТЬ КАКТУСОВ

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Резюме

Кактус есть связный неориентированный граф $G$ по меньшей мере с двумя вершинами, обладающий тем свойством, что каждое ребро из $G$ содержится по большей мере в одном контуре графа $G$. В статье исследованы доматическое число, идоматическое число и линейная древесность кактусов.